

Finite-time stability analysis of fractional order time delay systems: Bellman-Gronwall's approach

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The paper extends some basic results from the area of finite time and practical stability to nonlinear, perturbed, fractional order time-delay systems where a robust stability test procedure is proposed. The problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of nonlinear fractional order time delay systems is examined.

Key words: finite time stability, nonlinear systems, time delay, fractional order system.

Introduction

THE question of stability is of particular interest in the control theory. Also, the problem of investigation of time delay systems has been explored over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic, hydraulic; in networks, chemical processes, long transmission lines, etc., [1]. On the other hand, the existence of pure time delay, whether it is present in control or/and state, may cause undesirable system transient response or even instability. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method or using the idea of matrix measure, [2,3,4,5].

Another approach will be presented here i.e., system stability from the non-Lyapunov point of view will be investigated. In practice, system stability (e.g. in the sense of Lyapunov), is investigated along with the bounds of system trajectories. A system could be stable but still completely useless if it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem. For the more it is of particular significance to consider the behavior of dynamical systems only over a finite time interval. These boundedness properties of system responses, i.e., the solution of system models, are very important from the engineering point of view. Bearing this in mind, numerous definitions of the so-called technical and practical stability were introduced. The analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned. Motivated by "brief discussion" on practical stability the monograph of LaSalle and Lefschet [6], Weiss and Infante [7] have introduced various notations of stability over finite time interval for continuous-time systems and constant set trajectory bounds. A more general type of stability ("practical stability with settling time",

practical exponential stability, etc.) which includes many previous definitions of finite stability was introduced and considered by Grujić [8, 9]. The concept of finite-time stability called "final stability", was introduced by Lashirer and Story [10] and further development of these results was due to Lam and Weiss [11]. Also, analysis of linear time-delay systems in the context of finite and practical stability was introduced and considered in papers [12, 13].

Recently, there has been some development in control theory of fractional differential systems (FDS) for stability questions, where for linear FDS in state-space form, both internal and external stabilities are considered [14]. The fractional-order means that the delay differential equation order is *non-integer (fractional)*. However, for fractional order dynamic systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. At the moment, direct check of the stability of fractional order systems using polynomial criteria (e.g., Routh's or Jury's type) is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopolynomial function of fractional powers of the complex variable s . Thus, there remain only geometrical methods of complex analysis based on the so-called argument principle (e.g. Nyquist type) which can be used for the stability check in the BIBO sense (bounded-input bounded-output). Also, analytical approach is suggested by Chen and Moore, [15], where the analytical stability bound using Lambert function W for the case of the linear FDS is considered. Analysis and stabilization of fractional (exponential) delay systems of retarded and neutral type are presented by Bonnet and Partington [16, 17]. On the other hand, the proposed approach in this paper does not demand solving delay FDS, but is based on forming the corresponding criteria (criterion of practical stability and finite time stability) in which basis matrices of system exclusively appear.

For the first time, a finite time stability test procedure is proposed for the perturbed (non)linear non-autonomous fractional order time delay systems. Here, the problem of

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sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of (non)linear non-autonomous fractional order time delay systems is examined. To the best knowledge of the author, these problems have not yet been analyzed for this class of fractional order time-delay systems.

Motivation and preliminaries

Preliminaries on integer time-delay systems

A linear, multivariable time-delay system can be represented by differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t - \tau) \quad (1)$$

and with associated function of initial state:

$$x(t) = \psi_x(t), \quad -\tau \leq t \leq 0, \quad (2)$$

Equation (1) is referred to as homogenous state equation. Also, more generally a linear, multivariable time-delay system can be represented by the following differential equation:

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t - \tau) + B_0 u(t) \quad (3)$$

and with associated function of initial state and control:

$$x(t) = \psi_x(t), \quad -\tau \leq t \leq 0, \quad (4)$$

Eq. (3) is referred to as non-homogenous or unforced state equation, where $x(t)$ is the state vector, $u(t)$ control vector, A_0, A_1, B_0 constant system matrices of appropriate dimensions and τ is pure time delay, $\tau = const.$ ($\tau > 0$). Also, in case of multiple time delays in the state of fractional order systems can be presented as:

$$\frac{d^\alpha x(t)}{dt^\alpha} = A_0 x(t) + \sum_{i=1}^n A_i x(t - \tau_i), \quad (5)$$

$$0 \leq \tau_1 < \tau_2 < \tau_3 < \dots < \tau_i < \dots < \tau_n = \Delta$$

and with associated function of initial state (4). Moreover, a class of non-linear systems with time delay described by the state space equation is considered here:

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{i=1}^n A_i x(t - \tau_i) + B_0 u(t) + f_0(x(t)) + \sum_{i=1}^n f_i(x(t - \tau_i)) \quad (6)$$

$$0 \leq \tau_1 < \tau_2 < \tau_3 < \dots < \tau_i < \dots < \tau_n = \Delta$$

with the initial functions (4) of the system. Vector functions $f_0, f_i, i = 1, 2, \dots, n$ represent nonlinear parameter perturbations of the system with respect to $x(t)$ and $x(t - \tau_i), i = 1, 2, \dots, n$ respectively. Also, the following assumption is introduced:

$$\|f_0(x(t))\| \leq c_0 \|x(t)\|, \quad t \in [0, \infty) \quad (7)$$

$$\|f_i(x(t - \tau_i))\| \leq c_i \|x(t - \tau_i)\|, \quad t \in [0, \infty), \quad i = 1, 2, \dots, n$$

where $c_0, c_i, \in R^+, i = 1, 2, \dots, n$ are known real positive numbers. The dynamical behaviour of the system (1), (3), (5) or (6) with initial functions (2), or (4) is defined over time interval $J = (t_0, t_0 + T)$, where quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously. It is obvious that $J \in R$. Time invariant sets, used as bounds of the system trajectories, are assumed to be open, connected and bounded. Let index " ε " stand for the set of all allowable states of system and index " δ " for the set of all initial states of the system, such that the set $S_\delta \subseteq S_\varepsilon$. In general, it may be written:

$$S_\rho = \{x : \|x(t)\|_Q^2 < \rho\}, \quad \rho \in [\delta, \varepsilon], \quad (8)$$

where Q is assumed to be symmetric, positive definite, real matrix. Also, S_{α_u} denotes the set of all allowable control actions. Let $\|x\|_{(\cdot)}$ be any vector norm (e.g., $\cdot = 1, 2, \infty$) and $\|(\cdot)\|$ the matrix norm induced by this vector. Matrix measure has been widely used in the references when dealing with stability of time delay systems. The matrix measure μ for any matrix $A \in C^{n \times n}$ is defined as follows:

$$\mu(A) = \lim_{\omega \rightarrow 0} \frac{\|I + \omega A\| - 1}{\omega} \quad (9)$$

The matrix measure defined in (10) can be subdefined in three different ways, depending on the norm utilized in its definitions (see [18]). Expression (4) can be written in its general form as:

$$x(t_0 + \theta) = \psi_x(\theta), \quad -\tau \leq \theta \leq 0, \quad \psi_x(\theta) \in C[-\tau, 0], \quad (10)$$

where t_0 is the initial time of observation of the system (1) and $C[-\tau, 0]$ is a Banach space of continuous functions over a time interval of length τ , mapping the interval $[t - \tau, t]$ into R^n with the norm defined in the following manner:

$$\|\psi\|_C = \max_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|, \quad (11)$$

It is assumed that the usual smoothness condition is present so there is no difficulty with the matter of existence, uniqueness and continuity of solutions with respect to the initial data.

Definition 1 [12]: System given by (1) satisfying initial condition (2) is a finite stable w.r.t $\{\delta, \varepsilon, t_0, J, \}$, $\delta < \varepsilon$ if and only if:

$$\|\psi_x\|_C < \delta \quad (12)$$

implying:

$$\|x(t)\| < \varepsilon, \quad \forall t \in J \quad (13)$$

Definition 2 [13]: System given by (3), satisfying initial condition (4) is a finite stable w.r.t $\{\varepsilon(t), \varepsilon, \alpha_u, \tau, J, \mu(A_0) \neq 0\}$, if and only if:

$$\psi_x \in S_\delta, \quad \forall t \in [-\tau, 0] \quad (14)$$

and

$$\mathbf{u}(t) \in S_{\alpha_u}, \forall t \in J \tag{15}$$

implying:

$$\mathbf{x}(t; t_0, \mathbf{x}_0) \in S_\varepsilon, \forall t \in [0, T] \tag{16}$$

The illustration of the previous definition is given in Fig.1.

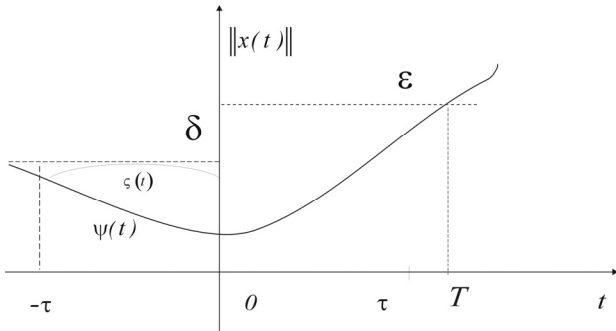


Figure 1. Finite time stability concept illustration

Results given in the continuation enable checking finite time stability of the (non)autonomous systems to be considered (1), (3), (5) or (6) and (2), (4) without finding the fundamental matrix or corresponding matrix measure.

Definition 3: System given by (3) satisfying initial condition (4) is finite stable w.r.t $\{t_o, J, \delta, \varepsilon, \beta\}$, $\delta < \varepsilon$ if and only if:

$$\|\psi_x\|_C < \delta \tag{17}$$

$$\|\mathbf{u}(t)\| < \beta, \forall t \in J \tag{18}$$

implying:

$$\|\mathbf{x}(t)\| < \varepsilon, \forall t \in J \tag{19}$$

Definition 4: System given by (5) satisfying initial condition (4) is finite stable w.r.t $\{t_o, J, \delta, \varepsilon, \Delta\}$, $\delta < \varepsilon$ if and only if:

$$\begin{aligned} \|\varphi_x\|_C &< \delta, \\ \forall t \in J_\Delta, \quad J_\Delta &= [-\Delta, 0] \in R \end{aligned} \tag{20}$$

implying:

$$\|\mathbf{x}(t)\| < \varepsilon, \forall t \in J, \tag{21}$$

Fundamentals of fractional calculus

Historical introduction and basic definitions

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with the correspondence between Leibniz and L'Hospital in 1695, where a differentiation of order one-half was discussed. Over the years great mathematicians such as Euler, Fourier, Abel and others did some work on the fractional calculus that, surprisingly, remained a sort of curiosity. Further, the theory of fractional-order derivative was developed mainly in the 19th century. In his 700 pages long book on calculus published

in 1819, Lacroix developed the formula for n -th derivative of $y = x^m$, where m is a positive integer, $D^n x^m = (m!/(m-n)!)x^{m-n}$ where $n(\leq m)$ is an integer. The modern epoch started in 1974 when a consistent formalism of the fractional calculus has been developed by K.B. Oldham and J. Spanier [19], where the fractional (non-integer) calculus is a *generalization* of the ordinary differential and integral calculus. Only in the last few decades, however, did scientists and engineers realize that such fractional differential equations provide a natural framework for the discussion of various kinds of questions modeled by fractional differential equations and fractional integrals, i.e., they provide more accurate models of systems under consideration. Moreover, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Many authors have demonstrated applications of fractional calculus in various fields such as physics, chemistry, and engineering [19-21]. From mathematical point of view, the fractional integrodifferential operators (fractional calculus) are a generalization of integration and derivation to non-integer order (fractional) operators. At first, the differential and integral operators can be generalized into one fundamental D_t^α operator t which is known as fractional calculus, [19-21]:

$${}_a D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0. \end{cases} \tag{22}$$

Two definitions generally used for the fractional differintegral are the Grunwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [18,19]. The original Grunwald-Letnikov definition of fractional derivative is given by a limit, i.e.,

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{23}$$

where a, t are the limits of operator and $[x]$ means the integer part of x . Integral version of GL is defined by

$${}_a D_t^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{24}$$

The RL definition of the fractional derivative is given by the expression:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{25}$$

for $(n-1 < \alpha < n)$ and $\Gamma(\cdot)$ is the well known Euler's gamma function in the following form:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z = x + iy, \quad \Gamma(z+1) = z\Gamma(z), \tag{26}$$

One of the basic properties of the gamma function is that

is satisfies the following functional equation:

$$\Gamma(z+1) = z\Gamma(z), \Rightarrow \Gamma(n+1) = n(n-1)! = n!, \quad (27)$$

The second important property of the gamma function is that it has simple poles at the points $z = -n$, ($n = 0, 1, 2, \dots$). Another important relation-ship for the gamma function is the Legendre formula:

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}2^{2z-1}\Gamma(2z), \quad 2z \neq 0, -1, -2, \dots \quad (28)$$

Taking $z = n+1/2$ from the previous relation a set of particular values of the gamma function can be obtained:

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}, \quad (29)$$

Closely related to the fractional-order differentiation is the fractional order integration, i.e., Riemann-Liouville fractional integral is defined as:

$${}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0 \quad (30)$$

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Fractional differentiation is a linear operation:

$$D^\alpha (\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t), \quad (31)$$

Also, the "chain rule" has the form:

$$\frac{d^\beta f(g(t))}{dt^\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k}_\Gamma \left(\frac{d^{\beta-k}}{dt^{\beta-k}} 1 \right) \frac{d^k}{dt^k} f(g(t)) \quad (32)$$

Where $k \in \mathbb{Z}$ and $\binom{\beta}{k}_\Gamma$ are the coefficients of the generalized binomial:

$$\binom{\beta}{k}_\Gamma = \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)}. \quad (33)$$

Also, Caputo [22] has proposed that the integer order (classical) derivative of function x should be incorporated, as they are commonly used in the initial value problems with integer-order equations. In that way, the derivatives of the Caputo type can be used:

$${}_0^C D_t^\alpha [U(t)] = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{U^{(n)}(\tau)}{(t-\tau)^{n-\alpha-1}} d\tau, \quad (34)$$

$$n-1 < p < n, \quad U^{(n)}(\tau) = d^n U / d\tau^n$$

For convenience, Laplace domain is usually used to describe the fractional integro-differential operation for solving engineering problems. The formula for the Laplace transform of the RL fractional derivative has the form:

$$\int_0^\infty e^{-st} {}_0^C D_t^\alpha f(t) dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k {}_0^C D_t^{\alpha-k-1} f(t)|_{t=0} \quad (35)$$

and Laplace transform of the Caputo fractional derivative is:

$$\int_0^\infty e^{-st} {}_0^C D_t^\alpha f(t) dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \quad (36)$$

where formula (36) involves the initial conditions $f^{(k)}(0)$ with integer derivatives $f^{(k)}(t)$. In pure mathematics, RL derivative is more commonly used than Caputo derivative. In practical applications, the initial conditions ${}_0^C D_t^{\alpha-k-1} f(t)|_{t=0}$ are frequently not available [19], so Caputo fractional derivative is considered here where derivatives of integer order of the function f as the initial conditions should be incorporated (eq. 36). Recently, in [23] Heymans and Podlubny gave some explanations for RL fractional-order initial values where it is possible to obtain initial values for such initial conditions by appropriate measurements or observations. The relation between the two fractional derivatives Riemann-Liouville and Caputo is:

$${}_a^C D_t^\alpha f(t) = {}_a^{RL} D_t^\alpha \left(f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!} \right), \quad (37)$$

Here, in order to provide a suitable mathematical treatment of the fractional derivative phenomena the following notation is introduced:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} f^{(n)}(t) & \text{if } \alpha = n \in \mathbb{Z} \\ {}_a^C D_t^\alpha f(t) & \text{if } n-1 < \alpha < n. \end{cases} \quad (38)$$

The Caputo and Riemann-Liouville formulations coincide when the initial conditions are zero. Lorenzo and Hartley [24] considered variable prehistories of $x(t)$ in $t < 0$, and these effects of the fractional derivative were taken into account in terms of the initialization function. Using short memory principle [20] and taking into account (2), the correct initial function can be obtained. Also, it is assumed that the usual smoothness condition is present so that there is no difficulty with the matter of existence, uniqueness, and continuity of solutions with respect to the initial data [25].

Previous results related to fractional order time-delay systems

Recently, there have been some developments in the control theory of FDS (with time-delay) for the stability questions [15-17, 20, 26]. In his PhD, Matignon [26] gives the model of pressure wave transmission through an air-filled tube with viscothermic perturbation and he discusses the stability of the transfer function as an example of a fractional delay system. Also, it is shown that PD^α control of Newcastle robot can be presented by the linear time delay fractional order of differential equation in the state space form, [28]:

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t-\tau) \quad (39)$$

$${}_0^C D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1,$$

and with associated function of initial state: $\mathbf{x}(t) = \psi_x(t)$, $-\tau \leq t \leq 0$ for ($0 < \alpha < 1$). Further more, for the first time, finite time stability analysis of (non)linear (non)autonomous fractional order systems with delayed state are presented by Lazarević *et al.* [27-30].

Also, more generally, non-homogenous, linear multivariable time-delay system can be represented by the

following differential equation:

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau) + B_0 \mathbf{u}(t) \quad (40)$$

and with associated function of the initial state (2).

Here, a class of fractional non-linear nonautonomous time delay systems with perturbations described by the state space equation is considered:

$$\begin{aligned} \frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} &= A_0 \mathbf{x}(t) + \sum_{i=1}^n A_i \mathbf{x}(t - \tau_i) + \\ &+ B_0 \mathbf{u}(t) + f_0(\mathbf{x}(t)) + \sum_{i=1}^n f_i(\mathbf{x}(t - \tau_i)) \quad (41) \\ 0 &\leq \tau_1 < \tau_2 < \tau_3 < \dots < \tau_i < \dots < \tau_n = \Delta \end{aligned}$$

with the initial function (4) of the system and vector functions $f_0, f_i, i = 1, 2, \dots, n$ satisfied (6).

Theorem 1: [29] Autonomous system given by (39) satisfying initial condition (2) is a finite time stable w.r.t. $\{\delta, \varepsilon, t_0, J\}$, $\delta < \varepsilon$, if the following condition is satisfied:

$$\left[1 + \frac{\sigma_{\max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)} \right] \cdot e^{\frac{\sigma_{\max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)}} \leq \varepsilon / \delta, \quad \forall t \in J. \quad (42)$$

where $\sigma_{\max}^A = \sigma_{\max}(A_0) + \sigma_{\max}(A_1)$ and $\Gamma(\cdot)$ is the Euler's gamma function [20].

Remark 1: If $\alpha = 1$, conditions same as those related to integer order time delay systems (1) as follows from [12] can be obtained:

$$\left[1 + \frac{\sigma_{\max}^A(t-t_0)^1}{1} \right] \cdot e^{\frac{\sigma_{\max}^A(t-t_0)^1}{1}} \leq \varepsilon / \delta, \quad \forall t \in J, \Gamma(2) = 1 \quad (43)$$

Theorem 2: [30] Nonautonomous system given by (40) satisfying initial condition (2) is finite time stable w.r.t. $\{\delta, \varepsilon, \alpha_u, \alpha_0, t_0, J\}$, $\delta < \varepsilon$, if the following condition is satisfied:

$$\begin{aligned} &\left[1 + \frac{\sigma_{\max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)} \right] \cdot e^{\frac{\sigma_{\max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)}} + \\ &\gamma^\square \frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)} \leq \varepsilon / \delta, \quad \forall t \in J. \quad (44) \end{aligned}$$

where $\gamma^\square = b_0 \alpha_u / \delta$, $\|B_0\| = b_0$ and $\Gamma(\cdot)$ Euler's gamma function.

Remark 2: If $\alpha = 1$, conditions same as those related to integer order time delay systems (3) as follows from [13] can be obtained:

$$\left[1 + \frac{\sigma_{\max}^A(t-t_0)^1}{1} \right] \cdot e^{\frac{\sigma_{\max}^A(t-t_0)^1}{1}} + \gamma^\square \frac{(t-t_0)^1}{1} \leq \varepsilon / \delta, \quad \forall t \in J, \Gamma(2) = 1 \quad (45)$$

Theorem 3: [29] Autonomous system $u(t) \equiv 0, f_i \equiv 0, i = 0, 1, \dots, n$ given by (41) satisfying initial condition (2) is finite time stable w.r.t. $\{\delta, \varepsilon, \Delta, t_0, J\}$, $\delta < \varepsilon$ if the following condition is satisfied:

$$\left[1 + \frac{\sigma_{\Sigma \max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)} \right] \cdot e^{\frac{\sigma_{\Sigma \max}^A(t-t_0)^\alpha}{\Gamma(\alpha+1)}} \leq \varepsilon / \delta, \quad \forall t \in J. \quad (46)$$

where $\sigma_{\Sigma \max}^A(\cdot) = \sum_i \sigma_i(A_i)$ of matrices $A_i, i = 0, 1, 2, \dots, n$.

Main results

Here, the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of nonlinear perturbed nonautonomous fractional order time-delay systems are examined.

Main theorem: Nonlinear nonautonomous system given by (41) satisfying initial condition (12) is finite time stable w.r.t. $\{\delta, \varepsilon, \alpha_u, \alpha_0, t_0, J\}$ $\delta < \varepsilon$, if the following condition is satisfied:

$$\begin{aligned} &\left(1 + \frac{\sigma_{\Sigma \max c}(t-t_0)^\alpha}{\Gamma(\alpha+1)} \right) e^{\frac{\sigma_{\Sigma \max c}(t-t_0)^\alpha}{\Gamma(\alpha+1)}} + \\ &\frac{\gamma_{u0}^*}{\Gamma(\alpha+1)} (t-t_0)^\alpha \leq \varepsilon / \delta, \quad \forall t \in J \quad (47) \end{aligned}$$

where $\gamma_{u0}^* = \alpha_u b_0 / \delta$, and $\Gamma(\cdot)$ Euler's gamma function.

Proof: In accordance with the property of the fractional order $0 < \alpha < 1$, the solution in the form of the equivalent Volterra integral equation can be obtained:

$$\begin{aligned} x(t) &= x(t_0) + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left(A_0 \mathbf{x}(s) + \sum_{i=1}^n A_i \mathbf{x}(s-\tau_i) + B_0 \mathbf{u}(s) + \right. \\ &\left. + \sum_{i=1}^n f_i(\mathbf{x}(s-\tau_i)) + f_0(\mathbf{x}(s)) \right) ds \quad (48) \end{aligned}$$

Applying the norm $\|(\cdot)\|$ to eq. (48) and using appropriate property of the norm, the following applies:

$$\begin{aligned} \|x(t)\| &\leq \|x(t_0)\| + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t |(t-s)^{\alpha-1}| \left\| \begin{aligned} &A_0 \mathbf{x}(s) + \sum_{i=1}^n A_i \mathbf{x}(s-\tau_i) + B_0 \mathbf{u}(s) + \\ &+ f_0(\mathbf{x}(s)) + \sum_{j=1}^n f_j(\mathbf{x}(s-\tau_j)) \end{aligned} \right\| ds \quad (49) \end{aligned}$$

Also, applying the norm $\|(\cdot)\|$ and taking into account assumption (7), for eq. (49), it can be obtained:

$$\begin{aligned} \left\| \frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} \right\| &\leq \|A_0\| \|x(t)\| + \sum_{i=1}^n \|A_i\| \|x(t-\tau_i)\| \\ &+ \|B_0\| \|u(t)\| + \|f_0(\mathbf{x}(t))\| + \sum_{i=1}^n \|f_i(\mathbf{x}(t-\tau_i))\| \\ &\leq (\sigma_{\max}(A_0) + c_o) \|x(t)\| + \\ &+ \left(\sum_{i=1}^n \sigma_{\max i}(A_i) + c_i \right) \|x(t-\tau_i)\| + \|B_0\| \|u(t)\| \quad (50) \end{aligned}$$

where $\|A\|$ denotes the induced norm of a matrix A, as well as,

$$\|x(t - \tau_i)\| \leq \sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\|, \quad i = 1, 2, \dots, n, \quad (51)$$

Applying this inequality, eq. (50) can be presented in the following manner:

$$\begin{aligned} \left\| \frac{d^\alpha x(t)}{dt^\alpha} \right\| &\leq \sigma_{A_0 \cos} \|x(t)\| + \sigma_{A_{icis}} \sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\| \\ + b_0 \|u(t)\| &\leq \sigma_{\Sigma \max c} \sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\|, \quad (52) \\ + b_o \|u(t)\|, \quad &t > t_0 + \Delta \end{aligned}$$

or:

$$\begin{aligned} \left\| A_0 x(t) + \sum_{i=1}^n A_i x(t - \tau_i) + B_0 u(t) \right. \\ \left. + f_o(x(t)) + \sum_{i=1}^n f_i(x(t - \tau_i)) \right\| &\leq \quad (53) \\ \leq \sigma_{\Sigma \max c} \left(\sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\| + \|\psi_x\|_C \right) &+ b_0 \|u(t)\|, \quad t > t_0 + \end{aligned}$$

where is $\sigma_{\Sigma \max c} = \sigma_{A_0 c_0 s} + \sigma_{A_{icis}}$. Combining (53) with (50), yields:

$$\begin{aligned} \|x(t)\| \leq \|x(t_0)\| + \\ + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t |(t-s)^{\alpha-1}| \left\{ \sigma_{\Sigma \max c} \left(\sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\| + \right) \right. \\ \left. + b_0 \|u(t)\| \right\} ds, \quad (54) \end{aligned}$$

or,

$$\begin{aligned} \|x(t)\| \leq \|\psi_x\|_C \left[1 + \frac{\sigma_{\Sigma \max c} (t - t_0)^\alpha}{\Gamma(\alpha + 1)} \right] + \\ + \frac{\sigma_{\Sigma \max c}}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\| ds +, \quad (55) \\ + \frac{1}{\Gamma(\alpha + 1)} (\alpha_u b_0) (t - t_0)^\alpha \end{aligned}$$

Obviously, a nondecreasing function $\mu(t)$ can be introduced:

$$\mu(t) = \|\psi_x\|_C \left(1 + \frac{\sigma_{\Sigma \max c} (t - t_0)^\alpha}{\Gamma(\alpha + 1)} \right), \quad (56)$$

when applying the known Bellman-Gronwall lemma, [31, 32] it is easy show that:

$$\begin{aligned} \|x(t)\| \leq \sup_{t - \Delta \leq t^* \leq t} \|x(t^*)\| \leq \mu(t) e^{\frac{\sigma_{\Sigma \max c}}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} ds} \\ = \mu(t) e^{\frac{\sigma_{\Sigma \max c} (t - t_0)^\alpha}{\Gamma(\alpha + 1)}} \end{aligned} \quad (57)$$

and

$$\begin{aligned} \|x(t)\| \leq \delta \left(1 + \frac{\sigma_{\Sigma \max c} (t - t_0)^\alpha}{\Gamma(\alpha + 1)} \right) e^{\frac{\sigma_{\Sigma \max c} (t - t_0)^\alpha}{\Gamma(\alpha + 1)}} \\ + \frac{1}{\Gamma(\alpha + 1)} (\alpha_u b_0) (t - t_0)^\alpha \end{aligned} \quad (58)$$

Finally, if the basic condition of the main theorem, namely relation (47) is used it yields:

$$\|x(t)\| < \varepsilon, \quad \forall t \in J. \quad (59)$$

which had to be proved.

Simulation results

Example 1: ([28]): A linear fractional, time-delay system is considered in the following state space description:

$$\begin{aligned} \frac{d^{1/2} x(t)}{dt^{1/2}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -17.8 & 0 & -12.8 & 0 \end{bmatrix} x(t) + \\ + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.04 & -0.04 & 0 & 0 \end{bmatrix} x(t - 0.1) \end{aligned} \quad (60)$$

The finite time stability w.r.t $\{t_0 = 0, J = \{0, 1\}, \delta = 0.06, \varepsilon = 100, \tau = 0.1\}$, has to be checked where $\psi_x(t) = (0.05, 0, 0, 0)^T$, $\forall t \in [-0.1, 0]$. From the initial data and the eq. (60), it can be easily obtained that:

$$\|\psi_x(t)\|_C < 0.06, \quad (61)$$

$$\sigma_{\max}(A_0) = 21.95, \quad \sigma_{\max}(A_1) = 0, \quad \sigma_{\max} = 21.95,$$

$$\Gamma(1 + 1/2) = \frac{\sqrt{\pi} \Gamma(2 + 1)}{2^2 \Gamma(1 + 1)} = \frac{\sqrt{\pi} (2)!}{2^2!} = \frac{\sqrt{\pi}}{2} = 0.886, \quad (62)$$

From Theorem 1, it immediately follows:

$$\begin{aligned} \left[1 + \frac{21.95 T_e^{1/2}}{0.866} \right] \cdot e^{\frac{21.95 T_e^{1/2}}{0.866}} \leq 100 / 0.06, \quad (63) \\ \Rightarrow T_e = 0.05 \text{ s} \end{aligned}$$

T_e being ‘‘estimated time’’ of finite time stability. Also, numerical results are presented in Figures 2 and 3 using eq. (23), to validate the analytical predictions (63). The calculation has been done for the following parameters:

$$\begin{aligned} \alpha = 1/2, \quad h = 0.01 \text{ s}, \quad \tau = 0.1 \text{ s}, \\ \psi_x(t) = (0.05, 0, 0, 0)^T, \quad \forall t \in [-0.1, 0] \end{aligned} \quad (64)$$

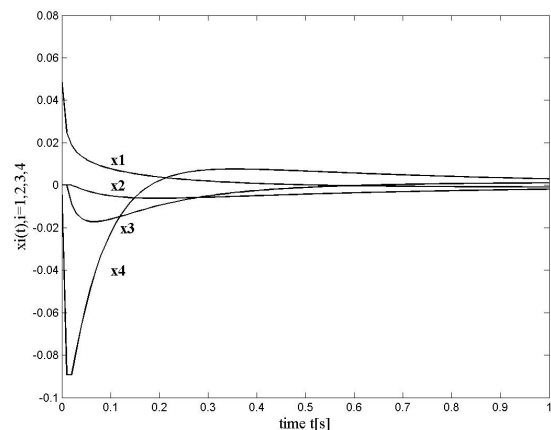


Figure 2. Time histories of the $x_i, i = 1, 2, 3, 4$

Moreover, the norm $\|x(t)\|$ on time interval $[0,1]$ as it is shown in Fig.3 can be obtained:

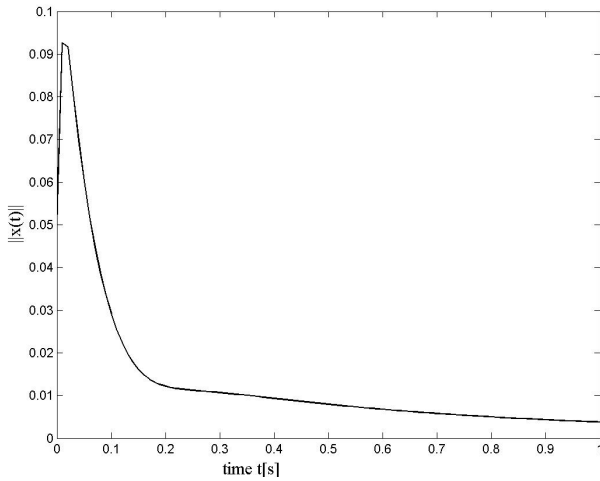


Figure 3. Trend norm values of $\|x(t)\|$

Example 2: Using Time-Delay PD^α compensator on a nonlinear system of equations with respect to the small perturbation $e(t) = y(t) - y_d(t)$ it can be obtained:

$$\begin{aligned} \dot{e}(t) + \lambda e(t) + f_p(e(t)) &= \\ &= K_p e(t - \tau) + K_D d e^{(\alpha)}(t - \tau) / dt^\alpha, \end{aligned} \quad (65)$$

where are:

$$\alpha = 1/2, \lambda = 2, K_p = 1, K_D = 1, \quad (66)$$

Also, all initial values are zeros. Introducing:

$$x_1(t) = e(t), \quad x_2(t) = d^{1/2} e(t) / dt^{1/2}, \quad (67)$$

expression (65) can be written in state-space form:

$$\begin{aligned} D_t^\alpha x_1(t) &= D_t^{1/2} e(t) = x_2(t), \\ D_t^\alpha x_2(t) &= D_t^{1/2} (D_t^{1/2} e(t)) = \dot{e}(t) = \\ &= -2x_1(t) + x_1(t - \tau) + x_2(t - \tau) - f_p(x_1(t)) \end{aligned} \quad (68)$$

or, in a condensed form, where $\mathbf{x}(t) = (x_1, x_2)^T$, the following expression can be obtained:

$$\begin{aligned} D_t^{1/2} \mathbf{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \\ &+ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(\mathbf{x}(t)) \end{aligned} \quad (69)$$

Therefore, the task is to check the finite time stability w.r.t $\{t_0 = 0, J = \{0, 2\}, \delta = 0.1, \varepsilon = 50, \tau = 0.1\}$ where $\psi_x(t) = 0, \forall t \in [-0.1, 0], c_0 = 0.1$. From the initial data and the Eqs. (54) and (4) it can be obtained:

$$\begin{aligned} \|\psi_x(t)\|_C &< 0.1, \quad \sigma_{\max}(\mathbf{A}_0) = 2, \sigma_{\max}(\mathbf{A}_1) = \sqrt{2} \\ \Rightarrow \mu_p &= \sigma_{\max}^A + c_0 = 3.41 + 0.1 = 3.51 \end{aligned} \quad (70)$$

Using condition (47) it follows:

$$\left[1 + \frac{3.51 T_e^{0.5}}{0.886} \right] \cdot e^{\frac{3.51 T_e^{0.5}}{0.886}} \leq 50 / 0.1 = 500 \Rightarrow T_e = 1.295 \text{ s.} \quad (71)$$

Conclusion

In this paper, stability results for (non)linear, perturbed (non)autonomous fractional order time delay systems are given in state space form. To the best knowledge of the author, these problems have not yet been analyzed for this class of nonlinear time-delay fractional order systems. Sufficient conditions of this kind of stability are derived by applying generalized Bellman-Gronwall's theorem. In this way, the system stability over finite time interval can be checked.

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Analiza stabilnosti na konačnom vremenskom intervalu sistema sa kašnjenjem necelobrojnog reda: Bellman-Gronwallov pristup

Ovaj rad proširuje neke osnovne rezultate iz oblasti praktične stabilnosti i stabilnosti na konačnom vremenskom intervalu na nelinearne, perturbovane, sisteme sa kašnjenjem necelobrojnog reda gde je predložen postupak testiranja robusne stabilnosti. Proučavan je problem dovoljnih uslova koji omogućavaju da trajektorije sistema ostanu unutar *a priori* zadatih skupova i to za posebnu klasu nelinearnih sistema sa kašnjenjem necelobrojnog reda.

Ključne reči: stabilnost na konačnom vremenskom intervalu, nelinearni sistem, sistem sa kašnjenjem, sistem necelobrojnog reda.

Анализ устойчивости на конечном временном промежутке системы со временной задержкой дробного порядка: подход Беллман-Гронвалла

Настоящая работа расширяет некоторые из основных результатов из области практической устойчивости и устойчивости на конечном временном промежутке на нелинейные и нарушенные системы, а в том роде и на системы со временной задержкой дробного порядка, где предложен поступок испытания живучей устойчивости. Здесь исследована и проблема достаточных условий, обеспечивающих чтобы траектории системы на приорив остались внутри заданных наборов, а именно для обособленного класса нелинейных систем со временной задержкой дробного порядка.

Ключевые слова: устойчивость на конечном временном промежутке, нелинейная система, система со временной задержкой, временная задержка, система дробного порядка.

Analyse de la stabilité chez l'intervalle temporel fini des systèmes à délai de l'ordre fractionnel: approche de Bellman-Gronwall

Ce travail reporte les résultats basiques du domaine de stabilité pratique chez l'intervalle temporel fini aux systèmes non-linéaires perturbés à délai de l'ordre fractionnel où on a proposé le procédé de test de la stabilité robuste. On a étudié le problème des conditions suffisantes qui permettent que les trajectoires du système restent a priori à l'intérieur des ensembles donnés et cela pour la classe particulière des systèmes non-linéaires à délai de l'ordre fractionnel.

Mots clés: stabilité chez l'intervalle temporel fini, système non-linéaire, système à délai, délai temporel, système de l'ordre fractionnel.