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The Results of Common Fixed Points in b-Metric Spaces

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Abstract: In this paper, we present some results on the existence and uniqueness of common fixed points on d^*-complete topological spaces. Our results generalize and improve upon earlier results in the literature. Finally, we give some examples in l_p spaces, ($p \in (0,1)$), where we use the obtained results.

Keywords: common fixed point; b-metric spaces; (SC) property

MSC: 54H25; 47H10; 54E25

1. Introduction

In the past 50 years, non-metric generalizations of Banach's fixed point theory and its applications have played an important role in nonlinear analysis; see [1–5]. There are many definitions of extended metric spaces (in these spaces, the distance need not satisfy the triangle inequality or need not be symmetric). Some examples of such spaces are d-complete L spaces or Kasahara spaces [6] (see also [7]). Hicks [8] first introduced the notion of d-complete topological spaces and obtained the topological properties of those spaces. In the paper [9], d-complete topological spaces were extended via the d^*-complete topological spaces. Fixed point theory for non-commutative mappings was introduced by Kannan [10] and further developed by Srivastava and Gupta [11], Wong [12] and Ćirić [13]. Classical results have been presented by Wong [12] and Ćirić [13], which were extended by George et al. [14] on b-metric spaces.

In this paper, we obtain some theorems that draw out significant results in [9], about the existence and uniqueness of fixed point obtained for d^*-complete topological spaces. The significance of our improvement is that we obtained results about common fixed points for two mappings that are not required to have a commutative property. Our results are shown at complete b-metric spaces $CbMS$ with an (SC) property.

Our results generalize previous results of Wong [12], Ćirić [13], Bianchini [15], Bryant [16], Caccioppoli [17], Marjanović [18], Reich [19], Tasković [20], Yen [21] and Zamfirescu [22] on b-metric spaces.

Moreover, we take into consideration some properties of b-spaces, a class of topological spaces which belong to E-spaces (spaces with regular écart) and include metric spaces. However, we draw the attention of the reader to the fact that we use $CbMS$ for a complete b-metric space and bMS for a b-metric space.

2. Preliminary Notes

In this section, we list several well-known definitions, remarks and lemmas.

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Definition 1. *Suppose that Ω and Γ are topological spaces. The mapping $f : \Omega \to \Gamma$ is sequentially continuous if for every sequence $(\omega_n) \subseteq \Omega$,*

$$\lim_{n\to+\infty} \omega_n = p \text{ implies } \lim_{n\to+\infty} f\omega_n = fp.$$

For $\omega_0 \in \Omega$, we affirm that the sequence (ω_n) defined by $\omega_n = f^n\omega_0$ is a sequence of Picard iterates for f at point ω_0 or that (ω_n) is the orbit of f at point ω_0.

Suppose that Ω is a nonempty set, $f : \Omega \to \Omega$ is a mapping and $x \in \Omega$ is a fixed point for f if $fx = x$.

The first part of the following statement was developed and proved by Adamović [23]. Its second part was presented in [24].

Lemma 1. *Let Ω be a nonempty set and $f : \Omega \to \Omega$ be a mapping. Let $l \in \mathbb{N}$, where f^l has a unique fixed point u_*. Then,*

(1) u_* is the unique fixed point for f;
(2) If Ω is the topological space and a Picard-iterated sequence defined by f^l converges to u_*, then the sequence of Picard iterates defined along f converges to u_*.

Definition 2 ([8,9])**.** *Suppose that Ω is a Hausdorff topological space and $d : \Omega \times \Omega \to [0, +\infty)$ is a mapping. We define the next three valuable properties:*

(a) For any $\omega, \theta \in \Omega$, $d(\omega, \theta) = 0$ if and only if $\omega = \theta$;
(b) For every sequence $(\omega_n) \subseteq \Omega$,

$$\sum_{n=0}^{\infty} d(\omega_n, \omega_{n+1}) < \infty \text{ implies that } (\omega_n) \text{ converges in } \Omega; \tag{1}$$

(c) For each sequence of $(\omega_n) \subseteq \Omega$, if there is $L > 0$ and $\lambda \in [0, 1)$, where

$$d(\omega_n, \omega_{n+1}) \le L\lambda^n, \tag{2}$$

for $n = 0, 1, 2, \ldots$, then (ω_n) converges in Ω.

If (Ω, d) satisfies conditions (a) and (b) ((a) and (c)), then we say that (Ω, d) is a d-complete topological space (d^*-complete topological space).

Remark 1. *Obviously, any d-complete topological space (Ω, d) is d^*-complete; however, the converse is not true (see [9]).*

In the article [25], Fréchet established the classes of metric spaces E-spaces. The historical development of the theory of b-spaces was revisited in the paper by Berinde and Păcurar [26].

Definition 3. *The triplet (B, ρ, s), where B is a nonempty set, $\rho : B \times B \to [0, +\infty)$ and $s > 0$, is a bMS with constant s if the following conditions hold:*

(B_1) $\rho(u, v) = 0$ if and only if $u = v$,
(B_2) $\rho(u, v) = \rho(v, u)$,
(B_3) $\rho(u, v) \le s[\rho(u, w) + \rho(w, v)]$,

for all $u, v, w \in B$.

Remark 2. *(i) It is clear that $(B, \rho, 1)$ is a metric space.*
(ii) If $v = w$ is put into (B_3), we obtain $\rho(u, v) \le s[\rho(u, v) + \rho(v, v)] = s\rho(u, v)$. So, in a b-metric space (B, ρ, s), we have $s \ge 1$.

Here are some results that can be seen in [27,28].

In every b-metric space (B, ρ, s), one can propose the topology τ_ρ on behalf of defining the family of closed sets as follows:

A set $A \subseteq B$ is closed if and only if for every $u \in B$, $\rho(u, A) = 0$ implies that $u \in A$, where

$$\rho(u, A) = \inf\{\rho(u, a) : a \in A\}.$$

The convergence of the sequence (u_n) in the topology τ_ρ is not necessarily implied $\rho(u_n, u) \to 0$, although, the converse is true (see [24]).

Many notions in bMS would be the same as those in metric spaces.

Definition 4. *A sequence $(u_n) \subseteq B$ is said to be a Cauchy sequence if for a given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $d(u_m, u_n) < \varepsilon$, for all $m, n \geq N_\varepsilon$. A bMS (B, ρ, s) is said to be complete if every Cauchy sequence converges to some $u \in \Omega$.*

Let $r > 0$ and $u \in B$. By

$$B(u, r) = \{v \in \Omega : \ \rho(u, v) < r\},$$

we denote an open ball with a center u and a radius r.

Many properties of bMS would be the same as those in metric spaces (but, it is not all because there is no triangle inequality). For example, each bMS is a Hausdorff space.

Further, An et al. [29] proved that every bMS is a semi-metrizable space (for a definition of a semi-metrizable space, see [24]), but there exists bMS in which open balls are not open sets. Additionally, every bMS satisfies the first axiom of countability, which implies that continuity and sequential continuity in bMS are equivalent notions.

In [30], Miculescu and Mihail proved the following result. Its simple and short proof was presented by Mitrović [31].

Lemma 2. *Suppose that (B, ρ, s) is a bMS and sequence $\{u_n\} \subseteq B$. If there exists $\lambda \in [0, 1)$ such that*

$$\rho(u_{n+1}, u_n) \leq \lambda \rho(u_n, u_{n-1}) \tag{3}$$

for all $n \in \mathbb{N}$, then $\{u_n\}$ is Cauchy.

Remark 3. *From Lemma 2, we have that CbMS is a d^* complete topological space.*

Definition 5. *We say that b-metric space (B, ρ, s) has the property (SC) if*

$$\lim_{n \to +\infty} \rho(u_n, u) = 0 \textit{ implies } \overline{\lim}_{n \to +\infty} \rho(u_n, v) = \rho(u, v), \tag{4}$$

where $(u_n) \subset B, u, v \in B$.

Remark 4. *In [24], it was proved that bMS has the property (SC) if all its open balls are open sets in topology τ_ρ.*

3. Main Results

In the following results, some properties of the finite product of $CbMS$ are given. We use the notation $[n] = \{1, \ldots, n\}$, where $n \in \mathbb{N}$.

Lemma 3. *Let $(B_i, \rho_i, s_i), i \in [n]$ be bMS, $B = B_1 \times \cdots \times B_n$ and $\rho : B^2 \to [0, +\infty)$ be defined by*

$$\rho((p_1, \ldots, p_n), (q_1, \ldots, q_n)) = \max\{\rho_i(p_i, q_i) : i \in [n]\}, \tag{5}$$

where $p_i, q_i \in B_i, i \in [n]$. Let $s = \max\{s_i : i \in [n]\}$. Then,

(1) (B, ρ, s) is a b-metric space;

(2) (B, ρ, s) is complete if and only if $(B_i, \rho_i, s_i), i \in [n]$ is complete;

(3) (B, ρ, s) *has the property (SC) if and only if* $(B_i, \rho_i, s_i), i \in [n]$ *has the property (SC).*

Proof. (1) Conditions (1), (2) and (3) are trivial-satisfied.

(2) Suppose that (B, ρ, s) is $CbMS$. Suppose that $k \in [n]$, $(u_i^k) \subset B_k$ is Cauchy sequence in (B_k, ρ_k, s_k) and $p^j \in B_j$ such that $j \in [n] \backslash \{k\}$. Then, sequence $(u_i) = (u_i^1, \dots, u_i^n) \in B$ defined by

$$u_i^j = \begin{cases} u_i^k, \text{ if } j = k, \\ p^j, \text{ if } j \neq k, \end{cases}$$

is a Cauchy sequence in (B, ρ). So, x_i converges. Hence, we have that (u_i^k) converges.

Now, assume that (B_k, ρ_k, s_k) is a $CbMS$ for each $k \in [n]$, and that sequence $(u_i) \subseteq B$ defined by $u_i = (u_i^1, \dots, u_i^n)$ is a Cauchy sequence in (B, ρ, s). We can see that then, $(u_i^k) \subset B_k$ is a Cauchy sequence in (B_k, ρ_k). Let $\lim_{i \to +\infty} x_i^k = p^k, k \in [n]$. Then, for each $\varepsilon > 0$ and every $k \in [n]$, there exists $m_0^k \in \mathbb{N}$, where $i \geq m_0^k$ implies $\rho_(u_i^k, p^k) < \varepsilon$. Let $p = (p^1, \dots, p^n)$. So, $i \geq m_0 = \max\{m_0^1, \dots, m_0^n\}$ implies $\rho(u_i, p) \leq \varepsilon$. Hence, $\lim_{i \to +\infty} x_i = p$.

(3) Let $U_i \subset B_i, i \in [n]$ be open balls and $U = U_1 \times \cdots \times U_n$. Then, U is an open set in topology τ_ρ if each $U_i, i \in [n]$ is an open set in topology τ_{ρ_i}. So, if U is open, then all U_i are open.

Further, if $B(p, r) \subseteq B$, where $p = (p_1, \dots, p_n)$ is an open set, then $B(p_i, r) \subseteq B_i$ is an open set as a projection of $B(p, r)$ to B_i. □

Next a common fixed point theorem extends previous fixed point results presented by Mitrović et al. [9] and Tasković [20].

Theorem 1. *Let* $(B_i, \rho_i, s_i), i \in [n]$ *be a CbMS with the (SC) property. Let* $B = B_1 \times \cdots \times B_n$ *and* $\rho : B^2 \to [0, +\infty)$ *be a mapping defined as*

$$\rho((\omega_1, \dots, \omega_n), (v_1, \dots, v_n)) = \max\{\rho_i(\omega_i, y_i) : i \in [n]\}. \tag{6}$$

Let $s = \max\{s_i : i \in [n]\}$, $f_i, g_i : B \to B_i, i \in [n]$ *and* $F, G : B \to B$ *be defined by* $F = (f_1, \dots, f_n)$ *and* $G = (g_1, \dots, g_n)$. *Suppose that*

$$\rho(F\omega, G\theta) \leq \lambda \rho(\omega, \theta) \tag{7}$$

for all $\omega, \theta \in \Omega$ *and some* $\lambda \in [0, 1)$. *Then, F and G have a unique common fixed point* $\Theta \in \Omega$. *Also,* Θ *is a unique limit of all Picard sequences defined by F and a unique limit of all Picard sequences defined by G.*

Proof. Suppose that $\Theta_0 = (\omega_1^0, \dots, \omega_n^0) \in B$ and (Θ_i) is a sequence defined by $\Theta_{2i+1} = F\Theta_{2i}$ and $\Theta_{2i+2} = G\Theta_{2i+1}$. We have that

$$\rho(\Theta_{i+1}, \Theta_{i+2}) \leq \lambda \rho(\Theta_i, \Theta_{i+1}) \tag{8}$$

for $i = 0, 1, 2, \dots$. Now, from (8), Lemmas 2 and 3, we conclude that there exists $\Theta \in B$ where $\Theta = \lim_{i \to +\infty} \Theta_i$. Also, we have

$$\Theta = \lim_{i \to +\infty} \Theta_i = \lim_{i \to +\infty} \Theta_{2i+1} = \lim_{i \to +\infty} F\Theta_{2i}$$

and

$$\Theta = \lim_{i \to +\infty} \Theta_i = \lim_{i \to +\infty} \Theta_{2i+2} = \lim_{i \to +\infty} G\Theta_{2i+1}.$$

From

$$\rho(F\Theta, \Theta_{2i+2}) \leq \lambda \rho(\Theta, \Theta_{2i+1}),$$

using the (SC) property, we obtain

$$\overline{\lim}_{i\to+\infty}\rho(F\Theta,\Theta_{2i+2}) \leq \lambda\overline{\lim}_{i\to+\infty}\rho(\Theta,\Theta_{2i+1}) = 0.$$

Therefore, $F\Theta = \lim_{i\to+\infty}\Theta_{2i+2} = \Theta$. Further, from

$$\rho(G\Theta,\Theta_{2i+1}) \leq \lambda\rho(\Theta,\Theta_{2i}),$$

again, using the (SC) property, we have

$$\overline{\lim}_{i\to+\infty}\rho(G\Theta,\Theta_{2i+1}) \leq \lambda\overline{\lim}_{i\to+\infty}\rho(\Theta,\Theta_{2i}) = 0.$$

So, $G\Theta = \lim_{i\to+\infty}\Theta_{2i+1} = \Theta$. Let $\omega \in B$, where $F\omega = \omega$ and $\omega \neq \Theta$. Then, we deduce that

$$\rho(\omega,\Theta) = \rho(F\omega,G\Theta) \leq \lambda\rho(\vartheta,\Theta).$$

This is a contradiction.

Similarly, let $\omega \in B$ where $G\omega = \omega$ and $\omega \neq \Theta$. Thereafter, we have

$$\rho(\omega,\Theta) = \rho(G\omega,F\Theta) \leq \lambda\rho(\omega,\Theta).$$

This is a contradiction.

So, Θ is a unique common fixed point for both F and G.

We obtain the convergence of Picard sequences defined by F and Picard sequences defined by G from

$$\rho(F^{n+1}\omega,G\Theta) \leq \lambda\rho(F^n\omega,\Theta) \leq \ldots \leq \lambda^{n+1}\rho(\vartheta,\Theta),$$

and

$$\rho(G^{n+1}\omega,F\Theta) \leq \lambda\rho(G^n\omega,\Theta) \leq \ldots \leq \lambda^{n+1}\rho(\omega,\Theta).$$

□

Remark 5. *In [32], the authors used additive metrics instead of* max *and obtained similar results.*

From Theorem 1, we obtain the next corollary which generalizes the well-known results initiated by Bryant [16].

Corollary 1. *Let (B,ρ,s) be a CbMS, $\lambda \in [0,1)$, $f,g : B \to B$ and $n \in \mathbb{N}$, where*

$$\rho(f^n\omega, g^n\theta) \leq \lambda\rho(\omega,\theta), \tag{9}$$

for all $u,v \in B$. Then, both f and g have a unique common fixed point $q \in B$. Also, q is a unique limit of all Picard sequences defined along f and a unique limit of all Picard sequences defined along g.

By Corollary 1, we arrive at the following common fixed point result that provides the theorem for Yen [21].

Corollary 2. *Let (B,ρ,s) be a CbMS, $\lambda \in [0,1)$, $f,g : B \to B$ and $m,n \in \mathbb{N}$, where*

$$\rho(f^m\omega, g^n\omega) \leq \lambda\rho(\omega,\theta), \tag{10}$$

for all $\omega,\theta \in \Omega$. Then, f and g have a unique common fixed point $q \in B$. Also, q is a unique limit of all Picard sequences defined by f and a unique limit of all Picard sequences defined by g.

Proof. Put $\omega = f^n\mu$ and $\theta = g^m\nu$, where $\mu,\nu \in B$. We have that f^{m+n} and g^{m+n} hold all conditions of Corollary 1. □

By Corollary 1, we obtain the next common fixed point, which expands upon the well-known theorem of R. Caccioppoli [17].

Corollary 3. *Let (B, ρ, s) be a CbMS, $\lambda \in [0, 1)$, $f, g : B \to B$, (c_n) be a sequence such that $\sum_{n=1}^{+\infty} c_n < +\infty, c_n \geq 0, n \in \mathbb{N}$ and*

$$\rho(f^n\omega, g^n\theta) \leq c_n\rho(\omega, \theta) \tag{11}$$

for all $\omega, \theta \in B, n \in \mathbb{N}$. Then, f and g have a unique common fixed point $q \in B$. Also, q is a unique limit for all Picard sequences defined along f and a unique limit for all Picard sequences defined along g.

Proof. For some positive integer n, we have $c_n < 1$. Now, the statement follows from Corollary 1. □

Lemma 4. *Let (B, ρ, s) be a CbMS with an (SC) property, $\lambda \in [0, 1)$, $f, g : B \to B$ and $\rho_* : B^2 \to [0, +\infty)$ be defined by $\rho_*(\omega, \theta) = 0$ for $\omega = \theta$ and*

$$\rho_*(\omega, \theta) = \max\{\rho(\omega, \theta), \rho(f\omega, g\theta), \ldots, \rho(f^n\omega, g^n\theta)\} \tag{12}$$

for $\omega \neq \theta$. Then, space (B, ρ_*, s) is a CbMS with an (SC) property.

Proof. The space (B, ρ_*, s) is bMS because conditions (B1), (B2) and (B3) are trivial-satisfied. Also, we have $\rho(\omega, \theta) \leq \rho_*(\omega, \theta)$ for any $\omega, \theta \in X$. Further, if $(\theta_j) \subseteq B$ is an arbitrary Cauchy sequence in (B, ρ_*), (θ_j) is a Cauchy sequence in (B, ρ), which implies that (B, ρ_*) is $CbMS$ because (B, ρ, s) is complete. Further, for every $k \in [n]$, we have $\overline{\lim}_{i\to+\infty}\rho(f^k\omega, g^k\theta_i) \leq \rho(f^k\omega, g^k\theta)$, which implies that $\overline{\lim}\rho_*(\omega, \theta_n) \leq \rho_*(\omega, \theta)$. Hence, (B, ρ, s) has the property (SC). □

The following theorem extends the previous results presented by M. Marjanović [18], from CMS to $CbMS$.

Theorem 2. *Suppose that (B, ρ, s) is a CbMS that satisfies the (SC) property, and $\lambda \in [0, 1)$ and $f, g : B \to B$ are two mappings such that*

$$\rho(f^{n+1}\omega, g^{n+1}\theta) \leq \lambda \max_{0\leq i\leq n}\{\rho(f^i\omega, g^i\theta)\}, \tag{13}$$

for all $\omega, \theta \in \Omega$. Then, f and g have a unique common fixed point $z \in B$, which is a unique limit for all Picard sequences defined along f and a unique limit for all Picard sequences defined along g.

Proof. By Lemma 4, space (B, ρ_*, s) is a $CbMS$ that has the (SC) property. Further, we have that

$$\rho_*(f^{n+1}\omega, g^{n+1}\theta) \leq \lambda\rho_*(\omega, \theta).$$

Based on the one-dimensional case of Theorem 1, it follows that f^{n+1} and g^{n+1} have a unique common fixed point, say q, which is a unique limit for all Picard sequences defined by f^{n+1} and a unique common fixed point that is a unique limit for all Picard sequences defined by f^{n+1}. By Lemma 1 we obtain that q is a unique fixed point for f and a unique limit for all Picard sequences defined along f, and q is a unique fixed point for g and a unique limit for all Picard sequences defined along g. Hence, q is a unique common fixed point for f and g. □

Next, the common fixed point theorem extends the results from George et al. [14].

Theorem 3. *Suppose that (B, ρ, s) is a CbMS, $f, g : B \to B$ and $\alpha, \beta \in [0, 1)$, where $\alpha + 2\beta < 1$ and*

$$\rho(fx, gy) \leq \max\left\{\rho(u, v), \rho(u, fu), \rho(v, gv), \frac{\rho(u, gv) + \rho(fu, v)}{2s}\right\} + \frac{\beta[\rho(u, gv) + \rho(fu, v)]}{s}, \tag{14}$$

for all $u, v \in B$. If one of the next conditions are satisfied,

(1) f and g are sequentially continuous or ρ is sequentially continuous;

(2) (B, ρ, s) has the (SC) property and $s(\alpha + \beta) < 1$,

then f and g have a unique common fixed point $z \in B$. Also, a sequence (u_n) defined by $u_{2n+1} = fx_{2n}, u_{2n+2} = gu_{2n+1}, n = 0, 1, 2, \ldots$, where $u_0 \in B$, converges to w.

Proof. Let $u_0 \in B$ be arbitrary and a (u_n) sequence defined by $u_{2n+1} = fu_{2n}$ and $u_{2n+2} = gu_{2n+1}$. Then, there exists $w \in \Omega$ such that $w = \lim u_n$; the proof is the same as that in [14] (Theorem 13).

Case (1): If f and g are sequentially continuous or ρ is sequentially continuous, then by Theorem 13 in [14], we obtain that f and g have a unique common fixed point $w \in \Omega$. The rest of the proof is like that in Case (2).

Case (2): Let (B, ρ, s) satisfy the (SC). From (14), we have

$$\rho(fw, gu_{2n+1}) \leq \alpha \max\{\rho(w, u_{2n+1}), \rho(w, fw), \rho(u_{2n+1}, gu_{2n+1}), \frac{\rho(u_{2n+1}, fw) + \rho(gu_{2n+1}, w)}{2s}\} + \frac{\beta[\rho(u_{2n+1}, fw) + \rho(gu_{2n+1}, w)]}{s},$$

which implies

$$\overline{\lim}\rho(fw, gu_{2n+1}) \leq \overline{\lim}[\alpha \max\{\rho(w, u_{2n+1}), \rho(w, fw), \rho(u_{2n+1}, u_{2n+2}), \frac{\rho(u_{2n+1}, fw) + \rho(gu_{2n+1}, w)}{2s}\} + \beta\frac{\rho(u_{2n+1}, fw)}{s} + \beta\frac{\rho(gu_{2n+1}, w)}{s}].$$

So, we have that

$$\overline{\lim}\rho(gu_{2n+1}, fw) \leq \alpha \max\{\rho(w, w), \rho(w, fw), \rho(w, w), \frac{\rho(w, fw) + \rho(w, w)}{2s}\} + \beta\frac{\rho(w, fw)}{s} + \beta\frac{\rho(w, w)}{s}].$$

Therefore,

$$\overline{\lim}\rho(gu_{2n+1}, fw) \leq (\alpha + \beta)\rho(w, fw).$$

Now, we have

$$\rho(w, fw) \leq s[\overline{\lim}\rho(gu_{2n+1}, fw) + \overline{\lim}\rho(gu_{2n+1}, w)] \leq s(\alpha + \beta)\rho(w, fw). \tag{15}$$

Since $s(\alpha + \beta) \in [0, 1)$ from (15), we have $w = f(w)$. Further, we have

$$\rho(fu_{2n}, gw) \leq \alpha \max\{\rho(u_{2n}, w),), \rho(u_{2n}, fu_{2n}), \rho(w, gw) \frac{\rho(u_{2n}, gw) + \rho(fu_{2n}, w)}{2s}\} + \beta\frac{\rho(u_{2n}, gw}{s} + \beta\frac{\rho(fu_{2n}, w)}{s},$$

which implies

$$\overline{\lim}\rho(fu_{2n}, gw) \leq \overline{\lim}[\alpha \max\{\rho(u_{2n}, w), \rho(u_{2n}, u_{2n+1}), \rho(w, gw), \frac{\rho((u_{2n}, gz) + \rho(u_{2n+1}, w))}{2s}\} + \beta\frac{\rho(u_{2n}, gw)}{s} + \beta\frac{\rho(fu_{2n}, w)}{s}].$$

So, we have that

$$\overline{\lim}\rho(fu_{2n}, gw) \leq \alpha \max\{\rho(w,w), \rho(w, fw), \rho(w,w), \frac{\rho(w, fw) + \rho(w,w)}{2s}\} + \beta\frac{\rho(w, fw)}{s} + \beta\frac{\rho(w,w)}{s}].$$

Therefore,

$$\overline{\lim}\rho(fu_{2n}, gw) \leq (\alpha + \beta)\rho(w, gw).$$

Now,

$$\rho(w, gw) \leq s[\overline{\lim}\rho(fu_{2n}, gw) + \overline{\lim}\rho(fu_{2n}, w)] \leq s(\alpha + \beta)\rho(w, gw),$$

so $w = g(w)$.

Further, we prove that the fixed point is unique. Suppose that there are two common fixed points w and w', i.e. $gw = fw = w$ and $gw' = fw' = w'$. Then, we obtain

$$\begin{aligned} d(w, w') = d(fw, gw') &\leq \alpha\{d(w, w'), d(w, w)d(w', w'), \frac{d(w, w') + d(w, w')}{2s}\} \\ &+ \beta\frac{d(w, w')}{2s} + \beta d(w, w') \leq (\alpha + 2\beta)d(w, w'), \end{aligned}$$

which implies that $w = w'$.

Finally, we prove the convergence of sequences of a corresponding Picard iteration:

$$\rho(f^{n+1}u, gw) \leq \lambda\rho(f^n u, w)) \leq \ldots \leq \lambda^{n+1}\rho(u, w),$$

and

$$\rho(g^{n+1}v, fw) \leq \lambda\rho(g^n v, w)) \leq \ldots \leq \lambda^{n+1}\rho(v, w).$$

□

The next Corollary extends the known results presented by Reich [19], Bianchini [15], Singh [33], Srivastava, and Gupta [11] and Ray [34] to b-metric spaces.

Corollary 4. *Suppose that (B, ρ, s) is a CbMS, $\lambda \in [0, 1)$ and $f, g : B \to B$. Suppose that*

$$d(f\omega, g\theta) \leq \lambda \max\{\rho(\omega, \theta), \rho(\omega, f\omega), \rho(\theta, g\theta)\}, \quad (16)$$

for all $\omega, \theta \in B$ and one of the following conditions is satisfied:

(1) f and g are sequentially continuous or ρ is sequentially continuous;
(2) (B, ρ, s) satisfies the (SC) property and $s\lambda < 1$.

Then, f and g have a unique common fixed point $w \in \Omega$. Also, w is a unique limit of all Picard sequences defined by f and a unique limit of all Picard sequences defined by g.

Finally, we prove the following statement, which extends earlier results presented by Marjanović [18] and Zamfirescu [22].

Corollary 5. *Let (B, ρ, s) be a CbMS with the (SC) property, $\lambda \in [0, 1)$ and $f, g : \Omega \to \Omega$. If there exist positive integers i, j and k such that $m = \max\{i, j, k\}$*

$$\rho(f^{m+1}\omega, g^{m+1}\theta) \leq \lambda \max\{\max_{0 \leq n \leq i}\{\rho(f^n\omega, g^n\theta)\}, \frac{\max\limits_{0 \leq n \leq j-1}\{\rho(f^n\omega, g^{n+1}\theta\} + \max\limits_{0 \leq n \leq k-1}\{\rho(g^n\omega, f^{n+1}\theta)\}}{2}\},$$

for all $\omega, \theta \in \Omega$, then f and g have a unique common fixed point $w \in X$. Also, w is a unique limit of all Picard sequences defined by f and a unique limit of all Picard sequences defined by g.

Proof. By Lemma 4, the space (B, ρ_*, s) is a $CbMS$ with the (SC) property. Further, we have that

$$\rho_*(f^{m+1}\omega, g^{m+1}\theta) \leq \lambda \max\{\rho_*(\omega, \theta), \rho_*(\omega, f\theta), \rho_*(g\omega, \theta))\}.$$

By Theorem 3, we obtain that f^{m+1} and g^{m+1} have a unique common fixed point q, and q is a unique limit of all Picard sequences defined by f^{n+1} and a unique limit of all Picard sequences defined by f^{n+1}. By Lemma 1, we obtain that q is a unique fixed point for f and a unique limit of all Picard sequences defined by f, and q is a unique fixed point for g and a unique limit of all Picard sequences defined by g. Hence, q is a unique common fixed point for f and g. □

Remark 6. *Note that Theorem 3 generalizes the classical results presented by Ćirić [13] and Wong [12] obtained in complete metric spaces.*

4. Some Examples

Example 1. *The space $l^p = \{\{u_n\} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |u_n|^p < +\infty, p \in (0,1)\}$, together with the function $d_p : l^p \times l^p \to \mathbb{R}$ defined by*

$$d_p(u, v) = \left(\sum_{n=1}^{+\infty} |u_n - v_n|^p \right)^{\frac{1}{p}}$$

where $u = \{u_n\}, v = \{v_n\} \in l^p$ is not a metric space (the function d_p does not satisfy the triangle inequality), but (l^p, d_p, s) is a b-metric space with $s = 2^{\frac{1}{p}-1}$ [35,36]. Let $f, g : l^p \to l^p, i = 1,2$ be defined by

$$f(u) = \begin{cases} (0, \frac{u_1}{4}, \frac{u_2}{4}, \frac{u_3}{4}, \ldots), & \text{if } x \neq (1,0,0,0,\ldots), \\ (\frac{1}{4}, 0, 0, 0, \ldots), & \text{if } u = (1,0,0,0,\ldots), \end{cases}$$

$$g(v) = (0, \frac{v_1}{4}, \frac{v_2}{4}, \frac{v_3}{4}, \ldots).$$

Then, we have

(1) If $u, v \in l^p \backslash \{(1,0,0,0,\ldots)\}$, then it is $d_p(fu, gv) \leq \frac{1}{4} d_p(u, v)$;
(2) If $u = (1,0,0,0,\ldots), v \in l^p \backslash \{(1,0,0,0,\ldots)\}$, then it is

$$\begin{aligned} d_p(fu, gv) &= d_p((\frac{1}{4}, 0, 0, 0, \ldots), (0, \frac{v_1}{4}, \frac{v_2}{4}, \frac{v_3}{4}, \ldots)) \\ &= \frac{1}{4}(1 + \sum_{i=1}^{+\infty} |y_1|^p)^{\frac{1}{p}} \\ &= \frac{1}{4} d_p(u, gv); \end{aligned}$$

(3) If $u \in l^p \backslash \{(1,0,0,0,\ldots)\}, v = (1,0,0,0,\ldots)$, then we obtain

$$\begin{aligned} d_p(fu, gv) &= d_p((0, \frac{u_1}{4}, \frac{u_2}{4}, \frac{u_3}{4}, \ldots), (0, \frac{1}{4}, 0, 0, \ldots)) \\ &= \left(\left| \frac{u_1 - 1}{4} \right|^p + \left| \frac{u_2}{4} \right|^p + \left| \frac{u_3}{4} \right|^p + \left| \frac{u_3}{4} \right|^p + \cdots \right)^{\frac{1}{p}} \\ &= \frac{1}{4} d_p(u, v); \end{aligned}$$

(4) If $u = v = (1,0,0,0,\ldots)$, we obtain

$$\begin{aligned} d_p(fu, gv) &= (\frac{1}{4}, 0, 0, 0, \ldots), (0, \frac{1}{4}, 0, 0, \ldots)) \\ &= \left(\frac{1}{4^p} + \frac{1}{4^p}\right)^{\frac{1}{p}} \\ &= \frac{2^{\frac{1}{p}}}{4}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_p(u, fu) &= d_p((1, 0, 0, 0, \ldots), (\frac{1}{4}, 0, 0, 0, \ldots)) \\ &= \frac{3}{4}. \end{aligned}$$

So, we conclude that the conditions of Corollary 4 are satisfied for $\lambda = \frac{2^{\frac{1}{p}}}{3} < 1$ *and* $p \in (\log_3 2, 1)$.

Example 2. *Let* $B = [0, 1]$ *and* $\rho(a, b) = (a - b)^2$ *for any* $a, b \in B$. *Then,* $(B, \rho, 2)$ *is CbMS and* ρ *is sequentially continuous. Let*

$$f(u) = \begin{cases} \frac{u}{2}, \text{ if } u \in [0, 1); \\ \frac{9}{10}, \text{ if } u = 1, \end{cases}$$

and $g(v) = \frac{v}{2}$ *for every* $v \in [0, 1]$. *We have that* $\rho(u, v) = \frac{1}{4}(u - v)^2$ *for* $u, v \in [0, 1)$. *Further, we have*

$$\rho(f1, g0) = \frac{81}{100} < 1 = \rho(1, 0)$$

and

$$\rho(f1, g1) = \frac{16}{100} < \frac{25}{100} = \rho(1, g0).$$

Since the conditions of Corollary 4 are satisfied for $\lambda = \frac{9}{10}$, *we conclude that* f *and* g *have a unique common fixed point* $w \in \Omega$, *which is a unique limit of all Picard sequences defined by* f *and a unique limit of all Picard sequences defined by* g.

5. Conclusions

We present some theorems that extended some results on the existence and uniqueness of a fixed point obtained for d^*-complete topological spaces in [9]. The significance of our improvement is that we obtained results about common fixed points for two mappings that do not possess the property of commutativity. Our results are given on *CbMS* with the (SC) property. Our results generalize previous results in the literature. Also, we considered some properties of b-spaces, a class of topological spaces that belong to E-spaces and includes metric spaces.

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