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REALIZATION OF THE BRACHISTOCHRONIC MOTION OF CHAPLYGIN SLEIGH IN A VERTICAL PLANE WITH AN UNILATERAL NONHOLONOMIC CONSTRAINT

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Abstract. The paper considers the procedure for determining the brachistochronic motion of the Chaplygin sleigh in a vertical plane, where the blade is such that it prevents the motion of the contact point in one direction only. The position of the sleigh mass center and orientation at the final positions is specified, as well as the initial value of mechanical energy. The simplest formulation of a corresponding optimal control problem is given and it is solved by applying Pontryagin's maximum principle. For some cases, analytical solutions of differential equations of the two-point boundary value problem (TPBVP) of the maximum principle were found. Numerical integration was carried out for other cases using the shooting method, where the assessment of missing terminal conditions was given and it was shown that the solution obtained represents the global minimum time for the brachistochronic motion. The method of the brachistochronic motion by means of a single holonomic and a single unilateral nonholonomic mechanical constraint is presented.

1. Introduction

Classical Bernoulli's problem of determining the brachistochrone for the particle in a vertical plane [1] has experienced many attempts at this problem generalizations for different, more complex, mechanical systems. A more detailed review of literature devoted to these generalizations can be found in the PhD dissertation [2], as well as in papers [3–14]. The results presented in [5, 7, 9, 11, 15, 16] represent the generalization of the classical brachistochrone problem to the case of the brachistochronic motion of a rigid body. Our paper is dedicated to determining the brachistochronic motion of Chaplygin sleigh in a vertical plane, which was also the research subject in [17]. However, our paper differs basically from the mentioned paper by a unilateral nonholonomic constraint, unlike classical bilateral

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constraint of the blade type, that is present in [17]. Details about this type of constraint can be found in paper [18], which presents an appropriate blade profile that corresponds to a unilateral constraint. Note that the results published so far for the Chaplygin sleigh motion are based on the assumption that Chaplygin sleigh moves on a horizontal plane (see e.g. [8, 19–27]), or on an inclined one [22].

Let us consider a Chaplygin sleigh of mass m and the radius of inertia i with respect to the central axis moving in a vertical plane as it is shown in Fig. 1. The sleigh position is defined by coordinates X and Y of the mass center C and angle φ between axis O_x and blade at point C. The sleigh position is specified at the initial and final instant, but the initial value of mechanical energy is also known, $E = mgL$, which is large enough for the sleigh to reach the final position, where g is acceleration of gravity. Let U and V be the projections of the velocity of point C onto a blade and perpendicular to a blade, respectively. If the constraint is bilateral [17], then $V = 0$, this paper will also consider the cases of unilateral constraints, when $V \geqslant 0$ or $V \leqslant 0$, depending on two possible cases, related to the side to which the constraint restrains. The same figure also shows profiles of the corresponding blades for all mentioned cases.

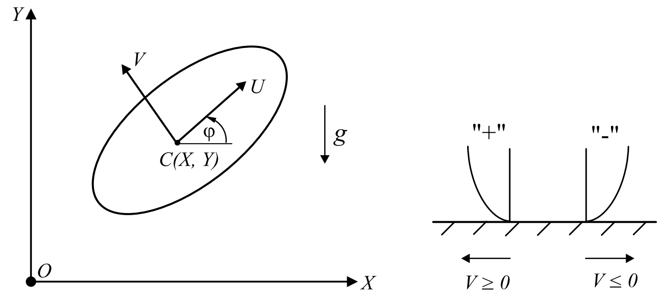

FIGURE 1. A Chaplygin sleigh in a vertical plane.

During brachistochronic motion, which in the original understanding of this problem is realized by ideal mechanical constraints, without action of the active forces, the mechanical energy remains unchanged:

$$\frac{1}{2}m(U^2 + V^2 + i^2\Omega^2) + mgY = mgL$$

where L is a given constant, which has the dimension of length.

In this paper, the brachistochronic motion is determined using only kinematic differential equations:

$$\begin{aligned}\dot{X} &= U\cos\varphi - V\sin\varphi\\ \dot{Y} &= U\sin\varphi + V\cos\varphi\\ \dot{\varphi} &= \Omega\end{aligned}$$

which significantly simplify this problem, considering the determination of the brachistochronic motion itself. The variable Ω represents the angular velocity of the body. It is only remarked herein that when the reaction R_c is defined from dynamic equations, it must be checked whether its direction corresponds to the unilateral constraint orientation.

To determine the brachistochronic motion, which corresponds to a minimum time between two specified positions, Pontryagin's maximum principle is applied [28, 29]. This paper is an expansion of [30], with a numerical example added on which we have merging of subintervals, where $V = 0$ or $V \neq 0$, as well as the possibility of analyzing realization of the brachistochronic motion, by the help of a single holonomic and a single unilateral nonholonomic constraint. The paper is organized as follows: Section 2 presents the simplest possible formulation of the optimal control problem. Section 3 describes the procedure of solving this optimization task with special reference to the possibility of obtaining analytical solutions. Illustration of the procedure given by using concrete numerical examples is reported in Section 4. Realization of the motion by means of ideal mechanical constraints is shown in Section 5.

2. Formulation of the optimal control problem

Introducing dimensionless quantities:

$$X = ix, \qquad \Omega = \omega\sqrt{\frac{g}{i}}, \qquad U = u\sqrt{gi}, \qquad L = il$$

$$Y = iy, \qquad V = \pm v^2\sqrt{gi}, \qquad t = \tau\sqrt{\frac{i}{g}},$$

the equations of state are obtained:

$$\begin{aligned} x' &= u\cos\varphi \mp v^2\sin\varphi \\ y' &= u\sin\varphi \pm v^2\cos\varphi \\ \varphi' &= \omega \end{aligned} \tag{2.1}$$

where the appropriate sign, "+" or "−", corresponds to the direction for the case of a corresponding blade. The notation $(\ldots)'$ represents differentiation with respect to dimensionless time τ. The principle of conservation of mechanical energy for the considered problem in dimensionless variables reads

$$u(\tau)^2 + v(\tau)^4 + \omega(\tau)^2 + 2y(\tau) - 2l = 0. \tag{2.2}$$

The initial conditions of motion are:

$$\tau_0 = 0, \quad x(\tau_0) = 0, \quad y(\tau_0) = 0, \quad \text{and} \quad \varphi(\tau_0) = 0 \tag{2.3}$$

whereas the final position is defined according to:

$$x(\tau_1) = x_1, \quad y(\tau_1) = y_1, \quad \varphi(\tau_1) = \varphi_1 \tag{2.4}$$

where τ_1 is unknown instant. The problem of determining the brachistochronic motion consists of determining the optimal controls:

$$u = u(\tau), \quad v = v(\tau), \quad \omega = \omega(\tau)$$

and their corresponding final equations of motion of this system so that the system is converted from the initial state (2.3) in a minimum time τ_1 into the state (2.4) with the constraint (2.2). The optimal control problem has the form:

$$(\min) J_0\big(u(\cdot), v(\cdot), \omega(\cdot), x(\cdot), y(\cdot), \varphi(\cdot)\big) = \int_0^{\tau_1} d\tau = \tau_1$$

subject to the state equations (2.1) and the conditions (2.2)–(2.4).

3. Solving the optimal control problem

In order to solve this problem, let us write the appropriate Pontryagin's function H for the case of time minimization [28, 29]:

$$\begin{aligned} H = -1 &+ \lambda_x(u\cos\varphi \mp v^2\sin\varphi) + \lambda_y(u\sin\varphi \pm v^2\cos\varphi) \\ &+ \lambda_\varphi\omega - \mu(u^2 + v^4 + \omega^2 + 2y - 2l) \end{aligned}$$

where μ is the multiplier corresponding to the constraint of mechanical energy (2.2) and λ_x, λ_y, λ_φ are co-state variables. The co-state system of differential equations [28, 29] reads:

$$\lambda_x' = 0\lambda_y' = 2\mu, \quad \lambda_\varphi' = -\big(\lambda_x(-u\sin\varphi \mp v^2\cos\varphi) + \lambda_y(u\cos\varphi \mp v^2\sin\varphi)\big). \tag{3.1}$$

Optimality conditions of the maximum principle [28, 29]:

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0, \quad \frac{\partial H}{\partial \omega} = 0$$

yield expressions for optimal controls:

$$\begin{aligned} u &= \frac{1}{2\mu}(\lambda_x\cos\varphi + \lambda_y\sin\varphi) \\ v^2 &= 0 \vee v^2 = \frac{\pm 1}{2\mu}(-\lambda_x\sin\varphi + \lambda_y\cos\varphi) \\ \omega &= \frac{1}{2\mu}\lambda_\varphi \end{aligned}$$

where the multiplier μ, in cases when independent variable τ at the end of motion is not specified, is defined from the condition:

$$H(\tau) = 0 \tag{3.2}$$

and it has the value:

$$\mu(\tau) = \frac{1}{4(l - y(\tau))} > 0$$

which is positive over the considered interval of motion.

In order to be the maximum of Pontryagin's function, the corresponding second-order derivatives must be negative:

$$\frac{\partial^2 H}{\partial u^2} < 0, \quad \frac{\partial^2 H}{\partial v^2} < 0, \quad \frac{\partial^2 H}{\partial \omega^2} < 0$$

wherefrom the criterion is finally obtained for the choice of an appropriate control:

$$
\begin{aligned}
&u = 2(l-y)(\lambda_x \cos\varphi + \lambda_y \sin\varphi) \\
&v^2 = \begin{cases} 0, \pm(-\lambda_x \sin\varphi + \lambda_y \cos\varphi) \leqslant 0 \\ \pm 2(l-y)(-\lambda_x \sin\varphi + \lambda_y \cos\varphi), \ \pm(-\lambda_x \sin\varphi + \lambda_y \cos\varphi) > 0 \end{cases} \\
&\omega = 2(l-y)\lambda_\varphi.
\end{aligned} \tag{3.3}
$$

It can be seen that, in choosing the solution of (3.3), the function $S = (-\lambda_x \sin\varphi + \lambda_y \cos\varphi)$ has the major role, whose sign is the basis for distinguishing segments of motion over which $V = 0$ or $V \neq 0$.

Differential equations of the two-point boundary value problem (TPBVP) are obtained by substituting (3.3) in (2.1) and (3.1). It is necessary to supplement boundary conditions (2.3) and (2.4) with condition (2.2) at the beginning or end of the interval of motion in order to, also, define unknown dimensionless time of motion τ_1.

Here, it should be noted that, in a general case, different solutions of (3.3) can be combined over different intervals of motion depending on the sign of a corresponding expression. In a general case, there are no analytical solutions of differential equations of the TPBVP if $v^2 = 0$ over the entire interval of motion or over some subintervals of motion. In those cases, the problem must be solved numerically. If the shooting method is applied and backward numerical integration is performed, by the choice of three parameters, λ_x, λ_{y1}, τ_1, three initial conditions are guessed (2.3), where, based on (2.2), (3.2) and (3.3):

$$\lambda_\varphi(\tau_1) = \pm\sqrt{\frac{1}{2(l-y_1)} - (\lambda_x \cos\varphi_1 + \lambda_{y1} \sin\varphi_1)^2}$$

and it should be checked which sign corresponds to the task concrete parameters.

Assessment of the interval of parameters' values is also obtained from (2.2):

$$
\begin{aligned}
|\lambda_x \cos\varphi_1 + \lambda_{y1} \sin\varphi_1| &\leqslant \frac{1}{\sqrt{2(l-y_1)}} \\
|\lambda_x| &\leqslant \frac{1}{\sqrt{2l}} \\
|\lambda_{y1}| &< \frac{1}{\sqrt{2(l-y_1)}}
\end{aligned} \tag{3.4}
$$

and it significantly assists in shooting as well as in seeking the global minimum if there are multiple solutions for TPBVP.

If the constraint is side-oriented so that the extremal solution over the entire interval is on an open set, differential equations of TPBVP have a simpler form:

$$
\begin{aligned}
x' &= 2(l-y)\lambda_x & \lambda_x' &= 0 \\
y' &= 2(l-y)\lambda_y & \lambda_y' &= \frac{1}{2(l-y)} \\
\varphi' &= 2(l-y)\lambda_\varphi & \lambda_\varphi' &= 0
\end{aligned}
$$

and have general solutions in the analytical form:

$$
\begin{aligned}
y &= l - \frac{1+\cos(pt+\alpha)}{p^2} \\
x &= \frac{2\lambda_x}{p^2}\Big(t + \frac{1}{p}\sin(pt+\alpha)\Big) + C_1 \\
\varphi &= \frac{2\lambda_\varphi}{p^2}\Big(t + \frac{1}{p}\sin(pt+\alpha)\Big) + C_2
\end{aligned}
$$

where $(p, \alpha, \lambda_x, \lambda_\varphi, C_1, C_2)$ are determined together with unknown moment τ_1 from (2.2), (2.3) and (2.4). It should be noted that these solutions also correspond to the case when the blade does not exist at all, consequently for the general case of the brachistochronic plane-parallel motion of a rigid body.

Form and structure of optimal controls will depend on whether the constraint is bilateral, unilateral and whether it exists at all. If unilateral, it will depend on corresponding direction, but also on initial conditions (2.3), final conditions (2.4) and quantity l, as well.

4. Numerical examples

Let us show for the concrete parameters of the task:

$$
l = 2, \quad x_1 = \varphi_1 = \frac{\pi+2}{2\sqrt{2}}, \quad y_1 = 1 \tag{4.1}
$$

The procedure is described in previous section, particularly for the constraints $V \geqslant 0$ (Case "+") and $V \leqslant 0$ (Case "−").

In the case "−", it is shown that over the entire interval of motion the solution is on an open set and that there are analytical solutions of the corresponding TPBVP:

$$
\begin{aligned}
&x = \frac{(\tau+\sin\tau)}{\sqrt{2}}, \qquad y = 1-\cos\tau, \qquad \varphi = \frac{(\tau+\sin\tau)}{\sqrt{2}} \\
&\lambda_x = \frac{1}{2\sqrt{2}}, \qquad \lambda_y = \frac{\sin\tau}{2(1+\cos\tau)}, \qquad \lambda_\varphi = \frac{1}{2\sqrt{2}}, \qquad \tau_1^- = \frac{\pi}{2}
\end{aligned} \tag{4.2}
$$

It can be also noticed that the mass center trajectory is a deformed cycloid with the coefficient $\frac{1}{\sqrt{2}}$.

In the case "+", it is shown that over the entire interval of motion the extremal solution is on the boundary $V = 0$. Based on the assessments (3.4) of all possible values of the missing parameters, considering their positive signs, graphical representation can be given of the surfaces that correspond to obtaining the appropriate initial conditions. Figure 2 shows three surfaces of different colors, each of which corresponds to the fulfillment of one of the initial conditions (2.3).

These surfaces have been obtained based on numerical calculations of the corresponding TPBVP in the program Wolfram Mathematica [31] using commands NDSolve[...],ContourPlot3D[...] and they represent numerical dependencies:

$$
x_0(\lambda_x, \lambda_{y1}, \tau_1) = 0, \quad y_0(\lambda_x, \lambda_{y1}, \tau_1) = 0, \quad \varphi_0(\lambda_x, \lambda_{y1}, \tau_1) = 0
$$

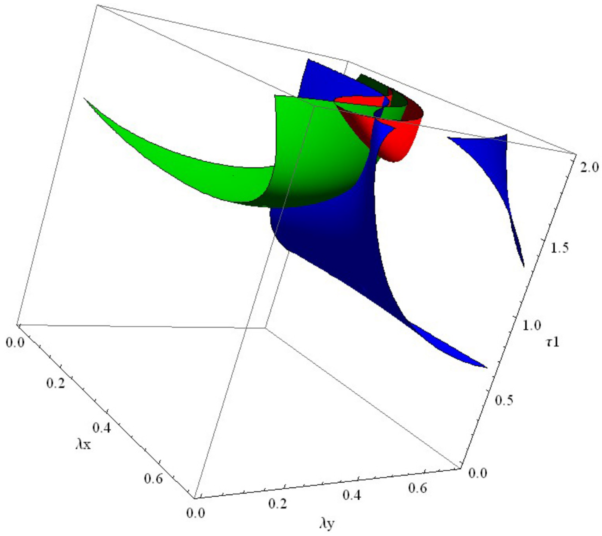

FIGURE 2. Representation of the global minimum time of motion.

The parametric values sought are at their intersection:

$$\lambda_x = 0.499037, \quad \lambda_{y1} = 0.369326, \quad \tau_1^+ = 1.76731 \tag{4.3}$$

and it can be also seen that the point of intersection is at the lowest position along the axis of dimensionless moment τ_1, whereby it is numerically shown that the obtained solution represents the global minimum time of motion. This method of graphical representation for any three-parameter shooting is very useful when TPBVP has multiple solutions, of which the one with the lowest time should be chosen [8].

Numerical solutions correspond to parametric values (4.3), as shown in Fig. 3, together with analytical solutions (4.2), which correspond to opposite orientation of a unilateral constraint.

It can be noticed that the time of motion in Case "$-$", τ_1^-, is lower compared to τ_1^+ in Case "+". The first case corresponds to the brachistochronic plane-parallel motion of a rigid body in the vertical plane, while the second case corresponds completely to the case of a classical double-sided blade, as analyzed in [17]. The same figure depicts the trajectory of the mass center for both cases. In addition, based on the numerical and analytical solution, it can be shown that the function $S = -\lambda_x \sin\varphi + \lambda_y \cos\varphi$ is negative over the entire interval of motion and that conditions (3.3) of the maximum principle are fully satisfied (see Fig. 4).

Such task parameters can be defined, contour conditions and initial mechanical energy, so that the brachistochronic motion contains segments where optimal solution is from an open set and segments on which $V = 0$. Let the numerical parameters (4.1) be slightly modified

$$l = 2, \quad x_1 = \varphi_1 = 1.25, \quad y_1 = 1$$

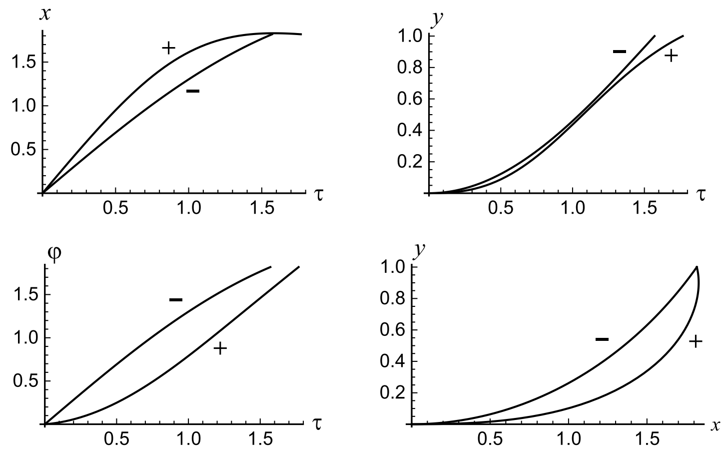

FIGURE 3. Final equations of motion and mass center trajectory of the Chaplygin sleigh.

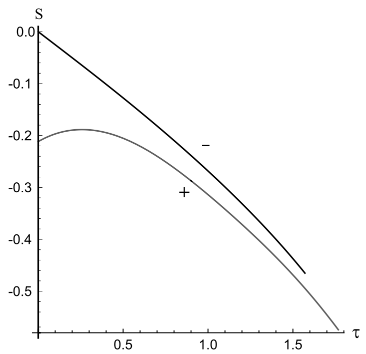

FIGURE 4. Function $S = -\lambda_x \sin\varphi + \lambda_y \cos\varphi$.

where only Case "+" will be considered. Solving the TPBVP defined by (2.1)–(2.4), (3.1) and (3.3) yields values of the parameters (4.3):

$$\lambda_x = 0.378358, \quad \lambda_{y1} = 0.475233, \quad \tau_1 = 1.18207$$

for which Fig. 5 shows the diagram of changes in dimensionless velocities and angular velocity, where it is seen that there are segments on which $V = 0$ or $V > 0$. The same figure gives the function $S = -\lambda_x \sin\varphi + \lambda_y \cos\varphi$, whose sign conditions the optimal change of velocity (3.3).

5. Realization of motion by means of ideal mechanical constraints

Also, it would be necessary to show the method of realization by the help of ideal constraints, which would also be the generalization of paper [24], for the case of guides, or paper [7], if motion is realized by rolling of the moving centroid along

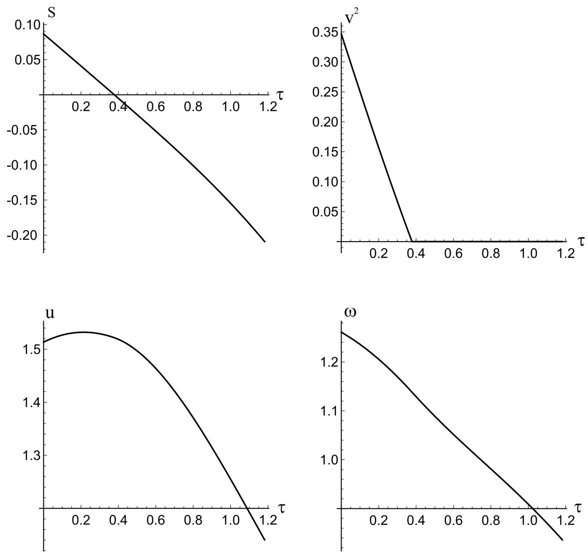

FIGURE 5. Diagrams of dimensionless velocities, angular velocity and function S.

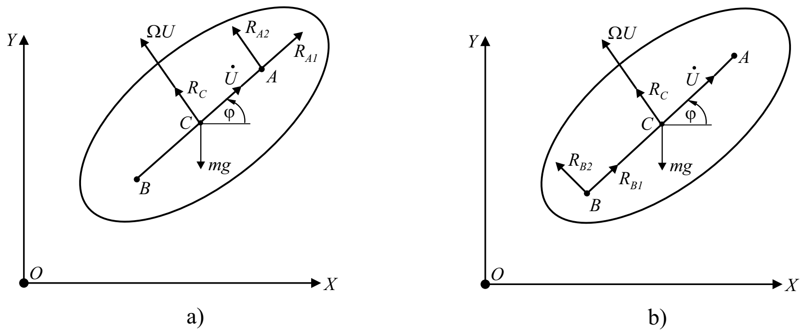

FIGURE 6. Constraint reactions for the case of an additional guide: a) at point A, b) at point B.

the stationary one. This additional research would require the usage of dynamical differential equations of the Chaplygin sleigh motion.

The brachistochronic motion (4.2) in case "$-$" can be fully realized like in paper [7].

In this section, we opt for the realization of the brachistochronic motion using an additional holonomic constraint of the guide type. Two cases are considered, when the guide is at point A or at point B, where $AC = BC = i$ (see Fig. 6).

Let's limit ourselves to the case "+" where the solutions of earlier considered TPBVP with parameters (4.3) are known now. On this solution, for the entire time of motion $V = 0$. If the guide is at point A, dynamical equations have dimensionless form ($R_C = mgr_C$, $R_A = mgr_A$, $R_B = mgr_B$):

$$u' = r_{A1} - \sin\varphi, \quad \omega u = r_{A2} + r_C - \cos\varphi, \quad \omega' = r_{A2}$$

whereas the same equations for the case of guides at point B are:

$$u' = r_{B1} - \sin\varphi, \quad \omega u = r_{B2} + r_C - \cos\varphi, \quad \omega = -r_{B2}$$

where u' and ωu are dimensionless projections of point C acceleration onto corresponding directions and a ω' is body's dimensionless angular acceleration.

From above mentioned equations, all reactions of constraints can be also defined, but for further analysis it is needed to determine the laws of change in reaction R_C in both cases:

$$r_C = \omega u + \cos\varphi - \omega' \tag{5.1}$$

$$r_C = \omega u + \cos\varphi + \omega' \tag{5.2}$$

and check its sign ($R_C > 0$).

Figure 7 presents the laws of change (5.1) and (5.2), which indicate that the brachistochronic motion is impossible to realize if the guide is at point A. At the initiation of motion, it is necessary that $R_C < 0$, which this constraint is unable to realize. In contrast, it is possible to position the guide at point B, because then it is $R_C > 0$ over the entire interval of motion.

This is the essential difference compared to the case of a classical blade considered in paper [17], where both methods of realization are possible because the nonholonomic constraint is bilaterally restraining. Also, note that in the mentioned paper [17] one of the two additionally imposed holonomic constraints is redundant. The brachistochronic planar motion of the body is possible to achieve only by means of two ideal mechanical constraints, the third one being unnecessary. In a unilateral restricting nonholonomic constraint of the blade type [18] the position of another constraint, the guide, must be chosen in such way that necessary force at the contact point must correspond to condition $R_C > 0$ over the entire time interval.

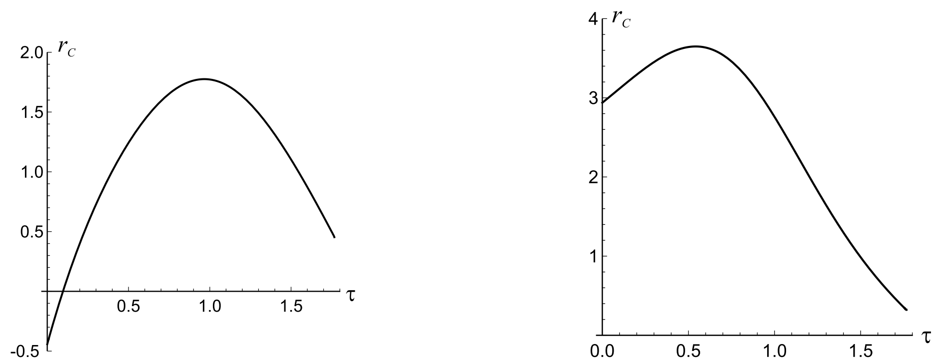

FIGURE 7. Dimensionless reaction of constraint for the case of guides at points A and B.

In the third case from Section 4, on different segments of brachistochronic motion, it is possible to combine guides at different points on the segment where $V = 0$, depending on the check of conditions $R_C > 0$. On the segments where $V > 0$, $R_C = 0$ the brachistochronic motion can be realized by the help of centroids [7] or guides at two different points, where the considered points A and B can be used.

6. Conclusions

The paper contributes to the unexplored area of optimal control of the motion of nonholonomic mechanical systems with unilateral restricted constraint reaction. The simplification of the optimal control task is also the originality of our work, based only on kinematic equations. A special contribution represents the analytical solution of differential equations for individual cases of unilateral constraints, when the solution is on an open set. It also entirely corresponds to the case of brachistochronic plane-parallel motion of a rigid body in the vertical plane, when the blade does not exist. The mass center trajectory is a deformed cycloid, unlike ordinary cycloid, in the case of a classical brachistochrone. It is shown that in the concrete case, with numerical solution of TPBVP, the obtained solution represents, for specified task parameters, the global minimum time of motion.

A special contribution of this work is a detailed analysis of the possibility of realizing planar motion of the rigid body by means of two ideal mechanical constraints, of which one is a unilateral restricting nonholonomic constraint.

Continuing research can take place in several directions. It is also possible, as indicated in a classical paper by Caratheodory [20], to impose maximum possible value of the unilateral constraint reaction of the blade. In that regard, the results of [24], which considered constrained classical nonholonomic bilateral constraint of the blade type, could be generalized to the motion in the vertical plane with unilateral constraint.

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РЕАЛИЗАЦИЈА БРАХИСТОХРОНОГ КРЕТАЊА ЧАПЛИГИНОВИХ САОНИЦА У ВЕРТИКАЛНОЈ РАВНИ СА ЈЕДНОСТРАНО ЗАДРЖАВАЈУЋОМ НЕХОЛОНОМНОМ ВЕЗОМ

Резиме. Дат је поступак одређивања брахистохроног кретања Чаплигинових саоница у вертикалној равни, где је сечиво такво да спречава бочно померање у тачки контакта само у једном смеру. Задати су положај центра масе саоница и угаона оријентација у крајњим положајима као и почетна вредност механичке енергије. Дата је најпростија могућа формулација одговарајућег задатка оптималног управљања, који је решаван применом Понтрјагиновог принципа максимума. У појединим случајевима нађена су аналитичка решења двотачкастог граничног проблема принципа максимума. У осталим случајевима извршено је нумеричко решавање методом погађања. Дате су процене недостајућих граничних услова, на основу којих је утврђено да добијено решење представља глобални минимум времена кретања. Анализирана је пасивна реализација брахистохроног кретања помоћу две идеалне механичке везе, једне обострано задржавајуће холономне и једне једнострано задржавајуће нехолономне.

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