1[st] International Congress of Serbian Society of Mechanics,
10-13[th] April, 2007, Kopaonik

THE OPTIMIZATION OF A THIN WALLED I-BEAM SUBJECTED TO DISPLACEMENT CONSTRAINTS

Nina M. Andjelic[1], Vesna O. Milosevic Mitic[2]

[1] Faculty of Mechanical Engineering,
The University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35
e-mail: nandjelic@mas.bg.ac.yu
[2] Faculty of Mechanical Engineering,
The University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35
e-mail: vmilosevic@mas.bg.ac.yu

Abstract

One approach to the optimization of a thin walled cantilever I-beam subjected to constrained torsion is considered. The aim of this paper is the determination of the minimum mass i.e. minimum cross sectional area of structural thin-walled I-beam elements for given loads, material and geometrical characteristics. That is why the area of the cross section is assumed to be the objective function. The displacement constraints are introduced: allowable angle of twist and allowable angle of twist per unit length. The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory from one side and the basic assumptions of the optimum design from the other. Applying the Lagrange multiplier method, the equation of the second order, which solutions represent the optimal values of the ratios of the parts of the chosen cross section, is formed. The obtained results are used for numerical calculation applying The Finite Element Method.

Key words: optimization, thin-walled beams, optimal dimensions, displacement constraint

1. Introduction

In most structures, it is possible to find the elements in which, depending on loading cases and the way of their introductions, the effect of constrained torsion is present and its consequences are particularly evident in the case of thin-walled profiles. Thin-walled open section beams are widely applied due to their low weight in many structures. Thin walled beams have a specific behavior and because of that, their optimization represents a particular problem. During the process of dimensioning of a structure, beside requested dimensions which are necessary to permit to the particular part of the structure to support the applied loads, it is also often very important to find the optimal values of the dimensions. Very often used types of cross sections, particularly in steel structures are the I-sections.

2. Definition of the problem

The considered cantilever beam, of the length l is subjected to the constrained torsion because of the fact that its one end is fixed and the other free end is loaded by a concentrated torsion moment M. The cross section (Fig. 1) is supposed to have flanges of mutually equal widths and thicknesses $b_1 = b_3$, $t_1 = t_3$. The aim of the paper is to determine the minimal mass of the beam or, in another way, to find the minimal cross-sectional area

$$A = A_{min} \tag{1}$$

for the given loads and material and geometrical properties of the considered beam.

In the considered problem the cross sectional area will be treated as an objective function and it is obvious from the Fig. 1 that

$$A = \sum b_i t_i, \quad \mathrm{i} = 1,2,3 \tag{2}$$

where b_i and t_i are widths and thicknesses of the parts of the considered cross section.

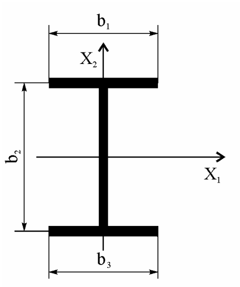


Fig. 1. I - cross section

3. Constraints

Only the displacement constraints will be taken into account in the calculations that follow. The ratio

$$z = b_2 / b_1 \tag{3}$$

will be the optimal relation of the dimensions of the considered cross section.

The considered displacement constraints are allowable angle of twist and allowable angle of twist per unit length, denoted by θ_0 and $\theta_0^{`}$ respectively.

The flexural-torsion cross section characteristic [2, 4] is given by the expression

$$k = \sqrt{GI_t / EI_\omega} \tag{4}$$

where:
- I_t - torsion constant,
- I_ω - sectorial moment of inertia,
- E - modulus of elasticity and
- G - shear modulus.

3.1 Displacement constraint - allowable angle of twist

In the case when the allowable angle of twist θ_0 is taken as the constraint, the constrained function can be written in the form (5) [2, 4]

$$\theta_{max} = \theta(l) = \frac{M\,l}{G\,I_t}\left(1 - \frac{Th\,kl}{kl}\right) \le \theta_0 \tag{5}$$

or in the form (6)

$$\varphi_1 = kl - Th\,kl - \frac{G\,I_t\,kl}{M\,l}\theta_0 \le 0\,. \tag{6}$$

3.2 Displacement constraint - allowable angle of twist per unit length

If the allowable angle of twist per unit length $\theta_0^{`}$ is taken as the constraint, the constrained function can be written in the form (7) [2, 4]

$$\theta_{max}^{`} = \theta^{`}(l) = \frac{M}{G\,I_t}\left(1 - \frac{1}{Ch\,kl}\right) \le \theta_0^{`} \tag{7}$$

or (8)

$$\varphi_2 = Ch\,kl\left(1 - \theta_0^{`}\frac{G\,I_t}{M}\right) - 1 \le 0\,. \tag{8}$$

4. Lagrange multiplier method

Lagrange's Multiplier Method [1, 3, 5-9] is the clasical approach to constraint optimization. Lagrange multiplier, which is labeled as λ, measures the change of the objective function with aspect to the constraint. Applying this method to the vector depending on two parameters b_i, (i = 1, 2), the system of equations (9) of the form $\varphi(b_i) = 0$, (i = 1, 2),

$$\frac{\partial}{\partial b_i}(A + \lambda\varphi) = 0, \quad \mathrm{i} = 1{,}2, \tag{9}$$

will be obtained and after the elimination of the multiplier λ, it will become (10)

$$\frac{\partial A}{\partial b_i}\frac{\partial \varphi}{\partial b_j} = \frac{\partial A}{\partial b_j}\frac{\partial \varphi}{\partial b_i}, \quad (\mathrm{i} \ne \mathrm{j}, \mathrm{i} = 1, \mathrm{j} = 2). \tag{10}$$

5. Analytical approach

The torsion constant and the sectorial moment of inertia for the considered I-section [2, 4] are given by the expressions (11) and (12) respectively

$$I_t = \frac{1}{3} b_1\, t_1^3 \left(2 + \psi^3 z\right) \tag{11}$$

$$I_\omega = \frac{1}{24} b_1^3\, b_2^2\, t_1, \tag{12}$$

where:

$$\psi = t_2 / t_1 \tag{13}$$

is the ratio of thicknesses of the parts of the cross section.

Applying the Lagrange multiplier method, after the differentiation of the expression (10) with respect to the variables b_1 and b_2, the expressions (6) and (8) take the form (14) and (15), respectively:

- for θ_0: $$2\frac{G\,l}{E\,k}\frac{t_1^2}{b_1^2 b_2^3}\left[-3Th^2 kl+\frac{G\,\theta_0}{M\,l}b_1 t_1^3\left(2+\psi^3 z\right)\right]\left(8-4\psi\,z+2\psi^3 z-3\psi^4 z^2\right)-$$
$$-\frac{G\,\theta_0}{M\,l}kl\,t_1^3\left(\psi^2-1\right)\psi\leq 0 \text{ and} \tag{14}$$

- for θ_0': $$2Th\,kl\frac{G\,l}{E\,k}\frac{1}{b_1^3\,b_2^2\,t_1}\left[3-\frac{G\theta_0'}{M}b_1\,t_1^3\left(2+\psi^3 z\right)\right]\left(-8+4\psi\,z-2\psi^3 z+3\psi^4 z^2\right)+$$
$$+\frac{G\theta_0'}{M}\left(1-\psi^2\right)\psi\,z\leq 0\,. \tag{15}$$

In the considered case when the I-beam is the object of the optimization, the equations (14) and (15), combined with (6) and (8), are reduced to the equation (16). The equation of the second order is obtained and its solutions represent the optimal ratios of the cross sectional dimensions for the chosen shape

$$\sum_{i=0}^{2} c_i z^i = 0\,, \tag{16}$$

where the coefficients c_i are given in the form (17) - if the constraint is allowable angle of twist:

$$c_0=8\,,$$
$$c_1=-2\psi\left[2-\psi^2+2\frac{\psi^2-1}{1-\dfrac{kl\,Th^2 kl}{kl-Th\,kl}}\right],$$
$$c_2=-3\psi^4\,, \tag{17}$$

i.e. in the form (16) - if the constraint is allowable angle of twist per unit length:

$$c_0=-8\,\,,$$
$$c_1=2\psi\left[2-\psi^2+2\frac{\psi^2-1}{\dfrac{kl\,Th\,kl}{1-Ch\,kl}}\right],$$
$$c_2=3\psi^4\,, \tag{18}$$

5.1 The results obtained by analytical approach

The following expressions will be introduced

$$D = \frac{\psi^2 - 1}{1 - \dfrac{kl\,Th^2 kl}{kl - Th\,kl}}, \quad D_1 = \frac{\psi^2 - 1}{\dfrac{kl\,Th\,kl}{1 - Ch\,kl}}. \tag{19}$$

The length of the considered cantilever beam is taken as $25 \le l \le 200$ (cm). The values (kl) are calculated using data for standard profiles and the ratio (13) is taken as ψ = 0.5; 0.75; 1. The results for the ratios (3) $z = b_2 / b_1$ obtained from the equations (10) are given in Tables 1 and 2:

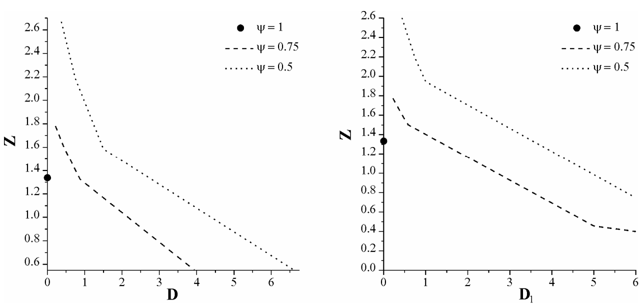
ψ	1	0.75				0.5			
D	0	0.22	0.44	0.88	3.94	0.38	0.75	1.5	6.76
z	1.33	1.78	1.60	1.32	0.55	2.67	2.19	1.56	0.52

Table 1: Displacement constraint θ_0 (Fig. 2a)

ψ	1	0.75				0.5			
D_1	0	0.22	0.44	0.58	437.5	0.38	0.75	1	750
z	1.33	1.78	1.60	1.50	0	2.67	2.19	1.94	0

Table 2: Displacement constraint $\theta_0^{'}$ (Fig. 2b)

The results are presented also graphically in Fig. 2a and 2b:



a b

Fig. 2. The optimal ratios z for a) θ_0, b) $\theta_0^{'}$

It is possible to draw the following conclusions (Tables 2 and 3 and Figure 2): when the values of ψ decrease (i.e. the values for D i D_1 increase) the value z decreases. Based on the performed calculation, the regions of optimal dimension values of the considered cantilever beam are defined:

- For displacement constraint θ_0:
  - $\psi = 1 \Rightarrow D = 0 \Rightarrow z = \text{const} = 1.33$,
  - $\psi = 0.75 \Rightarrow 0.22 \le D \le 3.94 \Rightarrow 1.78 \ge z \ge 0.55$,
  - $\psi = 0.5 \Rightarrow 0.38 \le D \le 6.76 \Rightarrow 2.67 \ge z \ge 0.52$.

- For displacement constraint $\theta_0^{'}$:
  - $\psi = 1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.33$,

- ψ=0.75 $\Rightarrow 0.22 \le D_1 \le 437.5 \Rightarrow 1.78 \ge z \ge 0$,
- ψ =0.5 $\Rightarrow 0.38 \le D_1 \le 750 \Rightarrow 2.67 \ge z \ge 0$.

The calculation shows that the optimal values of z obtained by using the criterion θ_0' are very small for the lengts $l > 100\,cm$. Because of that it is possible to say that the application of this criterion makes sense for following lengths:
- for ψ=0.75: $l \approx 95\,cm \Rightarrow z \ge 0.45$ and
- for ψ=0.5: $l \approx 90\,cm \Rightarrow z \ge 0.51$.

6. Application of the finite element method

Using the optimal values of z, obtained in the previous chapter by the θ_0' criterion, the numerical calculation applying The Finite Element Method was done.

As the numerical example the I - section cantilever beam having the length l=100 cm, fixed at one end and subjected to the concentrated torsion moment M = 10 *kNcm* at its free end (Fig. 3), will be considered by The Finite Element Method (FEM) using the software programme KOMIPS [10].

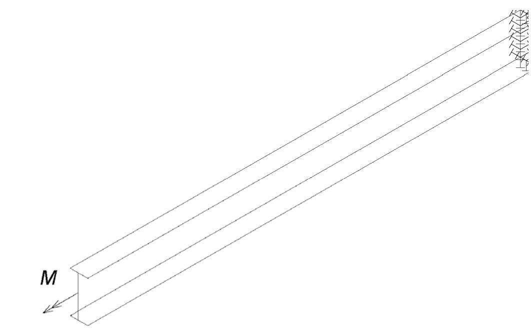


Fig. 3. I- beam – Middle surface, load, supports

6.1 The results obtained by Finite Element Method

As the example of the numerical calculation, the standard I - section I 10 (JUS C.B3.131) is considered. The problem is analyzed in three different ways:

a) Taking into account the initial dimensions of the standard I 10 - section: $b_{1\text{initial}} = b_{3\text{initial}}$ = 5 cm, $b_{2\text{initial}}$ = 9,32 cm, t_1 = 0,68 cm, t_2 = 0,45 cm (it represents the initial model), the initial ratio z_{initial} = 1,86 is obtained.

For the initial values t_1 and t_2 the optimal relation z_{optimal} = 1.65 is obtained from the expressions derived in this paper.

b) The optimal dimensions of the cross section $b_{1\text{optimal}}$ and $b_{2\text{optimal}}$ are obtained by equalizing initial and optimal areas ($A_{\text{initial}} = A_{\text{optimal}}$) and by using the calculated optimal relation z_{optimal} = 1.65 (it represents the optimal model no. 1),

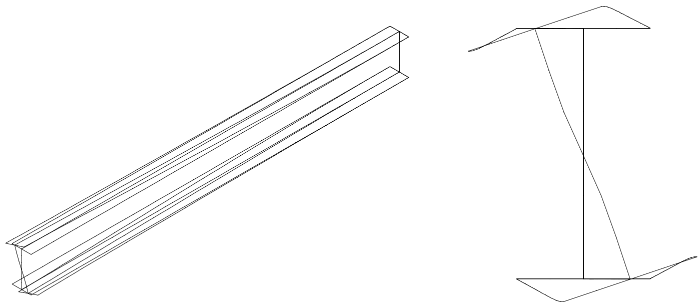
c) The optimal dimensions of the cross section $b_{1\text{optimal}}$ and $b_{2\text{optimal}}$ are obtained with the assumption $b_{2\text{optimal}} = b_{2\text{initial}}$ and by using the calculated optimal ratio z_{optimal} = 1.65 (it represents the optimal model no. 3).

For each model the cross sectional area is calculated and the results are given in Table 3:

Model	z	A (cm^2)	$\theta_0^{'}$ (°/m)
Initial	1.86	10.99	5.05
Optimal no. 1	1.65	10.99	4.79
Optimal no. 2	1.65	11.88	3.64

Table 3: Cross sectional areas and angles of twist per unit length $\theta_0^{'}$

Applying the FEM, the angle of twist per unit length $\theta_0^{'}$ is calculated for each model and the results obtained for the cantilever I - beam of the length l = 100 cm (Fig. 4), are also presented in Table 3.



a *b*

Fig. 4. Deformations (f_{max}=0.4cm): (a) isometric view, (b) xy – plane

Results obtained by applying KOMIPS program correspond to analytically obtained values for the initial model of 100 cm length: $\theta_{0\,analytical}^{'}$ = 5.15 °/m and $\theta_{0\,KOMIPS}^{'}$ = 5.05 °/m.

7. Conclusions

In this paper, one approach to the optimization of the thin-walled open channel section beams, using the Lagrange multiplier method is presented. Accepting the cross sectional area for the objective function and displacement constrains for the constrained functions, it is possible to find the optimal relation between the dimensions of the web and the flanges of the considered cross section. At first, the analytical calculation is made, and the obtained results are used for the calculations applying The Finite Element Method.

Based on the obtained results (17, 18), it can be seen that some differences exist between coefficients c_i calculated using criteria θ_0 or $\theta_0^{'}$, and minimum disagreement between obtained values for z is observed. Optimal values z obtained by using the criterion θ_0, are slightly higher than the values obtained by the $\theta_0^{'}$ criterion (Tables 1 and 2).

Results obtained by the Finite Element Method show that the initial and optimal model no. 1 (Table 4) have the same mass, but the optimal model no. 1 is better because the angle of twist per unit length $\theta_0^{'}$ has the lower value. Optimal model no. 2 has the lowest value of the $\theta_0^{'}$, but this is the optimum model with the highest mass. This model is the best regarding the displacement constraints, but it is also the heaviest one. As the conclusion it is possible to say that all optimal models are better than the initial one.

On the basis of the proposed optimization procedure it is possible to calculate the optimal ratios between the parts of the considered thinwalled profiles in a simple way.

References

[1] Vujanović, B., Spasić, D., *Optimization Methods (In Serbian)*, University of Novi Sad, 1997.
[2] Kollbruner, C.F., Hajdin, N., *Dunnwandige Stabe*, Springer Verlag, 1972.
[3] Onwubiko, C., *Introduction to Engineering Design Optimization*, Prentice Hall, New Jersey Inc., 2000.
[4] Ružić, D., *Strength of Structures (in Serbian)*, Faculty of Mechanical Engineering, University of Belgrade, 1995.
[5] Vanderplaats, G.N., *Numerical Optimization Techniques for Engineering Design*, McGraw-Hill, Inc., 1984.
[6] Farkas, J., *Optimum Design of Metal structures*, Akademiai KIADO, Budapest, 1984.
[7] Bathe, K-J., *Finite Element Procedures in Egineering Analysis*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632, 1982.
[8] Brousse, P., *Structural Optimization*, CISM, No.237, Springer-Verlag, Wien, 1975.
[9] Fox, R.L., *Optimization Methods for Engineering Design*, Addison-Wesley Publishing Company, Inc., 1971.
[10] Maneski, T., *Computer Modeling and Structure Calculation (in Serbian)*, Faculty of Mechanical Engineering, University of Belgrade,1998.

1st International Congress of Serbian Society of Mechanics,
10-13th April, 2007, Kopaonik

FINITE STRIP METHOD IN ANALYSIS OF OPTIMAL RECTANGULAR BENDING BRIDGE PLATES

Dragan Milašinović[1], Radomir Cvijić[2], Aleksandar Borković[3]

[1] Građevinski fakultet, Subotica
e-mail: ddmil@gf.su.ac.yu
[2] Arhitektonsko-građevinski fakultet, Banjaluka
e-mail: rcr@banjaluka.com
[3] Arhitektonsko-građevinski fakultet, Banjaluka
e-mail: aborkovic@gmail.com

Key words: finite strip method, free vibrations, bridge plates

Abstract

If the structure is moving then it is possible to reduce the dynamic problem to a static one by applying D'Alembert's principle of dynamic equilibrium in which an inertia force equal to the product of the mass and the acceleration is assumed to act on the structure in the direction of negative acceleration. For free vibration, the system is vibrating in a normal mode, and it is possible to transform equilibrium equation into a standard eigenvalue problem. Various schemes have been developed for solving eigenvalue equations such as the one by Bishop et al. [2]. In this paper finite strip method is used in analysis of natural frequencies and mode shapes of rectangular bending bridge plates. Point of our analysis was to calculate the lowest natural frequencies of different types of ribbed reinforced plates, so that we could compare them and determine which one of them is optimal. Optimal means that plate has lowest natural frequency for the given lenght.

The finite strip displacement functions in the problem of bending

Let us observe the problem of bending of a finite strip presented in Fig. 1. The approximative function must satisfy the partial differential equation of the 4th order

$$\Delta\Delta w(x,y)=0. \tag{1}$$

If both ends simply supported, the function of deflection will be presented in the form

$$w(x,y)=\sum_{m=1}^{\infty} w_m(x)\sin(m\pi y/a), \tag{2}$$

where *m* represents series term, or number of the harmonic. For any single series term we can anticipate the following polynome to represent the displacement amplitude *w*(*x*):

$$w(x)=C_1+C_2x+C_3x^2+C_4x^3, \tag{3}$$

where C_1-C_4 represent generalized displacements. This approximation enables the establishment of the compatibility of displacement *w* and first derivates *dw/dx* in the nodal lines of the discretizated structure presented in Fig. 1.

Using the condition: $\varphi = dw/dx$, after writing the polynome (3) for the nodal lines 1 and 2 with the coordinates x=0 and x=b respectively, we obtain