Error estimates for Gaussian quadrature formulae

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Happy birthday to Professor Gradimir Milovanović, in whose honor this conference is organized. ⌣̈

Gauss QF:

$$I(f) = \int_{-1}^{1} \omega(t) f(t)\, dt = \sum_{i=1}^{n} \omega_i f(\xi_i) + R_n(f) \qquad (1)$$

n nodes (ξ_i), $2n-1$ degree of exactness

Gauss-Legendre QF: associated with Legendre weight function.

Error:

When f is an analytic function, the remainder term can be represented as a contour integral with a complex kernel.

Error term

$$r_n(f;t) = f(t) - \sum_{i=1}^{n} l_i(t) f(\xi_i) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\Omega_n(t)}{(z-t)\Omega_n(z)}\, dz, \qquad (2)$$

where

$$\Omega_n(z) = c_n \prod_{i=1}^{n} (z - \xi_i) \quad (c_n \neq 0),$$

- ξ_i roots of of the corresponding orthogonal polynomial
- $\Omega_n(t)$
- l_i fundamental functions of Lagrange interpolation.
- as set $\Omega_n(t)$ we considered $\pi_n(t) = P_n^{(0,0)}(t)$ (special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$)

$$R_n(f) = I(f;\omega) - \sum_{i=1}^{n} \omega_i f(\xi_i) = \frac{1}{2\pi\, i} \oint_\Gamma K_n(z;\omega) f(z)\, dz, \tag{3}$$

where

$$I(f;\omega) = \int_{-1}^{1} \omega(t)\, f(t)\, dt, \quad \omega_i = \int_{-1}^{1} \omega(t)\, l_i(t)\, dt,$$

and the *kernel* $K_n(z) = K_n(z;\omega)$ can be expressed in the form

$$K_n(z;\omega) = \frac{\varrho_n(z;\omega)}{\Omega_n(z)}, \quad \varrho_n(z;\omega) = \int_{-1}^{1} \omega(t)\, \frac{\Omega_n(t)}{z-t}\, dt, \quad z \in \mathbb{C} \setminus [-1,1].$$

The integral representation (**??**) leads directly to the error bound

$$|R_n(f)| \le \frac{l(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_n(z)| \right) \left(\max_{z\in\Gamma} |f(z)| \right), \tag{4}$$

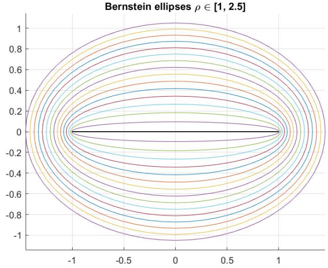
where $l(\Gamma)$ is the length of the contour.

Choice of the contour - Bernstein ellipse

A common choice for the contour Γ is

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \; z = \frac{1}{2}\left(u + u^{-1}\right), u = \rho e^{i\theta}, \; 0 \leq \theta < 2\pi \right\}.$$

- foci at the points ∓ 1
- sum of semi-axes $\rho > 1$
- major semi-axes $\frac{1}{2}(\rho + \rho^{-1})$
- minor semi-axes $\frac{1}{2}(\rho - \rho^{-1})$



Inspired by work:

H. Wang and L. Zhang, *Jacobi polynomials on tha Bernstein ellipse,* J. Sci. Comput. 75 (2018), pp. 457—477.

- Jacobi polynomials $P_n^{(\alpha,\beta)}$ considered
- Jacobi weight function $\omega(t) = (1-t)^\alpha (1+t)^\beta, \quad \alpha, \beta > -1.$

Problems:

- convergence rate of spectral interpolation
- spectral collocation method for solving integral and differential equations.

Results:

- explicit representation of $P_n^{(\alpha,\beta)}(t)$ in the variable of parametrization,
- the extrema of $|P_n^{(\alpha,\beta)}(z)|$ on the B.E.
- refined asymptotic estimate

Results that could be adopted from general case:

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k}, \quad z = \frac{1}{2}\left(u + u^{-1}\right), \tag{5}$$

where, in the case $\alpha = \beta > -1$,

$$d_{k,n} = \begin{cases} \dfrac{2^{2\alpha}\Gamma(n+\alpha+1)\Gamma\left(\frac{k+n+1}{2}+\alpha\right)\Gamma\left(\frac{n-k+1}{2}+\alpha\right)}{\sqrt{\pi}\Gamma(n+2\alpha+1)\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{n-k}{2}+1\right)\Gamma(\alpha+1/2)}, & n - \\ 0, & n - \end{cases} \tag{6}$$

Trigonometric representations of (**??**):

$$P_n^{(\alpha,\beta)}(\cos\theta) = d_{0,n} + 2\sum_{k=1}^{n} d_{k,n}\cos(k\theta). \tag{7}$$

Properties of the Chebyshev polynomials of the first and second kind

$$T_n(\cos\theta) = \cos(n\theta), \quad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad n \geq 0 \tag{8}$$

Representations of them when $z \in \mathcal{E}_\rho$,
$z = \frac{1}{2}(u + u^{-1}), u = \rho e^{i\theta}(0 \leq \theta < 2\pi)$

$$T_n(z) = \frac{1}{2}(u^n + u^{-n}), \quad U_n(z) = \frac{u^{n+1} - u^{-n-1}}{u - u^{-1}} \tag{9}$$

$$\int_{-1}^{1} \frac{p_n(t)}{z \mp t}\, dt, \quad |z| \neq 1, \tag{10}$$

with the p_n being any one of the Chebysev polynomials of degree n, can be computed explicitly

Modulus of the kernel

- The *kernel* is given by $K_n(z) = \frac{\varrho_n(z)}{\pi_n(z)}$, $z \notin [-1, 1]$,
- $\pi_n(z) = \Omega_n(z)$ is the Legendre polynomial of degree n

$$\varrho_n(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\,dt = \int_0^\pi \frac{\left(2\sum_{k=0}^{n}{}'' d_{k,n}\cos k\theta\right)\sin\theta}{z-\cos\theta}\,d\theta = 2\sum_{k=0}^{n}{}'' d_{k,n}\int_0^\pi \frac{\cos k\theta\ \sin\theta}{z-\cos\theta}\,d\theta. \tag{11}$$

$\alpha = \beta = 0$

$$d_{k,n} = \begin{cases} \dfrac{(n+k-1)!!\,(n-k-1)!!}{(n+k)!!\,(n-k)!!}, & n-k \quad \text{even}, \\[2ex] 0, & n-k \quad \text{odd}, \end{cases} \tag{12}$$

S. E. Notaris, *Integral formulas for Chebyshev polynomials and the error term of interpolatory quadrature formulae for analytic functions*, Math. Comp. 75 (2006), pp. 1217–1231.

$$I_k = \int_{-1}^{1} \frac{T_k(t)}{z-t}\, dt, \qquad (13a) \qquad\qquad I_k = \int_0^\pi \frac{\cos k\theta \, \sin\theta}{z - \cos\theta}\, d\theta \quad (13b)$$

$$I_k = T_k(z) \log \frac{z+1}{z-1} - 4 \sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{T_{k-2j+1}(z)}{2j-1} \quad (|z|>1), \qquad (14)$$

with the principal value of the logarithmic function
$\log z = \log|z| + i \arg z \quad (-\pi < \arg z \leq \pi)$

Expressions for modulus of the kernel

Inserting relation $z = (u+u)^{-1}/2 \quad (|u|>1)$ in (**??**),

$$I_k = \frac{u^k + u^{-k}}{2} \log \frac{z+1}{z-1} - 2 \sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{u^{k-2j+1} + u^{-(k-2j+1)}}{2j-1}. \tag{15}$$

Note that

$$\log \frac{z+1}{z-1} = \log \frac{(u+u^{-1})/2+1}{(u+u^{-1})/2-1} = \log \frac{(1+u^{-1})^2}{(1-u^{-1})^2} = 2\log \frac{1+u^{-1}}{1-u^{-1}}. \tag{16}$$

Since

$$\log(1+u^{-1}) = \sum_{j=0}^{\infty} \frac{(-1)^j u^{-j-1}}{j+1}, \quad \log(1-u^{-1}) = -\sum_{j=0}^{\infty} \frac{u^{-j-1}}{j+1} \quad (|u^{-1}|<1),$$

It follows that

$$\log \frac{z+1}{z-1} = 4 \sum_{j=0}^{\infty} \frac{u^{-2j-1}}{2j+1}. \tag{17}$$

$$I_k = 2\left[\left(u^k + \frac{1}{u^k}\right)\sum_{j=0}^{\infty}\frac{1}{(2j+1)u^{2j+1}} - \sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{u^{k-2j+1} + \frac{1}{u^{k-2j+1}}}{2j-1}\right]. \quad (18)$$

Proposition

It holds

$$I_k = \begin{cases} -4\sum_{j=0}^{\infty}\frac{(2j+1)u^{-2j-1}}{k^2-(2j+1)^2} = -4\left[\frac{u^{-1}}{k^2-1} + \frac{3u^{-3}}{k^2-9}\right] + o\left(u^{-4}\right), & k \quad \text{ev} \\ -4\sum_{j=1}^{\infty}\frac{2ju^{-2j}}{k^2-(2j)^2} = -4\left[\frac{2u^{-2}}{k^2-4} + \frac{4u^{-4}}{k^2-16}\right] + o\left(u^{-5}\right), & k \quad \text{od} \end{cases} \quad (19)$$

On the basis of (**??**), we have

$$\rho_n(z) = 2 \sum_{k=0}^{n}{}'' d_{k,n} I_k, \tag{20}$$

$$\rho_{2m}(z) = -16 \left(\sum_{l=0}^{m}{}'' \frac{d_{2l,2m}}{4l^2 - 1} \cdot \frac{1}{u} + 3 \sum_{l=0}^{m}{}'' \frac{d_{2l,2m}}{4l^2 - 9} \cdot \frac{1}{u^3} \right) + o \left(\frac{1}{u^4} \right) \quad (\rho \to \infty \tag{21}$$

and

$$\rho_{2m+1}(z) = -32 \left(\sum_{l=0}^{m} \frac{d_{2l+1,2m+1}}{(2l+1)^2 - 4} \cdot \frac{1}{u^2} + 2 \sum_{l=0}^{m} \frac{d_{2l+1,2m+1}}{(2l+1)^2 - 16} \cdot \frac{1}{u^4} \right) + o \Big(\quad (\rho \to \infty), \ m$$

which together with

$$\pi_n(z) = \sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k} = d_{n,n} u^n + d_{n-2,n} u^{n-2} + o\left(u^{n-3}\right) \quad (\rho \to \infty)$$

$$K_{2m}(z) = \frac{-16 \sum_{l=0}^{m}{}'' \dfrac{d_{2l,2m}}{4l^2-1}}{d_{2m,2m} u^{2m+1}} \cdot \frac{u^2 + 3 \dfrac{\sum_{l=0}^{m}{}'' \dfrac{d_{2l,2m}}{4l^2-9}}{\sum_{l=0}^{m}{}'' \dfrac{d_{2l,2m}}{4l^2-1}} + o\left(\frac{1}{\rho}\right)}{u^2 + \dfrac{d_{2m-2,2m}}{d_{2m,2m}} + o\left(\dfrac{1}{\rho}\right)} \quad (\rho \to \infty), \tag{23}$$

$$K_{2m+1}(z) = \frac{-32 \displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,2m+1}}{(2l+1)^2-4}}{d_{2m+1,2m+1} u^{2m+3}} \cdot \frac{u^2 + 2 \dfrac{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,2m+1}}{(2l+1)^2-16}}{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,2m+1}}{(2l+1)^2-4}} + o\left(\frac{1}{\rho}\right)}{u^2 + \dfrac{d_{2m-1,2m+1}}{d_{2m+1,2m+1}} + o\left(\dfrac{1}{\rho}\right)} \tag{24}$$

Observations

The second factor in the kernels in (**??**), (**??**) is of the form

$$I(\rho,\theta) = \frac{u^2 + A + o\left(\frac{1}{\rho}\right)}{u^2 + B + o\left(\frac{1}{\rho}\right)} \quad (\rho \to \infty),$$

where A and B are different real numbers, $u = \rho e^{i\theta}$, and

$$I(\rho,\theta) = I(\rho,\theta+\pi).$$

For each $X \in \mathbb{R}$ we have

$$\left| u^2 + X + o\left(\frac{1}{\rho}\right)\right|^2 = \rho^2\left(\rho^2 + 2X\cos 2\theta + o\left(\rho^{-1}\right)\right) \quad (\rho \to \infty).$$

For large enough ρ, we are interested in the situation when

$$I(\rho,\theta)=\frac{\rho^2+2A\cos 2\theta+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}{\rho^2+2B\cos 2\theta+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}\leq\frac{\rho^2-2A+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}{\rho^2-2B+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}$$

$$=\left|I\left(\rho,\frac{\pi}{2}\right)\right|=\left|I\left(\rho,\frac{3\pi}{2}\right)\right|\qquad \text{if}\quad A<B\quad \theta\in[0,2\pi) \tag{25}$$

(26)

$$I(\rho,\theta)=\frac{\rho^2+2A\cos 2\theta+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}{\rho^2+2B\cos 2\theta+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}\leq\frac{\rho^2+2A+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}{\rho^2+2B+o\left(\rho^{-1}\right)_{(\rho\to\infty)}}$$

$$=\left|I\left(\rho,0\right)\right|=\left|I\left(\rho,\pi\right)\right|\qquad \text{if}\quad A>B\quad \theta\in[0,2\pi)$$

That means

$$\left(\rho^2 + 2A\cos 2\theta + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)\left(\rho^2 - 2B + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)$$
$$-\left(\rho^2 - 2A + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)\left(\rho^2 + 2B\cos 2\theta + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right) \leq 0$$

for each $\theta \in [0, 2\pi)$ if $A < B$.
Analogously, if $A > B$ we have

$$\left(\rho^2 + 2A\cos 2\theta + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)\left(\rho^2 + 2B + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)$$
$$-\left(\rho^2 + 2A + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right)\left(\rho^2 + 2B\cos 2\theta + o\left(\rho^{-1}\right)_{(\rho\to\infty)}\right) \leq 0$$

for each $\theta \in [0, 2\pi)$.

The expressions on the left-hand sides of the sign $\leq$ in the previous two inequalities are of the form

$$2(A-B)(1+\cos 2\theta)\rho^2 + o(\rho) \quad {\scriptstyle(\rho\to\infty)}$$

and

$$2(B-A)(1-\cos 2\theta)\rho^2 + o(\rho) \quad {\scriptstyle(\rho\to\infty)},$$

when $A < B$ and $A > B$, respectively.

The last two inequalities hold always, except when in their left-hand sides

- $\theta = \pi/2$ and $\theta = 3\pi/2$ $(A < B)$
- or $\theta = 0$ and $\theta = \pi$ $(A > B)$, respectively.

$$D_n = 3\,\frac{\displaystyle\sum_{l=0}^{m}{}'' \frac{d_{2l,n}}{4l^2-9}}{\displaystyle\sum_{l=0}^{m}{}'' \frac{d_{2l,n}}{4l^2-1}} - \frac{d_{n-2,n}}{d_{n,n}} = 3\,\frac{\displaystyle\sum_{l=0}^{m}{}'' \frac{d_{2l,n}}{4l^2-9}}{\displaystyle\sum_{l=0}^{m}{}'' \frac{d_{2l,n}}{4l^2-1}} - \frac{n}{2n-1}. \tag{29}$$

if n is even, and

$$D_n = 2\,\frac{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,n}}{(2l+1)^2-16}}{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,n}}{(2l+1)^2-4}} - \frac{d_{n-2,n}}{d_{n,n}} = 2\,\frac{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,n}}{(2l+1)^2-16}}{\displaystyle\sum_{l=0}^{m} \frac{d_{2l+1,n}}{(2l+1)^2-4}} - \frac{n}{2n-1}, \tag{30}$$

if n is odd (if $D_n \neq 0$), i. e.

1) if $D_n < 0$, then the modulus of the kernel should attain its maximum at $\theta = \pi/2$ (and $\theta = 3\pi/2$);

2) if $D_n > 0$, then the modulus of the kernel should attain its maximum at $\theta = 0$ (and $\theta = \pi$).

Conjecture

It is conjectured that the maximum modulus of the kernel $K_n(z)$ *is attained at* $\theta = \pi/2$ *(and* $\theta = 3\pi/2$*), i. e.*

$$\max_{z \in \mathcal{E}_\rho} |K_n(z)| = \max_{\theta \in [0,2\pi)} |K_n(z)| = \left| K_n \left(\pm \frac{i}{2} (\rho - \rho^{-1}) \right) \right|,$$

for $\rho \geq \rho^*$, *i. e. for* ρ *large enough.*

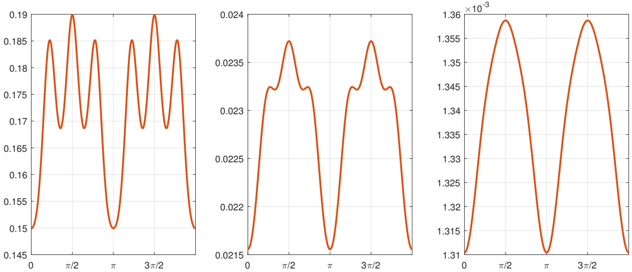
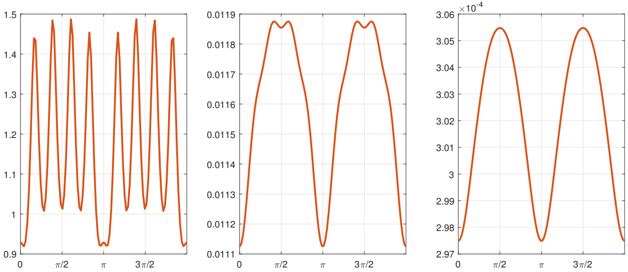


Figure: $|K_3(z)|$ in the cases $\rho = 1.5;\ 2;\ 3$, respectively.



Explicit expressions for the modulus of the kernel

In this section we will present the analytical representation of the modulus of the *kernel* $K_n(z)$. Since the *kernel* is given by

$$K_n(z) = \frac{\varrho_n(z)}{\pi_n(z)} = \frac{A + iB}{C + iD},$$

the explicit representation of its modulus can be expressed by

$$|K_n(z)| = \frac{\sqrt{A^2 + B^2}}{\sqrt{C^2 + D^2}},$$

Therefore it is easy to compute the modulus $|\varrho_n(z)| = \sqrt{A^2 + B^2}$.

Explicit expressions for the modulus of the kernel

$$\varrho_n(z) = 2\sum_{k=0}^{n}{}'' d_{k,n} I_k = 2\sum_{k=0}^{n}{}'' d_{k,n}\left(T_k(z) I_0 - 4\sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{T_{k-2j+1}(z)}{2j-1} \right)$$

$$= \underbrace{\sum_{k=0}^{n}{}'' 2d_{k,n}\big(Re(T_k)Re(I_0) - Im(T_k)Im(I_0) + Re(S)\big)}_{A}$$

$$+ \quad i\cdot \underbrace{\sum_{k=0}^{n}{}'' 2d_{k,n}\big(Re(T_k)Im(I_0) + Im(T_k)Re(I_0) + Im(S)\big)}_{B}.$$

$$Im(I_0) = \arctan\left(\frac{\gamma-1}{\delta}\right) - \arctan\left(\frac{\gamma+1}{\delta}\right). \quad Re(I_0) = -\frac{1}{2}\ln\frac{(\gamma-1)^2+\delta^2}{(\gamma+1)^2+\delta^2},$$

Explicit expressions for the modulus of the kernel

With introduced substitutions

$\gamma = \dfrac{\rho + \rho^{-1}}{2} \cos\theta$ and $\delta = \dfrac{\rho - \rho^{-1}}{2} \sin\theta$.

$$T_k = \frac{1}{2}\left(u^k + \frac{1}{u^k}\right) = \underbrace{\frac{\rho^k + \rho^{-k}}{2} \cos k\theta}_{Re(T_k)} + i \cdot \underbrace{\frac{\rho^k - \rho^{-k}}{2} \sin k\theta}_{Im(T_k)}.$$

$$S = -\sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{T_{k-2j+1}(z)}{2j-1} = Re(S) + i \cdot Im(S).$$

$$Re(S) = -\sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{1}{2j-1}\left(\frac{\rho^{k-2j+1} + \rho^{-(k-2j+1)}}{2} \cos(k-2j+1)\theta\right),$$

$$Im(S) = -\sum_{j=1}^{\left[\frac{k+1}{2}\right]}{}' \frac{1}{2j-1}\left(\frac{\rho^{k-2j+1} - \rho^{-(k-2j+1)}}{2} \sin(k-2j+1)\theta\right).$$

Explicit expressions for the modulus of the kernel

$I_0 = 2\log\frac{1+u^{-1}}{1-u^{-1}} \quad (|u|>1)$ for n odd (it is analogous when n is even)

$$\rho_n(z) = 2\sum_{k=0}^{n}{}'' d_{k,n} I_k = -8\sum_{k=0}^{n}{}'' d_{k,n}\sum_{j=1}^{\infty}\frac{2j}{k^2-(2j)^2}\cdot\frac{1}{u^{2j}} = \sum_{j=1}^{\infty}\frac{c_j}{u^{2j}},$$

with

$$c_j = -16j\sum_{k=0}^{n}{}''\frac{d_{k,n}}{k^2-(2j)^2}.$$

Finally, we derive

$$|\varrho_n(z)| = \sqrt{\sum_{j=1}^{\infty} c_j^2\rho^{-4j} + 2\sum_{i=2}^{\infty}\sum_{j=1}^{i-1} c_i c_j \rho^{-2i-2j}\cos 2(i-j)\theta}.$$

Similarly, in order to calculate $|\pi_n(z)| = \left|\sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k}\right| = \sqrt{C^2 + D^2}$ with $\alpha = \beta = 0$, we need to express its real and imaginary parts

$$C = \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} d_{n-2k,n} \left(\rho^{n-2k} + \rho^{-(n-2k)}\right) \cos(n-2k)\theta + d_{0,n}, & n \quad \text{even}, \\ \sum_{k=0}^{\frac{n-1}{2}} d_{n-2k,n} \left(\rho^{n-2k} + \rho^{-(n-2k)}\right) \cos(n-2k)\theta, & n \quad \text{odd}, \end{cases}$$

$$D = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} d_{n-2k,n} \left(\rho^{n-2k} - \rho^{-(n-2k)}\right) \sin(n-2k)\theta.$$

According to the previously introduced notation, under the assumption that f is analytic inside $\mathcal{E}_{\rho_{\max}}$, the error bound of the corresponding quadrature formula can be optimized by

$$|R_n(f)| \le r_n(f),$$

where, according to (**??**) with $\Gamma = \mathcal{E}_\rho$,

$$r_n(f) = \inf_{\rho^* < \rho < \rho_{\max}} \left[\frac{\ell(\mathcal{E}_\rho)}{2\pi} \left(\max_{z \in \mathcal{E}_\rho} |K_n(z)| \right) \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right) \right].$$

$$\ell(\mathcal{E}_\rho) \le 2\pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right),$$

where $a_1 = (\rho + \rho^{-1})/2$.

Error bound $r_n(f)$ reduces to

$$r_n(f,\omega) = \inf_{\rho^*<\rho<\rho_{\max}} \left[a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right) \left(\max_{z\in\mathcal{E}_\rho} |K_n(z)| \right) \left(\max_{z\in\mathcal{E}_\rho} |f(z)| \right) \right].$$

We made several tests and compared them with respect to the exact (actual) errors.

Example ($f_1(z) = e^{e^{\cos(\omega z)}}$, $\omega > 0$)

The function f_1 is entire and it is known that

$$\max_{z\in\mathcal{E}_\rho} |f_1(z)| = e^{e^{\cosh(\omega b_1)}}, \quad b_1 = (\rho - \rho^{-1})/2, \quad \rho_{\max} = +\infty.$$

Example 1 - $f_1(z) = e^{e^{\cos(\omega z)}}$, $\omega = 1$ results

Table: Error bounds $r_n(f_1)$, actual errors ($Error$) and exact value of integral (I_ω)

(n, ω)	$r_n(f_1)$	$Error$	ρ_{opt}	I_ω
(2,1)	7.328 (00)	1.592 (00)	2.071	21.7719...
(5,1)	2.289 (-02)	5.849 (-03)	2.560	21.7719...
(6,1)	6.700 (-03)	7.669(-04)	2.661	21.7719...
(7,1)	4.622 (-04)	9.527(-05)	2.747	21.7719...
(8,1)	1.195(-04)	1.131(-05)	2.821	21.7719...
(9,1)	7.364 (-06)	1.291(-06)	2.884	21.7719...
(10,1)	1.738(-06)	1.423(-07)	2.941	21.7719...
(11,1)	9.894 (-08)	1.523(-08)	2.992	21.7719...
(13,1)	1.166 (-09)	1.611(-10)	3.078	21.7719...
(15,1)	1.237(-11)	1.558(-12)	3.150	21.7719...
(25,1)	5.573(-22)	5.046(-23)	3.410	21.7719...

Example 1 - $f_1(z) = e^{e^{\cos(\omega z)}}$, $\omega = 2$ results

Table: Error bounds $r_n(f_1)$, actual errors ($Error$) and exact value of integral (I_ω)

(n,ω)	$r_n(f_1)$	$Error$	ρ_{opt}	I_ω
(3,2)	4.812 (00)	2.477 (00)	1.145	14.0805...
(6,2)	1.086 (00)	1.251(-01)	1.621	14.0805...
(7,2)	2.028 (-01)	4.261(-02)	1.657	14.0805...
(8,2)	1.448(-01)	1.401(-02)	1.691	14.0805...
(9,2)	2.488 (-02)	4.474(-03)	1.722	14.0805...
(10,2)	1.650(-02)	1.391(-03)	1.751	14.0805...
(11,2)	2.666 (-03)	4.224(-04)	1.774	14.0805...
(13,2)	2.570 (-04)	3.661(-05)	1.816	14.0805...
(15,2)	2.273(-05)	2.955(-06)	1.859	14.0805...
(17,2)	1.872 (-06)	2.245(-07)	1.861	14.0805...
(18,2)	1.050(-06)	6.065(-08)	1.901	14.0805...
(19,2)	1.449 (-07)	1.618(-08)	1.909	14.0805...
(20,2)	7.905(-08)	4.269(-09)	1.921	14.0805...
(25,2)	5.008(-11)	4.680(-12)	1.955	14.0805...

Example ($f_2(z) = e^{\omega z^2}, \ \omega > 0$.)

The function f_2 is entire and it is known that

$$\max_{z \in \mathcal{E}_\rho} |f_1(z)| = e^{\omega a_1^2}, \quad a_1 = (\rho + \rho^{-1})/2, \quad \rho_{\max} = +\infty.$$

$f_2(z) = e^{\omega z^2}, \ \ \omega > 0$ - Results

(n, ω)	$r_n(f_2)$	*Error*	ρ_{opt}	I_ω
(2,0.5)	1.068(-01)	2.719(-02)	4.015	2.3899...
(4,0.5)	2.060(-04)	3.785(-05)	6.176	2.3899...
(7,0.5)	1.243(-09)	3.637(-10)	7.481	2.3899...
(9, 0.5)	3.069(-13)	7.974(-14)	8.490	2.3899...
(10, 0.5)	8.098(-15)	1.000(-15)	8.719	2.3899...
(11, 0.5)	4.830(-17)	6.938(-18)	9.382	2.3899...
(15, 0.5)	4.261(-25)	8.566(-26)	10.930	2.3899...
(16, 0.5)	8.438(-27)	6.702(-28)	10.880	2.3899...
(7,1)	2.059(-07)	5.911(-08)	5.291	2.9253...
(9, 1)	2.072(-10)	5.196(-11)	5.961	2.9253...
(11, 1)	1.276(-13)	2.977(-14)	6.640	2.9253...
(13, 1)	1.112(-16)	2.081(-17)	7.211	2.9253...
(15, 1)	1.780(-20)	3.585(-21)	7.750	2.9253...
(16, 1)	5.755(-22)	5.612(-23)	7.699	2.9253...
(19, 1)	2.203(-27)	1.514(-28)	7.810	2.9253...

Example $f_3(z) = \dfrac{\cos(z)}{z^2+\omega^2}$, $\omega > 0$ and results

For function $f_3(z)$ it holds that

$$\max_{z\in\mathcal{E}_\rho} |f_3(z)| = \frac{\cosh(b_1)}{-a_1^2+\omega^2},$$

where $b_1 = (\rho - \rho^{-1})/2$. Here the infimum is calculated with respect to the interval $\rho \in (\rho^*, \rho_{\max})$, where $\rho_{\max} = \omega + \sqrt{1+\omega^2}$. We made a couple of experiments with intervals $(\rho^*, \rho_{\max})$, which exists, and obtained in each case effective bound.

(n,ω)	$r_n(f_3)$	*Error*	I_ω
(2,0.5)	1.281 (00)	1.031 (00)	4.3181...
(3,0.5)	5.793 (-01)	4.478 (-01)	4.3181...
(4,0.5)	1.913 (-01)	1.643 (-01)	4.3181...
(5,0.5)	7.059 (-02)	6.415 (-02)	4.3181...
(6,0.5)	2.560 (-02)	2.441 (-02)	4.3181...
(7,0.5)	1.139 (-02)	9.370 (-03)	4.3181...
(8,0.5)	5.068 (-03)	3.581 (-03)	4.3181...
(10,0.5)	1.004 (-03)	5.239 (-04)	4.3181...

(n,ω)	$r_n(f_3)$	*Error*	I_ω
(2,1)	2.330 (-01)	1.090 (-01)	1.3658...
(3,1)	2.749 (-01)	1.934 (-02)	1.3658...
(4,1)	3.741 (-02)	3.347 (-03)	1.3658...
(5,1)	2.501 (-02)	5.784 (-04)	1.3658...
(6,1)	7.203 (-03)	9.971 (-05)	1.3658...
(7,1)	1.717 (-04)	1.717 (-05)	1.3658...
(8,1)	1.316 (-05)	2.954 (-06)	1.3658...
(10,1)	7.928 (-07)	8.729 (-08)	1.3658...

Previously Published Results:

D.R. Jandrlic et al., *Error bounds of Gaussian quadrature formulae with Legendre weight function for analytic integrands*, ETNA. 46 (2022), pp. 424—437.

A.V. Pejcev *A note on "Error bounds of Gaussian quadrature formulae with Legendre weight function for analytic integrands" by M. M. Spalevic et al.*, ETNA. 59 (2023), pp. 89—98.

- derived effective error bounds for Gauss-Legendre QF.
- established asymptotic estimate of the kernel
- formulated conjecture which describes the behavior of the kernels
- performed numerical experiments to confirm that error bounds obtained in this way are very close to actual.

Preliminary Findings: Gegenbauer polynomials

- $\alpha = \beta \in \mathbb{N}$,
- Weight function

$$omega(t) = 1 - t^2$$

.

- $$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} d_{|n-2k|,n} u^{n-2k}, \quad z = \frac{1}{2}\left(u + u^{-1}\right), \qquad (32)$$

- where, in the case $\alpha = \beta > -1$,

$$d_{k,n} = \begin{cases} \dfrac{2^{2\alpha}\Gamma(n+\alpha+1)\Gamma\left(\frac{k+n+1}{2}+\alpha\right)\Gamma\left(\frac{n-k+1}{2}+\alpha\right)}{\sqrt{\pi}\Gamma(n+2\alpha+1)\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{n-k}{2}+1\right)\Gamma(\alpha+1/2)}, \\ 0, \end{cases} \qquad (33)$$

Properties of the Chebyshev polynomials of the second kind

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad n \geq 0, \quad U_n(z) = \frac{u^{n+1} - u^{-n-1}}{u - u^{-1}}$$

Connection with Chebyshev polynomials was established by Notaris:

$$\int_{-1}^{1} \frac{p_n(t)}{z \mp t}\, dt, \quad |z| \neq 1,$$

with the p_n being any one of the Chebysev polynomials of degree n, this integral can be computed explicitly

Modulus of the kernel

Here, $\pi_n(z) = P_n^{(\alpha,\alpha)}(z)$ represents the Gegenbauer polynomial of degree n. The function $\varrho_n(z)$ is given by the integral:

$$\varrho_n(z) = \int_{-1}^{1} \frac{P_n^{(\alpha,\alpha)}(t)}{z-t}(1-t^2)\,dt \tag{34}$$

Additionally from equation:

$$P_n^{(\alpha,\alpha)}(\cos\theta) = 2\sum_{k=0}^{n}{}' d_{k,n}\cos k\theta, \tag{35}$$

Now the function $\varrho_n(z)$ is

$$\varrho_n(z) = \int_0^{\pi} \frac{2\sum_{k=0}^{\prime n} d_{k,n}\cos k\theta}{z-\cos\theta}\sin\theta^{2\alpha+1}\,d\theta \tag{36}$$

Considering:

$$(\sin\theta)^{2s+1} = 2^{-2s}\sum_{i=0}^{s}(-1)^s\binom{2s+1}{s-i}\sin(2i+1)\theta, \qquad (37)$$

$$\begin{aligned}
\varrho_n(z) &= \int_0^\pi \frac{2\sum_{k=0}^{\prime n} d_{k,n}\cos k\theta \cdot 2^{-2\alpha}\sum_{i=0}^{\alpha}(-1)^\alpha\binom{2\alpha+1}{\alpha-i}\sin(2i+1)\theta}{z-\cos\theta}d\theta \\
&= \frac{1}{2^{2\alpha}}\sum_{k=0}^{n}{}' d_{k,n}\sum_{i=0}^{\alpha}(-1)^i\binom{2\alpha+1}{\alpha-i}\int_0^\pi \frac{\sin(k+1+2i)\theta-\sin(k-2i-1)\theta}{z-\cos\theta}d\theta \qquad (38) \\
&= \begin{cases} \frac{1}{2^{2\alpha}}\sum_{k=0}^{n}{}' d_{k,n}\sum_{i=0}^{\alpha}(-1)^i\binom{2\alpha+1}{\alpha-i}\Big[I_{k+2i}-I_{k-2i-2}\Big], & k\geq 2i+1, \\ \frac{1}{2^{2\alpha}}\sum_{k=0}^{n}{}' d_{k,n}\sum_{i=0}^{\alpha}(-1)^i\binom{2\alpha+1}{\alpha-i}\Big[I_{k+2i}+I_{2i-k}\Big], & k<2i+1. \end{cases}
\end{aligned}$$

The term I_k in the equation is obtained from the integral expression that appears in the previous calculations. Further, using results by Notaris, for the expressions

$$I_k = \int_0^\pi \frac{\sin(k+1)\theta}{z - \cos\theta} d\theta = \int_{-1}^1 \frac{U_k(t)}{z-t}\, dt, \tag{39}$$

it stands that

$$I_k = U_k(z) \log \frac{z+1}{z-1} - 4 \sum_{j=1}^{\left[\frac{k+1}{2}\right]} \frac{U_{k-2j+1}(z)}{2j-1} \quad (|z| > 1), \tag{40}$$

$$I_k = 4(u-u^{-1})^{-1}\left[\left(u^{k+1}-\frac{1}{u^{k+1}}\right)\sum_{j=0}^{\infty}\frac{1}{(2j+1)u^{2j+1}} - \sum_{j=1}^{\left[\frac{k+1}{2}\right]}\frac{u^{k-2j+2}-\frac{1}{u^{k-2j+2}}}{2j-1}\right], \quad (41)$$

$$I_k = 4\left[\sum_{t=0}^{k}u^{2t-k}\sum_{j=0}^{\infty}\frac{1}{(2j+1)u^{2j+1}} - \sum_{j=1}^{\left[\frac{k+1}{2}\right]}\frac{1}{2j+1}\sum_{t=0}^{k-2j+1}u^{2t-(k-2j+1)}\right]. \quad (42)$$

Proposition

It holds that

$$I_k = \begin{cases} 4\left[\frac{1}{k+1}u^{-1}+\frac{3k^2+4k-1}{(k-1)(k+1)(k+2)}u^{-3}\right]+o\left(u^{-5}\right), & k \quad even \\ 4\left[\frac{2k+2}{k(k+2)}u^{-2}+\frac{4(k+1)(k^2+2k-4)}{k(k+4)(k^2-4)}u^{-4}\right]+o\left(u^{-6}\right), & k \quad odd \end{cases} \quad (43)$$

$$\varrho_n(z) = \frac{c_1}{u} + \frac{c_3}{u^3} + o(u^5), \tag{44}$$

where c_1 is given by:

$$c_1 = \frac{1}{4^\alpha} {\sum_{l=0}^{m}}' d_{2l,2m} \sum_{i=0}^{\alpha} (-1)^i \binom{2\alpha+1}{\alpha-i} c^1(l,i), \tag{45}$$

and c_3 is given by:

$$c_3 = \frac{1}{4^\alpha} {\sum_{l=0}^{m}}' d_{2l,2m} \sum_{i=0}^{\alpha} (-1)^i \binom{2\alpha+1}{\alpha-i} c^3(l,i). \tag{46}$$

THANK YOU FOR YOUR ATTENTION!