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Research Article

Coupled Coincidence Points of Mappings in Ordered Partial Metric Spaces

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New coupled coincidence point and coupled fixed point results in ordered partial metric spaces under the contractive conditions of Geraghty, Rakotch, and Branciari types are obtained. Examples show that these results are distinct from the known ones.

1. Introduction

In recent years many authors have worked on domain theory in order to equip semantics domain with a notion of distance. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [2–9]) have proved fixed point theorems in partial metric spaces.

Many generalizations of Banach's contractive condition have been introduced in order to obtain more general fixed point results in metric spaces and their generalizations. We mention here conditions introduced by Geraghty [10], Rakotch [11], and Branciari [12].

The notion of a coupled fixed point was introduced and studied by Bhaskar and Lakshmikantham in [13]. In subsequent papers several authors proved various coupled and common coupled fixed point theorems in partially ordered metric spaces (e.g., [14–17]). These results were applied for investigation of solutions of differential and integral equations. In a recent paper [18], Berinde presented a method of reducing coupled fixed point results in ordered metric spaces to the respective results for mappings with one variable.

In this paper, we further develop the method of Berinde and obtain new coupled coincidence and coupled fixed point results in ordered partial metric spaces, under the contractive conditions of Geraghty, Rakotch, and Branciari types. Examples show that these results are distinct from the known ones. In particular, they show that using the order and/or the partial metric enables conclusions which cannot be obtained in the classical case.

2. Notation and Preliminary Results

2.1. Partial Metric Spaces

The following definitions and details can be seen in [1–9].

Definition 2.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$,

(P_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(P_2) $p(x,x) \leq p(x,y)$,

(P_3) $p(x,y) = p(y,x)$,

(P_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a *partial metric space*.

It is clear that if $p(x,y) = 0$, then from (P_1) and (P_2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$, with respect to τ_p, if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$. This will be denoted as $x_n \to x$, $n \to \infty$, or $\lim_{n\to\infty} x_n = x$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.1}$$

is metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.2}$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. The corresponding metric is

$$p^s(x,y) = 2\max\{x,y\} - x - y = |x - y|. \tag{2.3}$$

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [1, 19].

Remark 2.2. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x,y)$.

Definition 2.3. Let (X, p) be a partial metric space. Then,

(1) a sequence $\{x_n\}$ in (X, p) is called a *Cauchy sequence* if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite);

(2) the space (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 2.4. *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b) *The space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Definition 2.5. Let X be a nonempty set. Then $(X, \preceq, p)$ is called an *ordered partial metric space* if (i) $(X, \preceq)$ is a partially ordered set, and (ii) (X, p) is a partial metric space.

We will say that the space $(X, \preceq, p)$ satisfies the *ordered-regular condition* (abr. (ORC)) if the following holds: if $\{x_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 2.6 (see [13, 14]). Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$, and $g : X \to X$.

(1) F is said to have *g-mixed monotone property* if the following two conditions are satisfied:

$$\begin{aligned} &(\forall x_1, x_2, y \in X)\ gx_1 \preceq gx_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y), \\ &(\forall x, y_1, y_2 \in X) gy_1 \preceq gy_2 \Longrightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned} \tag{2.4}$$

If $g = i_X$ (the identity map), we say that F has the *mixed monotone property*.

(2) A point $(x, y) \in X \times X$ is said to be a *coupled coincidence point* of F and g if $F(x, y) = gx$ and $F(y, x) = gy$ and their *common coupled fixed point* if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

Definition 2.7 (see [20]). Let (X, d) be a metric space, and let $F : X \times X \to X$ and $g : X \to X$. The pair (F, g) is said to be *compatible* if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \qquad \lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0 \tag{2.5}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g\ x_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g\ y_n = y$ for some $x, y \in X$.

2.2. Some Auxiliary Results

Lemma 2.8. (i) *Let $(X, \preceq, p)$ be an ordered partial metric space. If relation $\sqsubseteq$ is defined on X^2 by*

$$Y \sqsubseteq V \Longleftrightarrow x \preceq u \wedge y \succeq v, \qquad Y = (x, y), \qquad V = (u, v) \in X^2, \tag{2.6}$$

and $P : X^2 \times X^2 \to \mathbb{R}^+$ *is given by*

$$P(Y, V) = p(x, u) + p(y, v), \qquad Y = (x, y), \qquad V = (u, v) \in X^2, \tag{2.7}$$

then $(X^2, \sqsubseteq, P)$ *is an ordered partial metric space. The space* $(X^2, \sqsubseteq, P)$ *is complete iff* $(X, \preceq, p)$ *is complete.*

(ii) If $F : X \times X \to X$ *and* $g : X \to X$*, and* F *has the* g*-mixed monotone property, then the mapping* $T_F : X^2 \to X^2$ *given by*

$$T_F Y = (F(x, y), F(y, x)), \qquad Y = (x, y) \in X^2 \tag{2.8}$$

is T_g*-nondecreasing with respect to* $\sqsubseteq$*, that is,*

$$T_g Y \sqsubseteq T_g V \Longrightarrow T_F Y \sqsubseteq T_F V, \tag{2.9}$$

where $T_g Y = T_g(x, y) = (gx, gy)$*.*

(iii) If g *is continuous in* (X, p) *(i.e., with respect to* τ_p*), then* T_g *is continuous in* (X^2, P) *(i.e., with respect to* τ_P*). If* F *is continuous from* (X^2, P) *to* (X, p) *(i.e.,* $x_n \to x$ *and* $y_n \to y$ *imply* $F(x_n, y_n) \to F(x, y)$*), then* T_F *is continuous in* (X^2, P)*.*

Proof. (i) Relation $\sqsubseteq$ is obviously a partial order on X^2. To prove that P is a partial metric on X^2, only conditions (P_1) and (P_4) are nontrivial.

(P_1) If $Y = V \in X^2$, then obviously $p(Y, Y) = p(Y, V) = p(V, V)$ holds. Conversely, let $p(Y, Y) = p(Y, V) = p(V, V)$, that is,

$$p(x, x) + p(y, y) = p(x, u) + p(y, v) = p(u, u) + p(v, v). \tag{2.10}$$

We know by (P_2) that $p(x, x) \leq p(x, u)$ and $p(y, y) \leq p(y, v)$. Adding up, we obtain that $p(x, x) + p(y, y) \leq p(x, u) + p(y, v)$, and since in fact equality holds, we conclude that $p(x, x) = p(x, u)$ and $p(y, y) = p(y, v)$. Similarly, we get that $p(u, u) = p(x, u)$ and $p(v, v) = p(y, v)$. Hence,

$$p(x, x) = p(x, u) = p(u, u), \qquad p(y, y) = p(y, v) = p(v, v), \tag{2.11}$$

and applying property (P_1) of partial metric p, we get that $x = u$ and $y = v$, that is, $Y = V$.

(P_4) Let $Y = (x, y), V = (u, v), Z = (w, z) \in X^2$. Then

$$\begin{aligned} P(Y, V) &= p(x, u) + p(y, v) \\ &\leq (p(x, w) + p(w, u) - p(w, w)) + (p(y, z) + p(z, v) - p(z, z)) \\ &= p(x, w) + p(y, z) + p(w, u) + p(z, v) - [p(w, w) + p(z, z)] \\ &= P(Y, Z) + P(Z, V) - P(Z, Z). \end{aligned} \tag{2.12}$$

(ii, iii) The proofs of these assertions are straightforward. □

Remark 2.9. Let p^s be the metric associated with the partial metric p as in (2.1). It is easy to see that, with notation as in the previous lemma,

$$P^s(Y,V) = p^s(x,u) + p^s(y,v), \qquad Y = (x,y), \qquad V = (u,v) \in X^2 \tag{2.13}$$

is the associated metric to the partial metric P on X^2. We note, however, that when we speak about continuity of mappings, we always assume continuity in the sense of the partial metric p, that is, in the sense of the respective topology τ_p. This should not be confused with the approach given by O'Neill in [3] where both p- and p^s-continuity were assumed.

It is easy to see that (using notation as in the previous lemma), the mappings F and g are p^s-compatible (in the sense of Definition 2.7) if and only if the mappings T_F and T_g are P^s-compatible in the usual sense (i.e., $\lim_{n\to\infty} d(T_F T_g Y_n, T_g T_F Y_n) = 0$, whenever $\{Y_n\}$ is a sequence in X^2 such that $\lim_{n\to\infty} T_F Y_n = \lim_{n\to\infty} T_g Y_n$).

Assertions similar to the following lemma (see, e.g., [21]) were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 2.10. *Let (X,d) be a metric space, and let $\{x_n\}$ be a sequence in X such that $\{d(x_{n+1},x_n)\}$ is decreasing and*

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{2.14}$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$d(x_{2m_k}, x_{2n_k}), \qquad d(x_{2m_k}, x_{2n_k+1}), \qquad d(x_{2m_k-1}, x_{2n_k}), \qquad d(x_{2m_k-1}, x_{2n_k+1}). \tag{2.15}$$

As a corollary (applying Lemma 2.10 to the associated metric p^s of a partial metric p, and using Lemma 2.4) we obtain the following.

Lemma 2.11. *Let (X,p) be a partial metric space, and let $\{x_n\}$ be a sequence in X such that $\{p(x_{n+1},x_n)\}$ is decreasing and*

$$\lim_{n\to\infty} p(x_{n+1},x_n) = 0. \tag{2.16}$$

If $\{x_{2n}\}$ is not a Cauchy sequence in (X,p), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$p(x_{2m_k}, x_{2n_k}), \qquad p(x_{2m_k}, x_{2n_k+1}), \qquad p(x_{2m_k-1}, x_{2n_k}), \qquad p(x_{2m_k-1}, x_{2n_k+1}). \tag{2.17}$$

3. Coupled Coincidence and Fixed Points under Geraghty-Type Conditions

Let $\mathcal{G}$ denote the class of real functions $\gamma : [0,+\infty) \to [0,1)$ satisfying the condition

$$\gamma(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0. \tag{3.1}$$

An example of a function in $\mathcal{G}$ may be given by $\gamma(t) = e^{-2t}$ for $t > 0$ and $\gamma(0) \in [0,1)$. In an attempt to generalize the Banach contraction principle, Geraghty proved in 1973 the following.

Theorem 3.1 (see [10]). *Let (X,d) be a complete metric space, and let $T : X \to X$ be a self-map. Suppose that there exists $\gamma \in \mathcal{G}$ such that*

$$d(Tx,Ty) \leq \gamma(d(x,y))d(x,y) \tag{3.2}$$

holds for all $x,y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to z when $n \to \infty$.

Subsequently, several authors proved such results, including the very recent paper of Đukić et al. [22].

We begin with the following auxiliary result.

Lemma 3.2. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete. Let $T,S : X \to X$ be self-maps such that S is continuous, $TX \subset SX$, and one of these two subsets of X is closed. Suppose that T is S-nondecreasing (with respect to $\preceq$) and there exists $x_0 \in X$ with $Sx_0 \preceq Tx_0$ or $Tx_0 \preceq Sx_0$. Assume also that there exists $\gamma \in \mathcal{G}$ such that*

$$p(Tx,Ty) \leq \gamma(p(Sx,Sy))p(Sx,Sy) \tag{3.3}$$

holds for all $x,y \in X$ such that Sx and Sy are comparable. Assume that either 1°: T is continuous and the pair (T,S) is p^s- compatible or 2°: X satisfies (ORC). Then, T and S have a coincidence point in X.

Proof. The proof follows the lines of proof of [22, Theorems 3.1 and 3.5].

Take $x_0 \in X$ with, say, $Sx_0 \preceq Tx_0$, and using that T is S-nondecreasing and that $TX \subset SX$ form the sequence $\{x_n\}$ satisfying $Tx_n = Sx_{n+1}$, $n = 0,1,2,\ldots$, and

$$Sx_0 \preceq Tx_0 = Sx_1 \preceq Tx_1 = Sx_2 \preceq \cdots \preceq Tx_n \preceq Sx_{n+1} \preceq \cdots . \tag{3.4}$$

Since Sx_{n-1} and Sx_n are comparable, we can apply the contractive condition to obtain

$$\begin{aligned} p(Sx_{n+1},Sx_n) = p(Tx_n,Tx_{n-1}) &\leq \gamma\big(p(Sx_{n-1},Sx_n)\big)p(Sx_{n-1},Sx_n) \\ &\leq p(Sx_{n-1},Sx_n). \end{aligned} \tag{3.5}$$

Consider the following two cases:

(1) $p(Sx_{n_0+1},Sx_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$;

(2) $p(Sx_{n+1},Sx_n) > 0$ for each $n \in \mathbb{N}$.

Case 1. Under this assumption, we get that

$$\begin{aligned} p(Sx_{n_0+2}, Sx_{n_0+1}) &= p(Tx_{n_0+1}, Tx_{n_0}) \le \gamma(p(Sx_{n_0+1}, Sx_{n_0}))p(Sx_{n_0+1}, Sx_{n_0}) \\ &= \gamma(0) \cdot 0 = 0, \end{aligned} \tag{3.6}$$

and it follows that $p(Sx_{n_0+2}, Sx_{n_0+1}) = 0$. By induction, we obtain that $p(Sx_{n+1}, Sx_n) = 0$ for all $n \ge n_0$ and so $Sx_n = Sx_{n_0}$ for all $n \ge n_0$. Hence, $\{Sx_n\}$ is a Cauchy sequence, converging to Sx_{n_0}, and x_{n_0} is a coincidence point of S and T.

Case 2. We will prove first that in this case the sequence $\{p(Sx_{n+1}, Sx_n)\}$ is strictly decreasing and tends to 0 as $n \to \infty$.

For each $n \in \mathbb{N}$ we have that

$$\begin{aligned} 0 < p(Sx_{n+2}, Sx_{n+1}) &= p(Tx_{n+1}, Tx_n) \le \gamma(p(Sx_{n+1}, Sx_n))p(Sx_{n+1}, Sx_n) \\ &< p(Sx_{n+1}, Sx_n). \end{aligned} \tag{3.7}$$

Hence, $p(Sx_{n+1}, Sx_n)$ is strictly decreasing and bounded from below, thus converging to some $q \ge 0$. Suppose that $q > 0$. Then, it follows from (3.7) that

$$\frac{p(Sx_{n+2}, Sx_{n+1})}{p(Sx_{n+1}, Sx_n)} \le \gamma(p(Sx_{n+1}, Sx_n)) < 1, \tag{3.8}$$

wherefrom, passing to the limit when $n \to \infty$, we get that $\lim_{n\to\infty}\gamma(p(Sx_{n+1}, Sx_n)) = 1$. Using property (3.1) of the function γ, we conclude that $\lim_{n\to\infty}p(Sx_{n+1}, Sx_n) = 0$, that is, $q = 0$, a contradiction. Hence, $\lim_{n\to\infty}p(Sx_{n+1}, Sx_n) = 0$ is proved.

In order to prove that $\{Sx_n\}$ is a Cauchy sequence in (X, p), suppose the contrary. As was already proved, $p(Sx_{n+1}, Sx_n) \to 0$ as $n \to \infty$, and so, using (P_2), $p(Sx_n, Sx_n) \to 0$ as $n \to \infty$. Hence, using (2.1), we get that $p^s(Sx_{n+1}, Sx_n) \to 0$ as $n \to \infty$. Using Lemma 2.11, we obtain that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$p(Sx_{2m_k}, Sx_{2n_k}), \qquad p(Sx_{2m_k}, Sx_{2n_k+1}), \qquad p(Sx_{2m_k-1}, Sx_{2n_k}), \qquad p(Sx_{2m_k-1}, Sx_{2n_k+1}). \tag{3.9}$$

Putting in the contractive condition $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that

$$\begin{aligned} p(Sx_{2m_k}, Sx_{2n_k+1}) &\le \gamma(p(Sx_{2m_k-1}, Sx_{2n_k}))p(Sx_{2m_k-1}, Sx_{2n_k}) \\ &< p(Sx_{2m_k-1}, Sx_{2n_k}). \end{aligned} \tag{3.10}$$

Hence,

$$\frac{p(Sx_{2m_k}, Sx_{2n_k+1})}{p(Sx_{2m_k-1}, Sx_{2n_k})} \le \gamma(p(Sx_{2m_k-1}, Sx_{2n_k})) < 1 \tag{3.11}$$

and $\lim_{k\to\infty}\gamma(p(Sx_{2m_k-1},Sx_{2n_k}))=1$. Since $\gamma\in\mathcal{G}$, it follows that $\lim_{k\to\infty}p(Sx_{2m_k-1},Sx_{2n_k})=0$, which is in contradiction with $\varepsilon>0$.

Thus, $\{Sx_n\}$ is a Cauchy sequence, both in (X,p) and in (X,p^s). Hence, it converges (in p and in p^s) to a point $Sz\in SX$ (we suppose SX to be closed, that is, complete; the case when TX is closed is treated similarly) such that

$$p(Sz,Sz)=\lim_{n\to\infty}p(Sx_n,Sz)=\lim_{n,m\to\infty}p(Sx_n,Sx_m). \tag{3.12}$$

Also, it follows easily that

$$\lim_{n\to\infty}p(Sx_n,Sz)=p(Sz,Sz)=0. \tag{3.13}$$

We will prove that S and T have a coincidence point.

(i) Suppose that $T:(X,p)\to(X,p)$ is continuous and that (T,S) is a p^s-compatible pair. We have that

$$\begin{aligned}p^s(TSz,SSz)&\le p^s(TSz,TSx_n)+p^s(TSx_n,STx_n)+p^s(STx_n,SSz)\\&\longrightarrow p^s(TSz,TSz)+0+p^s(SSz,SSz)=0,\quad\text{as } n\longrightarrow\infty.\end{aligned} \tag{3.14}$$

It follows that $T(Sz)=S(Sz)$ and Sz is a coincidence point of T and S.

(ii) If (X,p) satisfies (ORC), since $\{Sx_n\}$ is an increasing sequence tending to Sz, we have that $Sx_n\preceq Sz$ for each $n\in\mathbb{N}$. So we can apply (P_4) and the contractive condition to obtain

$$\begin{aligned}p(Sz,Tz)&\le p(Sz,Sx_{n+1})+p(Tx_n,Tz)\\&\le p(Sz,Sx_{n+1})+\gamma(p(Sx_n,Sz))p(Sx_n,Sz)\\&\le p(Sz,Sx_{n+1})+p(Sx_n,Sz).\end{aligned} \tag{3.15}$$

Letting $n\to\infty$ we get $p(Sz,Tz)=0$. Hence, we obtain that $Tz=Sz$ and z is a coincidence point. □

Now, we are in the position to prove the main result of this section.

Theorem 3.3. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete, and let $g:X\to X$ and $F:X\times X\to X$ be such that F has the g-mixed monotone property. Suppose that*

(i) *there exists $\gamma\in\mathcal{G}$ such that*

$$\begin{aligned}&p(F(x,y),F(u,v))+p(F(y,x),F(v,u))\\&\quad\le\gamma(p(gx,gu)+p(gy,gv))(p(gx,gu)+p(gy,gv))\end{aligned} \tag{3.16}$$

holds for all $x,y,u,v\in X$ satisfying ($gx\preceq gu$ and $gy\succeq gv$) or ($gx\succeq gu$ and $gy\preceq gv$);

(ii) *g is continuous, $F(X\times X)\subset g(X)$, and one of these two subsets of X is closed;*

(iii) *there exist* $x_0, y_0 \in X$ *such that* ($gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$) *or* ($gx_0 \succeq F(x_0, y_0)$ *and* $gy_0 \preceq F(y_0, x_0)$);

(iv) F *is continuous and* (F, g) *is compatible in the sense of Definition 2.7, or*

(iv′) $(X, \preceq, p)$ *satisfies (ORC).*

Then there exist $\overline{x}, \overline{y} \in X$ *such that*

$$g\overline{x} = F(\overline{x}, \overline{y}), \qquad g\overline{y} = F(\overline{y}, \overline{x}), \tag{3.17}$$

that is, g and F have a coupled coincidence point.

Proof. Let relation $\sqsubseteq$, partial metric P, and mappings T_F, T_g on X^2 be defined as in Lemma 2.8. Then $(X^2, \sqsubseteq, P)$ is an ordered partial metric space which is complete and T_F is a T_g-nondecreasing self-map on X^2. Moreover,

(i) there exists $\gamma \in \mathcal{G}$ such that

$$P(T_F Y, T_F V) \leq \gamma(P(T_g Y, T_g V)) P(T_g Y, T_g V) \tag{3.18}$$

holds for all $\sqsubseteq$-comparable $Y, V \in X^2$;

(ii) T_g is continuous, $T_F(X^2) \subset T_g(X^2)$, and one of these two subsets of X^2 is closed;

(iii) there exists $Y_0 \in X^2$ such that $T_g Y_0$ and $T_F Y_0$ are comparable;

(iv) T_F is continuous and the pair (T_F, T_g) is P^s-compatible, or

(iv′) $(X^2, \sqsubseteq, P)$ satisfies (ORC).

Thus, all the conditions of Lemma 3.2 are satisfied (with $T = T_F$ and $S = T_g$). Hence, there exists $\overline{Y} \in X^2$ such that $T_g\overline{Y} = T_F\overline{Y}$, that is, there exist $\overline{x}, \overline{y} \in X$ such that $g\overline{x} = F(\overline{x}, \overline{y})$ and $g\overline{y} = F(\overline{y}, \overline{x})$. Therefore, g and F have a coupled coincidence point. □

Putting $g = i_X$ (the identity map) in Theorem 3.3, we obtain the following.

Corollary 3.4. *Let $(X, \preceq, p)$ be an ordered partial metric space which is complete, and let $F : X \times X \to X$ have the mixed monotone property. Suppose that*

(i) *there exists* $\gamma \in \mathcal{G}$ *such that*

$$\begin{aligned} &p(F(x, y), F(u, v)) + p(F(y, x), F(v, u)) \\ &\quad \leq \gamma(p(x, u) + p(y, v))(p(x, u) + p(y, v)) \end{aligned} \tag{3.19}$$

holds for all $x, y, u, v \in X$ *satisfying* ($x \preceq u$ *and* $y \succeq v$) *or* ($x \succeq u$ *and* $y \preceq v$);

(ii) *there exist* $x_0, y_0 \in X$ *such that* ($x_0 \preceq F(x_0, y_0)$ *and* $y_0 \succeq F(y_0, x_0)$) *or* ($x_0 \succeq F(x_0, y_0)$ *and* $y_0 \preceq F(y_0, x_0)$);

(iii) F *is continuous, or*

(iii′) $(X, \preceq, p)$ *satisfies (ORC).*

Then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F(\overline{x}, \overline{y}), \qquad \overline{y} = F(\overline{y}, \overline{x}), \tag{3.20}$$

that is, F has a coupled fixed point.

If $p = d$ is a standard metric, this reduces to [15, Corollary 2.3]. The following example shows how Corollary 3.4 can be used.

Example 3.5. Let $X = [0, 1/2]$ be ordered by the standard relation $\leq$. Consider the partial metric p on X given by $p(x, y) = \max\{x, y\}$. Let $F : X \times X \to X$ be given as

$$F(x, y) = \begin{cases} \dfrac{1}{6}(x - y), & x \geq y, \\ 0, & x < y. \end{cases} \tag{3.21}$$

Finally, take $\gamma \in \mathcal{G}$ given as $\gamma(t) = e^{-t}/(t+1)$ for $t > 0$ and $\gamma(0) \in [0, 1)$. We will show that conditions of Corollary 3.4 hold true.

Take arbitrary $x, y, u, v \in X$ satisfying $x \geq u$ and $y \leq v$ (the other possible case is treated symmetrically), and denote $L = p(F(x, y), F(u, v)) + p(F(y, x), F(v, u))$ and $R = \gamma(p(x, u) + p(y, v))(p(x, u) + p(y, v))$. Consider the six possible cases: 1°: $0 \leq y \leq u \leq v \leq x \leq 1/2$, 2°: $0 \leq y \leq v \leq u \leq x \leq 1/2$, 3°: $0 \leq u \leq y \leq x \leq v \leq 1/2$, 4°: $0 \leq u \leq x \leq y \leq v \leq 1/2$, 5°: $0 \leq y \leq u \leq x \leq v \leq 1/2$, and 6°: $0 \leq u \leq y \leq v \leq x \leq 1/2$. It is easy to check that in all these cases $L \leq (1/6)(x + v)$ and that $R = (e^{-(x+v)}/(x + v + 1))(x + v)$. Since

$$\frac{1}{6}t \leq \frac{e^{-t}}{t+1} \cdot t \tag{3.22}$$

holds for each $t \in [0, 1]$ (as far as it holds for $t = 1$), we obtain that $L \leq R$. The conditions of Corollary 3.4 are satisfied and F has a coupled fixed point (which is $(0, 0)$).

4. Coupled Coincidence and Fixed Points under Rakotch-Type Conditions

Let $\mathcal{R}$ denote the class of real functions $\rho : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\limsup_{s \to t} \rho(s) < 1 \quad \text{for each } t > 0. \tag{4.1}$$

Rakotch proved in 1962 the following.

Theorem 4.1 (see [11]). *Let (X, d) be a complete metric space, and let $T : X \to X$ be a self-map. Suppose that there exists $\rho \in \mathcal{R}$ such that*

$$d(Tx, Ty) \leq \rho(d(x, y))d(x, y) \tag{4.2}$$

holds for all $x, y \in X$. Then T has a unique fixed point $z \in X$.

We will prove the respective result for the existence of a coupled coincidence point in the frame of partial metric spaces. We begin with the following auxiliary result, which may be of interest on its own.

Lemma 4.2. *Let $(X, \preceq, p)$ be an ordered partial metric space which is complete. Let $T, S : X \to X$ be self-maps such that S is continuous, $TX \subset SX$, and one of these two subsets of X is closed. Suppose that T is S-nondecreasing (with respect to $\preceq$) and there exists $x_0 \in X$ with $Sx_0 \preceq Tx_0$ or $Tx_0 \preceq Sx_0$. Assume also that there exists $\rho \in \mathcal{R}$ such that*

$$p(Tx, Ty) \leq \rho(p(Sx, Sy))p(Sx, Sy) \tag{4.3}$$

holds for all $x, y \in X$ such that Sx and Sy are comparable. Assume that either 1°: T is continuous and the pair (T, S) is p^s-compatible or 2°: X satisfies (ORC). Then, T and S have a coincidence point in X.

Proof. The proof is similar to the proof of Lemma 3.2, so we note only the basic step.

With γ replaced by ρ, (3.7) shows that $p(Sx_{n+1}, Sx_n)$ is strictly decreasing, thus converging to some $q \geq 0$. Suppose that $q > 0$. Then, it follows that

$$q \leq \limsup_{n \to \infty} \rho(p(Sx_{n+1}, Sx_n)) < q, \tag{4.4}$$

a contradiction. Hence, $\lim_{n \to \infty} p(Sx_{n+1}, Sx_n) = 0$ is proved. The rest of the proof is identical to that of Lemma 3.2. □

The following example shows that the existence of order may be crucial.

Example 4.3. Let $X = [0, 2]$ be endowed with the partial metric defined by

$$p(x, y) = \begin{cases} |x - y|, & x, y \in [0, 1], \\ \max\{x, y\}, & x \in (1, 2] \vee y \in (1, 2]. \end{cases} \tag{4.5}$$

The order $\preceq$ is given by

$$x \preceq y \Longleftrightarrow x = y \vee (x, y \in (1, 2] \wedge x \leq y). \tag{4.6}$$

Then it is easy to check that $(X, \preceq, p)$ is an ordered partial metric space which is complete and satisfies (ORC). Consider the mappings $S, T : X \to X$ given as

$$Sx = x, \qquad Tx = \begin{cases} 1 - x, & x \in [0, 1], \\ \dfrac{x}{2}, & x \in (1, 2]. \end{cases} \tag{4.7}$$

Take the function $\rho \in \mathcal{R}$ given by $\rho(t) = 1/2$. It is easy to check that the contractive condition (4.3) holds. Indeed, if $Sx = x$ and $Sy = y \in X$ are comparable, say $x \succeq y$, then either $x = y \in [0, 1]$ or $(x, y \in (1, 2]$ and $x \geq y)$. In the first case, $p(Sx, Sy) = p(x, x) = 0$ and

$p(Tx,Ty) = |(1-x)-(1-x)| = 0$; hence, $p(Tx,Ty) \leq \rho(0)\cdot 0$ trivially holds (whichever function $\rho \in \mathcal{R}$ is chosen). In the second case, $p(Sx,Sy) = p(x,y) = x$ and

$$p(Tx,Ty) = p\left(\frac{x}{2},\frac{y}{2}\right) = \frac{x-y}{2} < \frac{x}{2} \tag{4.8}$$

and $p(Tx,Ty) \leq (1/2)\cdot x = \rho(p(Sx,Sy))p(Sx,Sy)$. All the conditions of Lemma 4.2 are fulfilled and mappings T and S have a coincidence point ($x = 1/2$).

On the other hand, consider the same problem, but without order. Then the contractive condition does not hold and the conclusion about the coincidence point cannot be obtained in this way. Indeed, take any $x, y \in [0,1]$ with $x \neq y$. Then, $p(Sx,Sy) = p(x,y) = |x-y|$ and $p(Tx,Ty) = |(1-x)-(1-y)| = |x-y|$. Hence, if $p(Tx,Ty) \leq \rho(p(Sx,Sy))p(Sx,Sy)$, then $\rho(p(Sx,Sy)) \geq 1$ and $\rho \notin \mathcal{R}$.

Now, we are in the position to give

Theorem 4.4. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete, and let $g : X \to X$ and $F : X\times X \to X$ be such that F has the g-mixed monotone property. Suppose that*

(i) *there exists $\rho \in \mathcal{R}$ such that*

$$\begin{aligned} &p(F(x,y),F(u,v)) + p(F(y,x),F(v,u)) \\ &\quad\leq \rho(p(gx,gu)+p(gy,gv))(p(gx,gu)+p(gy,gv)) \end{aligned} \tag{4.9}$$

holds for all $x,y,u,v \in X$ satisfying ($gx \preceq gu$ and $gy \succeq gv$) or ($gx \succeq gu$ and $gy \preceq gv$) and conditions (ii), (iii), (iv), respectively, (iv′) of Theorem 3.3 hold true. Then there exist $\overline{x},\overline{y} \in X$ such that

$$g\overline{x} = F(\overline{x},\overline{y}), \qquad g\overline{y} = F(\overline{y},\overline{x}), \tag{4.10}$$

that is, g and F have a coupled coincidence point.

Proof. The proof is analogous to the proof of Theorem 3.3, and so is omitted. □

Putting $g = i_X$ (the identity map) in Theorem 4.4, we obtain the following.

Corollary 4.5. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete, and let $F : X\times X \to X$ have the mixed monotone property. Suppose that*

(i) *there exists $\rho \in \mathcal{R}$ such that*

$$\begin{aligned} &p(F(x,y),F(u,v)) + p(F(y,x),F(v,u)) \\ &\quad\leq \rho(p(x,u)+p(y,v))(p(x,u)+p(y,v)) \end{aligned} \tag{4.11}$$

holds for all $x, y, u, v \in X$ *satisfying* ($x \preceq u$ *and* $y \succeq v$) *or* ($x \succeq u$ *and* $y \preceq v$) *and conditions (ii), (iii), respectively, (iii′) of Corollary 3.4 hold true. Then there exist* $\overline{x}, \overline{y} \in X$ *such that*

$$\overline{x} = F(\overline{x}, \overline{y}), \qquad \overline{y} = F(\overline{y}, \overline{x}), \tag{4.12}$$

that is, F has a coupled fixed point.

5. Coupled Coincidence and Fixed Points under Integral Conditions

Denote by Φ the set of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ that are Lebesgue integrable and summable (having finite integral) on each compact subset of $\mathbb{R}^+$, and satisfying condition $\int_0^\varepsilon \varphi(t)\, dt > 0$ for each $\varepsilon > 0$. Branciari [12] was the first to use integral-type contractive conditions in order to obtain fixed point results. He proved the following.

Theorem 5.1 (see [12]). *Let* (X, d) *be a complete metric space, and let* $T : X \to X$ *be a self-map satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \tag{5.1}$$

for some $c \in (0, 1)$ *and some* $\varphi \in \Phi$. *Then T has a unique fixed point* $z \in X$ *and for each* $x \in X$ *the Picard sequence* $\{T^n x\}$ *converges to z when* $n \to \infty$.

Subsequently, several authors proved such results, including the very recent paper of Liu et al. [23].

We begin with the following.

Lemma 5.2. *Let* $(X, \preceq, p)$ *be an ordered partial metric space which is complete. Let* $T, S : X \to X$ *be self-maps such that S is continuous,* $TX \subset SX$, *and one of these two subsets of X is closed. Suppose that T is S-nondecreasing (with respect to* $\preceq$*) and there exists* $x_0 \in X$ *with* $Sx_0 \preceq Tx_0$ *or* $Tx_0 \preceq Sx_0$. *Assume also that there exists* $\rho \in \mathcal{R}$ *and* $\varphi \in \Phi$ *such that*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \leq \rho(p(Sx, Sy)) \int_0^{p(Sx,Sy)} \varphi(t)dt \tag{5.2}$$

holds for all $x, y \in X$ *such that Sx and Sy are comparable. Assume that either* 1°: *T is continuous and the pair* (T, S) *is* p^s*-compatible or* 2° : *X satisfies (ORC). Then, T and S have a coincidence point in X.*

Proof. Take $x_0 \in X$ with, say, $Sx_0 \preceq Tx_0$, and using that T is S-nondecreasing and that $TX \subset SX$ form the sequence $\{x_n\}$ satisfying $Tx_n = Sx_{n+1}$, $n = 0, 1, 2, \ldots$, and

$$Sx_0 \preceq Tx_0 = Sx_1 \preceq Tx_1 = Sx_2 \preceq \cdots \preceq Tx_n \preceq Sx_{n+1} \preceq \cdots . \tag{5.3}$$

Since Sx_{n-1} and Sx_n are comparable, we can apply the contractive condition (5.2) to obtain

$$\begin{aligned}\int_0^{p(Sx_{n+1},Sx_n)}\varphi(t)dt &= \int_0^{p(Tx_n,Tx_{n-1})}\varphi(t)dt\\ &\le \rho\big(p(Sx_n,Sx_{n-1})\big)\int_0^{p(Sx_n,Sx_{n-1})}\varphi(t)dt \le \int_0^{p(Sx_n,Sx_{n-1})}\varphi(t)dt.\end{aligned} \tag{5.4}$$

If $p(Sx_{n_0},Sx_{n_0-1})=0$ for some $n_0\in\mathbb{N}$, then it easily follows that $Sx_{n_0-1}=Sx_{n_0}=Sx_{n_0+1}=\cdots$ and x_{n_0} is a point of coincidence of S and T. Suppose that $p(Sx_n,Sx_{n+1})>0$ for each $n\in\mathbb{N}$ (and, hence, the strict inequality holds in (5.4)). We will prove that $\{p(Sx_n,Sx_{n-1})\}$ is a nonincreasing sequence. Suppose, to the contrary, that $p(Sx_{n_0+1},Sx_{n_0})>p(Sx_{n_0},Sx_{n_0-1})$ for some $n_0\in\mathbb{N}$. Then, since $\varphi\in\Phi$, we get that

$$\begin{aligned}0 &< \int_0^{p(Sx_{n_0},Sx_{n_0-1})}\varphi(t)dt \le \int_0^{p(Sx_{n_0+1},Sx_{n_0})}\varphi(t)dt = \int_0^{p(Tx_{n_0},Tx_{n_0-1})}\varphi(t)dt\\ &\le \rho\big(p(Sx_{n_0},Sx_{n_0-1})\big)\int_0^{p(Sx_{n_0},Sx_{n_0-1})}\varphi(t)dt < \int_0^{p(Sx_{n_0},Sx_{n_0-1})}\varphi(t)dt,\end{aligned} \tag{5.5}$$

a contradiction.

Denote by $q\ge 0$ the limit of the nonincreasing sequence $\{p(Sx_n,Sx_{n-1})\}$ of positive numbers. Suppose that $q>0$. Then, using Lemmas 2.1 and 2.2 from [23], the contractive condition (5.2), and property (4.1) of function ρ, we get that

$$\begin{aligned}0 &< \int_0^q\varphi(t)dt = \limsup_{n\to\infty}\int_0^{p(Tx_n,Tx_{n-1})}\varphi(t)dt\\ &\le \limsup_{n\to\infty}\left(\rho\big(p(Sx_n,Sx_{n-1})\big)\int_0^{p(Sx_n,Sx_{n-1})}\varphi(t)dt\right)\\ &\le \limsup_{n\to\infty}\rho\big(p(Sx_n,Sx_{n-1})\big)\limsup_{n\to\infty}\int_0^{p(Sx_n,Sx_{n-1})}\varphi(t)dt\\ &\le \left(\limsup_{s\to q}\rho(s)\right)\int_0^q\varphi(t)dt < \int_0^q\varphi(t)dt,\end{aligned} \tag{5.6}$$

a contradiction. We conclude that $\lim_{n\to\infty}p(Sx_{n+1},Sx_n)=0$.

Now we prove that $\{Sx_n\}$ is a Cauchy sequence in (X,p) (and in (X,p^s)). If this were not the case, as in the proof of Lemma 3.2, using Lemma 2.11, we would get that there exist $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k\to\infty$:

$$p(Sx_{2m_k},Sx_{2n_k}),\qquad p(Sx_{2m_k},Sx_{2n_k+1}),\qquad p(Sx_{2m_k-1},Sx_{2n_k}),\qquad p(Sx_{2m_k-1},Sx_{2n_k+1}). \tag{5.7}$$

Putting in the contractive condition $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that

$$\int_0^{p(Tx_{2m_k-1},Tx_{2n_k})} \varphi(t)\ dt \leq \rho\big(p(Sx_{2m_k-1}, Sx_{2n_k})\big) \int_0^{p(Sx_{2m_k-1},Sx_{2n_k})} . \tag{5.8}$$

Passing to the upper limit as $k \to \infty$ and using properties of functions $\varphi \in \Phi$, $\rho \in \mathcal{R}$ as well as [23, Lemma 2.1], we get a contradiction

$$0 < \int_0^{\varepsilon} \varphi(t)dt \leq \limsup_{n\to\infty}\ \rho\big(p(Sx_{2m_k-1}, Sx_{2n_k})\big) \int_0^{\varepsilon} \varphi(t)dt < \int_0^{\varepsilon} \varphi(t)dt. \tag{5.9}$$

Thus, $\{Sx_n\}$ is a Cauchy sequence, converging to some $Sz \in SX$ (which we suppose to be closed in X) such that $p(Sz, Sz) = 0$. We will prove that T and S have a coincidence point.

(i) Suppose that $T : (X, p) \to (X, p)$ is continuous and (T, S) is p^s-compatible. As in the proof of Lemma 3.2, we get that $T(Sz) = S(Sz)$ and Sz is a coincidence point of T and S.

(ii) If (X, p) satisfies (ORC), since $\{Sx_n\}$ is an increasing sequence tending to Sz, we have that $Sx_n \preceq Sz$ for each $n \in \mathbb{N}$. So we can apply (P_4) and the contractive condition to obtain

$$\begin{aligned} p(Sz, Tz) &\leq p(Sz, Sx_{n+1}) + p(Tx_n, Tz) \\ &\leq p(Sz, Sx_{n+1}) + \rho\big(p(Sx_n, Sz)\big) \int_0^{p(Sx_n,Sz)} \varphi(t)dt. \end{aligned} \tag{5.10}$$

Passing to the upper limit as $n \to \infty$, using properties of functions $\rho \in \mathcal{R}$, $\varphi \in \Phi$, as well as [23, Lemma 2.1], we get that $p(Sz, Tz) = 0$. Hence, we obtain that $Tz = Sz$. □

Putting $S = i_X$ and $\rho(t) = c \in (0, 1)$ in Lemma 5.2, we obtain an ordered partial metric extension of Branciari's Theorem 5.1. The following example shows that this extension is proper.

Example 5.3. Let $X = [0, 1]$ be endowed with the standard metric and order, and consider $T, S : X \to X$ defined by $Tx = x/(1 + x)$, and $Sx = x$, $x \in X$. Take $\varphi \in \Phi$ given by $\varphi(t) = 1$. Then condition (5.1) does not hold. Indeed,

$$\int_0^{d(Tx,Ty)} 1 \cdot dt \leq c \int_0^{d(x,y)} 1 \cdot dt \tag{5.11}$$

would imply that

$$\frac{|x - y|}{(1 + x)(1 + y)} \leq c|x - y| \tag{5.12}$$

for all $x, y \in X$. But taking $x \neq y$ and $x, y \to 0+$ would give that $1 \leq c$, a contradiction.

On the other hand, consider the same problem in an (ordered) partial metric space, with the partial metric given by $p(x,y)=\max\{x,y\}$. Then, condition (5.2) reduces to

$$\int_0^{p(Tx,Ty)} 1\cdot dt \le c\int_0^{p(x,y)} 1\cdot dt, \tag{5.13}$$

which is, for $x\ge y$, equivalent to $x/(1+x)\le cx$ and holds true (for each $x\in X$) if $c\in[1/2,1)$. Obviously, T has a (unique) fixed point $x=0$.

The following theorem is obtained from Lemma 5.2 in a similar way as Theorems 3.3 and 4.4 from respective lemmas.

Theorem 5.4. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete, and let $g: X\to X$ and $F: X\times X\to X$ be such that F has the g-mixed monotone property. Suppose that*

(i) *there exists $\rho\in\mathcal{R}$ and $\varphi\in\Phi$ such that*

$$\begin{aligned}&\int_0^{p(F(x,y),F(u,v))+p(F(y,x),F(v,u))}\varphi(t)\ dt\\ &\quad\le\rho\big(p(gx,gu)+p(gy,gv)\big)\int_0^{p(gx,gu)+p(gy,gv)}\varphi(t)\ dt\end{aligned} \tag{5.14}$$

holds for all $x,y,u,v\in X$ satisfying ($gx\preceq gu$ and $gy\succeq gv$) or ($gx\succeq gu$ and $gy\preceq gv$); and conditions (ii), (iii), (iv), respectively, (iv′) of Theorem 3.3 hold true. Then there exist $\overline{x},\overline{y}\in X$ such that

$$g\overline{x}=F(\overline{x},\overline{y}),\qquad g\overline{y}=F(\overline{y},\overline{x}), \tag{5.15}$$

that is, g and F have a coupled coincidence point.

Putting $g=i_X$ in Theorem 5.4, we obtain the following.

Corollary 5.5. *Let $(X,\preceq,p)$ be an ordered partial metric space which is complete, and let $F: X\times X\to X$ have the mixed monotone property. Suppose that*

(i) *there exists $\rho\in\mathcal{R}$ and $\varphi\in\Phi$ such that*

$$\begin{aligned}&\int_0^{p(F(x,y),F(u,v))+p(F(y,x),F(v,u))}\varphi(t)dt\\ &\quad\le\rho\big(p(x,u)+p(y,v)\big)\int_0^{p(x,u)+p(y,v)}\varphi(t)dt\end{aligned} \tag{5.16}$$

holds for all $x, y, u, v \in X$ *satisfying* ($x \preceq u$ *and* $y \succeq v$) *or* ($x \succeq u$ *and* $y \preceq v$) *and conditions (ii), (iii), respectively, (iii′) of Corollary 3.4 hold true. Then there exist* $\overline{x}, \overline{y} \in X$ *such that*

$$\overline{x} = F(\overline{x}, \overline{y}), \qquad \overline{y} = F(\overline{y}, \overline{x}), \tag{5.17}$$

that is, F has a coupled fixed point.

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