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Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach

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ABSTRACT

In this paper, a stability test procedure is proposed for linear nonhomogeneous fractional order systems with a pure time delay. Some basic results from the area of finite time and practical stability are extended to linear, continuous, fractional order time-delay systems given in state-space form. Sufficient conditions of this kind of stability are derived for particular class of fractional time-delay systems. A numerical example is given to illustrate the validity of the proposed procedure.

1. Introduction

Stability analysis is one of the most important issues for control systems, although this problem has been investigated for time-delay systems over many years [1]. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method, or on using the idea of matrix measure, (see [2–5]). Here, another approach is presented, i.e system stability from the non-Lyapunov point of view (finite and practical stability) is studied [6–9]. Also, analysis of the linear time-delay systems in the context of finite and practical stability was introduced and considered, [10–13]. Recently there have been some advances in control theory of fractional (non-integer order) dynamical systems for stability questions. For example, regarding linear fractional differential systems of finite dimensions in state-space form, both internal and external stabilities are investigated by Matignon [14,15]. A condition based on the argument principle has been established to guarantee the asymptotic stability of the fractional order system. Some properties and (robust) stability results for linear, continuous, (uncertain) fractional order state-space systems are presented and discussed [16,17]. An analytical approach was suggested by Chen and Moore, [18,19], who considered the analytical stability bound using Lambert function W for a class of ordinary/fractional order of delay differential equations. Further, analysis and stabilization of fractional (exponential) delay systems of retarded/neutral type are considered [20,21], and BIBO stability [22]. Recently, for the first time, finite-time stability analysis of fractional time-delay systems is presented and reported on paper [23]. The main contribution of this paper is to propose finite-time stability test procedure linear (non)autonomous time-invariant delay fractional order systems (LTID FOS). Here, a Bellman–Gronwall's approach is proposed, using a recently obtained *generalized* Gronwall inequality reported in [24] as a starting point. The problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of linear (non)autonomous fractional order time-delay systems has been examined.

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2. Preliminaries on integer time-delay systems

A linear, multivariable time-delay system can be represented by the following differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t), \tag{1}$$

and with the associated function of the initial state: $x(t) = \psi_x(t), -\tau \le t \le 0$. In Eq. (1), $x(t) \in \mathrm{R}^n$ is a state vector, $u(t) \in \mathrm{R}^m$ is a control vector, A_0, A_1, B_0 are constant system matrices of appropriate dimensions, and τ is a pure time delay, $\tau = const.$ $(\tau > 0)$. Dynamical behaviour of system (1), with a given initial function is defined over time interval $J = [t_o, t_o + T], J \subset R$ where quantity T may be either a positive real number or a symbol $+\infty$, so finite-time stability and practical stability can be treated simultaneously. Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded. Let index "ε" stands for the set of all allowable states of the system and index "δ" for the set of all initial states of the system, such that the set $S_\delta \subseteq S_\varepsilon$. In general, one may write: $S_\rho = \left\{x : \ \|x(t)\|_Q^2 < \rho\right\}, \rho \in [\delta, \varepsilon]$, where Q will be assumed to be a symmetric, positive definite, real matrix. S_{α_u} denotes the set of all allowable control actions. Let $|x|_{(.)}$ be any vector norm (e.g., $. = 1, 2, \infty$) and $\|(.)\|$ the matrix norm induced by this vector. The initial function can be written in its general form as: $x(t_o + \theta) = \psi_x(\theta), \ -\tau \le \theta \le 0, \psi_x(\theta) \in C[-\tau, 0]$, where t_0 is the initial time of observation of the system (1) and $C[-\tau, 0]$ is a Banach space of continuous functions over a time interval of length τ, mapping the interval $[t-\tau, t]$ into R^n with the norm defined in the following manner: $\|\psi\|_C = \max_{-\tau \le \theta \le 0} \|\psi(\theta)\|$. It is assumed that the usual smoothness condition is present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data.

Definition 2.1. The system given by homogeneous state equation (1) $(u(t) \equiv 0, \forall t)$, satisfying initial condition $x(t) = \psi_x(t), -\tau \le t \le 0$ is finite stable w.r.t. $\{\delta, \varepsilon, t_o, J, \}, \delta < \varepsilon$ if and only if

$$\|\psi\|_C < \delta \tag{2}$$

imply:

$$\|x(t)\| < \varepsilon, \quad \forall t \in J \tag{3}$$

where t_0 denotes the initial time of observation of the system and J denotes time interval $J = [t_o, t_o + T], J \subset R$.

Definition 2.2. System given by (1) satisfying initial condition $x(t) = \psi_x(t), -\tau \le t \le 0$ is finite stable w.r.t. $\{\delta, \varepsilon, \alpha_u, t_o, J\}, \ \delta < \varepsilon$ if and only if:

$$\|\psi\|_C < \delta \tag{4}$$

and

$$\|u(t)\| < \alpha_u, \quad \forall t \in J \tag{5}$$

imply:

$$\|x(t)\| < \varepsilon, \quad \forall t \in J. \tag{6}$$

3. Fundamentals of fractional calculus

The fractional integro-differential operators (fractional calculus–(FC)) is a generalization of integration and derivation to non-integer order (fractional) operators. The idea of FC has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and Marquis de l'Hôpital in 1695. With time, great mathematicians such as Euler, Fourier, Abel and others did some work on the FC that, surprisingly, remained as a sort of curiosity. Further, the theory of FC was developed mainly in the 19th century. In fact, in his 700-page-long book on Calculus published in 1819, Lacroix [25] devoted two pages to FC, showing that $d^{1/2}[v]/dv^{1/2} = 2\sqrt{v}/\sqrt{\pi}$. Moreover, applications of FC are very wide nowadays in rheology, viscoelasticity, acoustics, optics, chemical physics, robotics, control theory of dynamical systems, electrical engineering, bioengineering and so on [26–28].The main reason for the success of applications FC is that these new fractional order models are more accurate than integer order models, i.e. there are more degrees of freedom in the fractional order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a "memory" term in a model. This memory term ensures the history and its impact on the present and future. Three definitions are generally used for the fractional differintegral. The first one is the Grunwald definition, given as [26,28]:

$$D_a^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{7}$$

where a, t are the limits of operator and $[x]$ means the integer part of x. The Riemann–Liouville (RL) definition of fractional derivative and integral are defined as:

Definition 3.1 (*[29]*). Let $\Omega = [a, b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis R and $f(t)$ be a continuous function defined on $[a, b]$. The *left-sided Riemann–Liouville fractional derivative* of order $(\Re(\alpha) \geq 0)$, $\alpha \in \mathrm{C}$, is:

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad (n = [\Re(\alpha)] + 1, t > a). \tag{8}$$

Definition 3.2 (*[29]*). Let $f(t)$ be a continuous function defined on $[a, b]$. The *left-sided Riemann–Liouville fractional integral* of order $\alpha \in \mathrm{C}$, $(\Re(\alpha) > 0)$, is:

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad (t > a, \Re(\alpha) > 0). \tag{9}$$

Without loss of generality we denote $D_{a+}^{\alpha} f(t)$ by $D_a^{\alpha} f(t)$ and $I_{a+}^{\alpha} f(t)$ by $I_a^{\alpha} f(t)$, respectively. Here, $\Gamma(.)$ is the well-known Euler's gamma function which is defined by the so-called *Euler integral of the second kind*:

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t, \quad z \in \mathrm{C}. \tag{10}$$

For this function the *reduction formula* holds:

$$\Gamma(z+1) = z\Gamma(z), \Rightarrow \Gamma(n+1) = n(n-1)! = n! \quad n \in \mathrm{N}_0. \tag{11}$$

From previous definitions, it can be seen that fractional derivative represents a *global* property of the function, (the property of the function on the finite interval), in contrast to the integer order derivative, which is a *local* property, (the property of the function at a single value of the independent variable).

Lemma 3.1 (*[29]*). *If $(\Re(\alpha) > 0)$ and $f(t) \in L_p(a, b)$ $(1 \leq p \leq \infty)$, (where $L_p(a, b)$ are set of those Lebesgue complex-valued measurable functions f on $\Omega = [a, b]$, $(-\infty < a < b < \infty)$ for which $\|f\|_p < \infty$) then the following equality:*

$$D_{a+}^{\alpha} \left(I_{a+}^{\alpha} f(t) \right) = f(t), \tag{12}$$

holds almost everywhere on $[a, b]$. There is also another definition of fractional differintegral introduced by Caputo [30]. *Namely, left-sided Caputo fractional derivative of order $\alpha \in \mathrm{C}$, $(\Re(\alpha) \geq 0)$ on $[a, b]$ are defined via the above Riemann–Liouville fractional derivative by*

$${}^{C}D_{a+}^{\alpha} f(t) = D_{a+}^{\alpha} \left(\left[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (s-a)^k \right] \right)(t), \tag{13}$$

where

$$n = \begin{cases} [\Re(\alpha) + 1] & \text{for } \alpha \notin \mathrm{N}_0 = \{0, 1, 2 \ldots\} \\ \alpha, & \text{for } \alpha \in \mathrm{N}_0. \end{cases} \tag{14}$$

If $\alpha \notin \mathrm{N}_0$ and for function $f(t)$ there exist Caputo and RL fractional derivatives then the following relation between the two fractional derivatives holds:

$${}^{C}D_{a+}^{\alpha} f(t) = D_{a+}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)!} (t-a)^{k-\alpha}, \quad n = [\Re(\alpha) + 1]. \tag{15}$$

The Caputo and Riemann–Liouville formulation coincide when the initial conditions are zero. In particular, Caputo fractional derivative is defined for function $f(t)$ which belongs to the space $AC^n[a, b] = \{f : \mathrm{d}^{n-1} f(t)/\mathrm{d}t^{n-1} \in AC[a, b]$ and $[a, b] \to \mathrm{C}\}$, $n \in \mathrm{N} := \{1, 2, 3, \ldots\}$ of absolutely continuous functions, [31]. Thus the following statement holds.

Theorem 3.1 (*[31]*). *Let $(\Re(\alpha) \geq 0)$ and let n be given by* (14). *If $f(t) \in AC^n[a, b]$ then the Caputo fractional derivative ${}^{C}D_{a+}^{\alpha} f(t)$ exists almost everywhere on $[a, b]$.*

(a) *If $\alpha \notin \mathrm{N}_0$, then*

$${}^{C}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s. \tag{16}$$

(b) *If $\alpha = n \in \mathrm{N}_0$, then*

$${}^{C}D_{a+}^{\alpha} f(t) = f^{(n)}(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}. \tag{17}$$

Theorem 3.2 (*[24] Generalized Gronwall Inequality*). *Suppose $x(t)$, $a(t)$ are nonnegative and local integrable on $0 \le t < T$, some $T \le +\infty$, and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M = const.$, $\alpha > 0$ with*

$$x(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1} x(s)\mathrm{d}s \tag{18}$$

on this interval. Then

$$x(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1} a(s)\right]\mathrm{d}s, \quad 0 \le t < T. \tag{19}$$

Corollary 3.1 (*Of Theorem 3.2 [24]*). *Under the hypothesis of Theorem 3.2, let $a(t)$ be a nondecreasing function on $[0, T)$. Then holds:*

$$x(t) \le a(t)E_\alpha\left(g(t)\Gamma(\alpha)t^\alpha\right) \tag{20}$$

where E_α is the Mittag-Leffler function defined by $E_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(k\alpha+1)$.

4. Main results

Recently, in [32] the authors considered a variable prehistory of $x(t)$ in $t < 0$, and its effects were taken into account in defining the fractional derivative in terms of the initialization function. Moreover, using short memory principle [26] and taking into account initial function one can obtain the correct initial function where it is assumed that there is no difficulty with questions of continuity of solutions with respect to initial data (function). Also, a new theory of electroviscoelasticity describes the behaviour of electrified liquid–liquid interfaces in fine dispersed systems, and is based on a new constitutive model of liquids [33]. Taking into account small transport time delay τ, electromagnetic oscillation of the "*continuum*" particle can be obtained by the linear time delay fractional order of differential equation, [34]. Here, it is considered a class of fractional linear autonomous system with time delay described by the state-space equation:

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = A_0x(t) + A_1x(t-\tau) \tag{21}$$

and nonautonomous

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = A_0x(t) + A_1x(t-\tau) + B_0u(t), \tag{22}$$

with associated function of initial state: $x(t) = \psi_x(t), -\tau \le t \le 0$. In this paper, we discuss the case $n = 1,\ 0 < \alpha < 1$. Here, we examine the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of linear (non)autonomous fractional order time-delay systems.

Theorem 4.1. *The linear nonautonomous system given by (22) satisfying initial condition $x(t) = \psi_x(t), -\tau \le t \le 0$ is finite-time stable w.r.t. $\{\delta, \varepsilon, \alpha_u, J_0, \}$, $\delta < \varepsilon$, if the following condition is satisfied:*

$$\left(1 + \frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\sigma_{\max 01}t^\alpha\right) + \frac{\gamma_{u0}^{\bullet}t^\alpha}{\Gamma(\alpha+1)} \le \varepsilon/\delta, \quad \forall t \in J_0 = \{0, T\}, \tag{23}$$

where $\gamma_{u0}^{\bullet} = \alpha_u b_0/\delta$, and $\sigma_{\max}^{(.)}$ being the largest singular value of matrix (.), where:

$$\sigma_{\max 01} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1).$$

Proof of Theorem 4.1. In accordance with the property of the fractional order $0 < \alpha < 1$, one can obtain a solution in the form of the equivalent Volterra integral equation:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(A_0x(s) + A_1x(s-\tau) + B_0u(s)\right)\mathrm{d}s. \tag{24}$$

Applying the norm $\|(.)\|$ on Eq. (24) and using appropriate property of the norm, it follows that:

$$\|x(t)\| \le \|x(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right| \|A_0x(s) + A_1x(s-\tau) + B_0u(s)\|\,\mathrm{d}s. \tag{25}$$

Also, applying the norm $\|(.)\|$ on Eq. (22), one can obtain:

$$\left\|\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right\| \leq \|A_0\|\,\|x(t)\| + \|A_1\|\,\|x(t-\tau)\| + \|B_0\|\,\|u(t)\| \\ \leq (\sigma_{\max}(A_0))\,\|x(t)\| + (\sigma_{\max}(A_1))\,\|x(t-\tau)\| + \|B_0\|\,\|u(t)\|, \tag{26}$$

where $\|A\|$ denotes the induced norm of a matrix A, as well as,

$$\|x(t-\tau)\| \leq \sup_{t-\tau\leq t^\bullet \leq t}\|x(t^\bullet)\|. \tag{27}$$

Applying this inequality, Eq. (26) can be presented in the following manner:

$$\left\|\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right\| \leq \sigma_{Ao}\,\|x(t)\| + \sigma_{A1}\sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\| + b_0\,\|u(t)\| \\ \leq \sigma_{\max 01}\sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\| + b_o\,\|u(t)\|, \quad t > t_0+\tau, \tag{28}$$

or

$$\|A_0x(t) + A_1x(t-\tau) + B_0u(t)\| \leq \sigma_{\max 01}\left(\sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\| + \|\psi_x\|_C\right) + +b_0\,\|u(t)\|, \quad t > t_{0^+}. \tag{29}$$

Taking into account (29) and (25), it yields:

$$\|x(t)\| \leq \|x(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)|^{\alpha-1}\left\{\sigma_{\max 01}\left(\sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\| + \|\psi_x\|_C\right) + b_0\,\|u(t)\|\right\}\mathrm{d}s, \tag{30}$$

or

$$\|x(t)\| \leq \|\psi_x\|_C\left[1 + \frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\right] + \frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\|\,\mathrm{d}s + \frac{1}{\Gamma(\alpha+1)}(\alpha_u b_0)t^\alpha. \tag{31}$$

Obviously, one can introduce nondecreasing function $a(t)$ such as:

$$a(t) = \|\psi_x\|_C\left(1 + \frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\right). \tag{32}$$

Now, one may apply generalized Gronwall inequality, [24], here, Corollary 3.1 of (Theorem 3.2) (20). Obviously, it is easy to show:

$$\|x(t)\| \leq \sup_{t-\tau\leq t^\bullet\leq t}\|x(t^\bullet)\| \leq a(t)\,E_\alpha\left(\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\Gamma(\alpha)\,t^\alpha\right) = a(t)\,E_\alpha\left(\sigma_{\max 01}t^\alpha\right), \tag{33}$$

and

$$\|x(t)\| \leq \delta\left(1 + \frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\sigma_{\max 01}t^\alpha\right) + \frac{1}{\Gamma(\alpha+1)}(\alpha_u b_0)t^\alpha. \tag{34}$$

Hence, using the basic condition of Theorem 4.1, relation (23) yields:

$$\|x(t)\| < \varepsilon, \quad \forall t \in J_0. \tag{35}$$

This is a proof of the theorem. □

Remark 4.1. If $\alpha = 1$, see (22), one can obtain same conditions which related to integer order time-delay systems (3) as follows [13]:

$$\left[1 + \frac{\sigma^A_{\max}(t-t_0)^1}{1}\right]\cdot \mathrm{e}^{\frac{\sigma^A_{\max}(t-t_0)^1}{1}} + \gamma\cdot\frac{(t-t_0)^1}{1} \leq \varepsilon/\delta, \quad \forall t \in J, \Gamma(2) = 1, \\ E_{\alpha=1}(z) = \mathrm{e}^z. \tag{36}$$

Theorem 4.2. *The linear autonomous system given by Eq. (21) satisfying initial condition $x(t) = \psi_x(t)$, $-\tau \leq t \leq 0$ is finite-time stable w.r.t. $\{\delta, \varepsilon, J_0, \}$, $\delta < \varepsilon$, if the following condition is satisfied:*

$$\left(1 + \frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha\left(\sigma_{\max 01}t^\alpha\right) \leq \varepsilon/\delta, \quad \forall t \in J_0. \tag{37}$$

Proof of Theorem 4.2. The proof immediately follows from the proof of Theorem 4.1 applying the same procedure taking into account Eqs. (20) and (37).

Remark 4.2. If $\alpha = 1$, see (21), one can obtain same conditions which relate to integer order time-delay systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0 x(t) + A_1 x(t-\tau), \tag{38}$$

as follows (see [12]):

$$\left[1 + \frac{\sigma_{\max 01} t^1}{1}\right] \cdot \mathrm{e}^{\frac{\sigma_{\max 01} t^1}{1}} \leq \varepsilon/\delta, \quad \forall t \in J_0 = [0, T]\,. \tag{39}$$

5. An illustrative example

Using a time-delay PD^α compensator on a linear system of equations with respect to the small perturbation $e(t) = y(t) - y_d(t)$,one can obtain:

$$\dot{e}(t) + \omega e(t) = K_P e(t-\tau) + K_D de^{(\alpha)}(t-\tau)/\mathrm{d}t^\alpha + u(t), \tag{40}$$

where $\alpha = 1/2,\ \omega = 2,\ K_p = 3,\ K_D = 4,\ u\,(t)$-feedforward control

Also, all initial values are zeros. Introducing:

$$x_1(t) = e(t), \qquad x_2(t) = d^{1/2}e(t)/\mathrm{d}t^{1/2}, \tag{41}$$

one can write (39) in state-space form:

$$\begin{aligned} &D_t^\alpha x_1(t) = D_t^{1/2} e(t) = x_2(t), \\ &D_t^\alpha x_2(t) = D_t^{1/2}\left(D_t^{1/2} e(t)\right) = \dot{e}(t) = -2x_1(t) + 3x_1(t-\tau) + +4x_2(t-\tau) + u(t) \end{aligned} \tag{42}$$

or, in condensed form, where $x(t) = (x_1, x_2)^\mathrm{T}$, one can obtain this as:

$$D_t^{1/2} x(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{43}$$

or

$$D_t^{1/2} x(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t), \tag{44}$$

with an associated function of the initial state:

$$x(t) = \psi_x(t) = 0, \quad -\tau \leq t \leq 0. \tag{45}$$

Also, there is the task of checking the finite-time stability w.r.t.
$\{t_0 = 0,\ J = \{0, 2\}\,,\ \delta = 0.1,\ \varepsilon = 100,\ \tau = 0.1, \alpha_u = 1\}$, where $\psi_x(t) = 0, \forall t \in [-0.1, 0]$.
From the initial data and Eqs. (43) and (22) one can obtain:

$$\begin{aligned} &\|\psi_x(t)\|_C < 0.1, \\ &\sigma_{\max}\,(A_0) = 2, \qquad \sigma_{\max}\,(A_1) = \sqrt{3^2 + 4^2} = 5, \Rightarrow\ \sigma_{\max 0,1} = 7. \end{aligned} \tag{46}$$

Applying the condition of Theorem 4.2 (22) one can get:

$$\left[1 + \frac{7T_\mathrm{e}^{0.5}}{0.886}\right] \cdot E_{0.5}\left(7T_\mathrm{e}^{0.5}\right) + \frac{10 \cdot T_\mathrm{e}^{0.5}}{0.886} \leq 100/0.1 \Rightarrow T_e \approx 0.1\ \mathrm{s}. \tag{47}$$

T_e being "estimated time" of finite time stability.

6. Conclusion

In this paper, finite-time stability analysis for a class of linear fractional order systems with time invariant delay was considered. New stability criteria for this class of fractional order systems were derived by applying generalized Gronwall inequality. In that way, one can check system stability over a finite time. Finally, an illustrative example for this class of system was presented.

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