BENDING OF A THIN PLATE SUBJECTED TO STRONG UNIFORM MAGNETIC FIELD

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Abstract. The problem of an elastic plate, originally unmagnetized, in a uniform magnetic field is considered in the paper. The magnetic forces are of two kinds: the force of the magnetic field on conducting currents in the material, induced by its motion in the static field and the force between the magnetic field and the magnetized material (independent of the motion).The general equations are linearized by assuming linear constitutive equations and that all electromagnetic variables in the deformed body may be divided into two parts: a rigid body state and a perturbation state. Maxwell's stress is calculated and involved in differential equation related to bending. Obtained equation is solved in analytical form using the integral-transformation technique (Double Fourier finite-sine transformation and Laplace transformation). Discussion of the obtained solution is done using Kirchhoff's hypothesis.

1. Introduction

The theory of electro-magneto-thermoelasticity investigates the interaction between the straian and the electromagnetic field in a solid elastic body. A propagation of an elastic field in presence of magnetic field was considered by L. Knopoff (1955), J.W. Dunkin i A.C. Eringen (1963). F.W. Brown (1966) developed a rigorous phenomenological theory for ferromagnetic materials on the basis of the large deformation theory and the clasical theory of feromagnetisam. H.F. Tiersten (1964) developed an analogous theory based on a microscopic model. Since the general nonlinear theory is complicated, Y.W. Pao and C.S. Yeh [1] derived a set of linear equations and boundary conditions for soft feromagnetic elastic materials. They applied linear theory to investigate magnetoelastic buckling of an isotropic plate. The same problem was treated on the other way by F.C. Moon i Y.H. Pao [2]. Basic general information about the theory of magneto-thermoelasticity was presented in monographs by H. Parkus [3]. A great contribution of a research in this scientific field was given by W. Nowacki, S.A. Ambarcumian [4], M. Krakowski [5]. From 1975 on Michigan Technological University (N.S. Christopherson, M.O. Peach, J.M. Dalrymple, L.G. Viegelahn [6]), a set of experiments were done to reconsider theoretical results. Because of a disagreement in analytical and experimental results methods of numerical analyses were involved in consideration (K. Miya, T. Takagi, Y. Ando (1980), X. Tian and Y. Shen [7]). Sharma and Pal investigated the propagation of magnetic-thermoelastic plane wave in homogeneous isotropic conducting plate under uniform static magnetic field [8]. The problem with high-frequncy electromagnetic waves was presented in [9] and the problem with low-frequency electromagnetic field was descused in [10].

2. Basic equations

Electro-magneto-thermoelastic problem considered in the paper shows one type of interaction between electromagnetic and strain field in a solid plate. It is assumed that the plate material is elastic and isotropic, possessing a good electric conductivity. Many nickel-iron alloys used for building the magnetic circuits of motors, generators, inductors, transformers are of this type.

As a result of time changing electromagnetic field conducting currents appear in electric conductors. This problem is mathematically described by the system of Maxwell's equations with the relations for slowly moving media and modified Ohm's low [6]:

$$rot\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \qquad rot\vec{K} = -\frac{\partial \vec{B}}{\partial t},$$

$$div\vec{D} = 0, \quad div\vec{B} = 0, \tag{1}$$

$$\vec{D} = \varepsilon_0\left(\vec{K} + \dot{\vec{u}} \times \vec{B}\right), \qquad \vec{B} = \mu\left(\vec{H} - \dot{\vec{u}} \times \vec{D}\right),$$

$$\vec{J} = \sigma\left(\vec{K} + \dot{\vec{u}} \times \vec{B}\right),$$

where the following notation is applied: H – intensity of the magnetic field, K – intensity of the electric field, B – magnetic flux density (magnetic induction), D – electric induction, J – current density, u – deflection, μ_0 – permeability of vacuum, σ – electric conductivity, ε_0 – dielectric constant of vacuum, t – time.

In the consideration of the plate vibrations, we shall take the assumption that the longitudinal vibrations are independent of transversal vibrations. Transversal vibrations can be obtained by using the following differential equation [11]:

$$D\nabla_1^4 w - \rho h \ddot{w} + \frac{\rho h^3}{12}\ddot{w} = \left(\sigma_{33}^+ - \sigma_{33}^-\right) + \left(T_{33}^+ - T_{33}^-\right) + \frac{h}{2}\frac{\partial}{\partial x_i}\left(\sigma_{i3}^+ + \sigma_{i3}^-\right) +$$

$$+ \frac{h}{2}\frac{\partial}{\partial x_i}\left(T_{i3}^+ + T_{i3}^-\right) + \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(X_{i,i} + f_{i,i}\right)x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} X_3 dx_3, \qquad (i=1,2), \tag{2}$$

where: D – flexural rigidity of the plate, X – mechanical force, f – Lorenz force. σ_{ij} and T_{ij} denote mechanical and magnetic stress tensors (σ_{ij}^+, T_{ij}^+ are stress components on the upper and σ_{ij}^-, T_{ij}^- on the lower side of the plate), where h is the plate thickness and ∇_1^4 is the four-dimension Laplace operator.

Of course, presented systems of equations has to be accomplished with the appropriate set of boundary and initial conditions.

Presented system of differential equations is complicated to be solved in analytical form. So, at first we have to simplify mathematical model of the discusted problem. In this paper only the problem of the plate subjected to homogenius magnetic field will be presented.

In this case, electromagnetic energy which is converting in thermal energy is very small, so Joules heat can be neglected. Mahwell's equations (1) can be linearysed, too. Deformation field is coupled with the magnetic field.

3. Thin metallic plate in a stationary homogenius magnetic field

Let the thin simply supported plate made of homogenius, isotropic linear magnetic material is subjected transversal to an external uniform homogenius magnetic field induction $\vec{B}_0 = \mu_0 \vec{H}_0$ in air. The magnetic permeability of the plate material is $\mu = \mu_0 \mu_r$ and at the beginning the plate is quite non-magnetized (Figure 1).

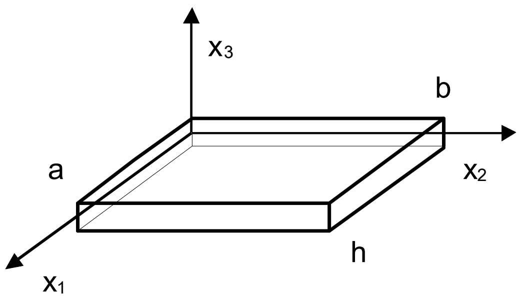

Figure1. Coordinate system and plate dimensions

In that case we have two kinds of forces:

1. forces of interaction between magnetic field and magnetized material, independent of vibrations - motion and
2. forces of the magnetic field and the conducting currents in the plate material, induced by the plate vibrations in the stationary magnetic field.

Electromagnetic field forming in the plate appeares on the action of magnetized material and macroscopic conducting current, as can be presented in next way

$$\vec{H}_p = \vec{H}_{p0} + \vec{h}_p^1 + \vec{h}_p^2 , \tag{3}$$

where $\vec{h}_p^1$ and $\vec{h}_p^2$ are small fluctuations of the magnetic field generated on the presented actions. In the mathematical model their influence on each other can be neglected.

If the thin metallic plate is placed in strong homogeneous magnetic field $H_0 = const.$ appropriate of the law of the line refraction, in the plate is forming magnetic field intensity H_{p0}. In the moment $t=0$ plate is losing stability place, apropos the plate has initial deformation conditions. It can be presented in the next form

$$w(x_1,x_2,0)=\varphi_0(x_1,x_2), \qquad \left.\frac{\partial w(x_1,x_2,t)}{\partial t}\right|_{t=0}=\varphi_1(x_1,x_2). \tag{4}$$

On the coupling between the electromagnetic field and the deformation field there are small fluctuations of the electric field $\vec{e}$ and the magnetic field in the plate $\vec{h}_p^2$, defined as

$$\vec{H}_p(x_1,x_2,x_3,t)=\vec{H}_{po}+\vec{h}_p^2(x_1,x_2,x_3,t), \tag{5}$$

$$\vec{K}(\vec{x},t)=\vec{e}(\vec{x},t).$$

As the change of the magnetic field under a deformation is small, for ferromagnetic materials can be accepted that the magnetic permeability is nearly constant $\mu \cong const.$. Neglecting of the productions between the values h_{pi}^2, e_i , v_i and the productions of their derivatives, relations (1) have the next form [6]

$$\vec{D}=\varepsilon\left(\vec{e}+\mu\vec{v}\times\vec{H}_{p0}\right), \qquad \vec{B}=\mu\left(\vec{H}_{p0}+\vec{h}_p^2\right),$$

$$\vec{J}=\sigma\left(\vec{e}+\mu\vec{v}\times\vec{H}_{p0}\right), \tag{6}$$

and the Maxwell's equations are

$$rot\vec{h}_p^2=\sigma\left(\vec{e}+\mu\vec{v}\times\vec{H}_{p0}\right)+\varepsilon\left(\dot{\vec{e}}+\mu\dot{\vec{v}}\times\vec{H}_{p0}\right), \qquad rot\vec{e}=-\mu\dot{\vec{h}}_p^2,$$

$$\varepsilon div\vec{e}+\varepsilon\mu div\left(\vec{v}\times\vec{H}_{p0}\right)=0, \qquad div\vec{h}_p^2=0. \tag{7}$$

Lorient's force has a form

$$\vec{f}=J\times B=\mu\left(rot\vec{h}_p^2-\dot{\vec{D}}\right)\times\vec{H}_{p0}. \tag{8}$$

By eliminating value $\vec{e}$ we can form differential equation for the magnetic field [6] as

$$\nabla_1^2\vec{h}_p^2-\sigma\mu\dot{\vec{h}}_p^2-\varepsilon\mu\ddot{\vec{h}}_p^2=\sigma\mu rot\left(\vec{v}\times\vec{H}_{p0}\right)-\varepsilon\mu rot\left(\dot{\vec{v}}\times\vec{H}_{p0}\right). \tag{9}$$

In the case of the quazistationry electromagnetic field we have $\vec{D}=0$, $\frac{\partial\vec{D}}{\partial t}=0$, or $\varepsilon \equiv 0$, so presented equations have simplified form

$$\vec{f}=\mu rot\vec{h}_p^2\times\vec{H}_{p0}, \tag{10}$$

$$\left(\nabla_1^2-\sigma\mu\partial_t\right)\vec{h}_p^2=-\sigma\mu rot\left(\dot{\vec{u}}\times\vec{H}_{p0}\right). \tag{11}$$

The whole equation system of this coupled magnetoelastic bending problem is consisted of the equations (11) and (2) together with the appropriate boundary and initial conditions and relation (10).

Let us discus one special case when the magnetic field is transversal to the middle surface of the thin plate. The plate is rectangular with the material density ρ (Figure 1.). Using the condition of the equality of the perpendicularity components on the magnetic induction on the boundary surfaces, the field forming in the plate is

$$H_{p0} = \frac{\mu_0}{\mu} H_0 , \qquad \vec{H}_{p0} = \left(0,0,-H_{p0}\right). \tag{12}$$

If the plate material has high electric conductivity we can say that

$$\vec{h}_p^2 \cong rot\left(\vec{u} \times \vec{H}_{p0}\right) = \left(w_{,1} H_{p0}\right)\vec{i} + \left(w_{,2} H_{p0}\right)\vec{j} - \left(\nabla_1^2 w x_3 H_{p0}\right)\vec{k} , \tag{13}$$

$$\vec{f} \cong -\left(\mu \nabla_1^2 w_{,1} x_3 H_{p0}^2\right)\vec{i} - \left(\mu \nabla_1^2 w_{,2} x_3 H_{p0}^2\right)\vec{j} . \tag{14}$$

Appropriate deferential equation of the transversal vibrations, using (2), is

$$D\nabla_1^4 w + \rho h \ddot{w} + \frac{\rho h^3}{12} \nabla_1^2 \ddot{w} = \left(T_{33}^+ - T_{33}^-\right) + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(f_{1,1} + f_{2,2}\right) x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} X_3 dx_3 . \tag{15}$$

As induced tangential components of the field are small compared to the normal component, appropriate stresses on the upper and the lower side of the plate are formed under the influence of the normal components intensities

$$H_p = H_{p0}\left(1 + x_3 \nabla_1^2 w\right), \qquad H_{0v} = H_0\left(1 + x_3 \nabla_1^2 w\right). \tag{16}$$

Using Maxwell's formula for the stress on the splitted surface of the two magnetics, and the fact that the stress is directing to the medium of the smaller magnetic permeability, we can take next relation

$$T_{33}^+ - T_{33}^- = \left(\mu - \mu_0\right) \frac{\mu_0}{\mu} h H_0^2 \nabla_1^2 w . \tag{17}$$

Equation (15) has the form

$$\left(D + \frac{\mu_0^2}{\mu} H_0^2 \frac{h^3}{12}\right) \nabla_1^4 w - \left(\mu - \mu_0\right) \frac{\mu_0}{\mu} h H_0^2 \nabla_1^2 w + \rho h \ddot{w} - \frac{\rho h^3}{12} \nabla_1^2 \ddot{w} = -\rho g h . \tag{18}$$

Boundary conditions for the simply supported plate are

$$w_{x_1 = 0,a} = 0 , \qquad \left.\frac{\partial^2 w}{\partial x_1^2}\right|_{x_1 = 0,a} = 0 , \tag{19}$$

$$w_{x_2 = 0,b} = 0 , \qquad \left.\frac{\partial^2 w}{\partial x_2^2}\right|_{x_2 = 0,b} = 0 .$$

System of the equations (18) and (19) with the initial conditions (4) can be solved using integral transform technique – double sine Fourier transformation and the Laplace transformation. Obtained solution has next form

$$w(x_1, x_2, t) = \frac{4}{ab} \sum_{m,n=1,3,5...}^{\infty}\sum w_{mn}(t) \sin\alpha_n x_1 \sin\alpha_m x_2 \ ,$$

$$w_{mn}(t) = \varphi_{0nm} \cos\omega_{nm} t + \varphi_{1nm} \frac{\sin\omega_{nm} t}{\omega_{nm}} - \frac{4\rho g h}{\alpha_n \alpha_m \left[\rho h + \frac{\rho h^3}{12}\left(\alpha_n^2 + \alpha_m^2\right)\right]} \frac{1-\cos\omega_{nm} t}{\omega_{nm}^2} \ ,$$

$$\omega_{nm}^2 = \frac{\left(D + \frac{\mu_0^2}{\mu} H_0^2 \frac{h^3}{12}\right)\left(\alpha_n^2 + \alpha_m^2\right)^2 + (\mu - \mu_0)\frac{\mu_0}{\mu} h H_0^2 \left(\alpha_n^2 + \alpha_m^2\right)}{\rho h + \frac{\rho h^3}{12}\left(\alpha_n^2 + \alpha_m^2\right)} ,$$

$$\alpha_n = \frac{n\pi}{a} , \qquad \alpha_m = \frac{m\pi}{b} , \tag{20}$$

$$\varphi_{knm} = \int_0^a \int_0^b \varphi_k(x_1, x_2) \sin\alpha_n x_1 \sin\alpha_m x_2 dx_1 dx_2 \qquad (k=0,1).$$

4. The conclusion

The analysis of the obtained results can be done using relation for ω_{nm}. In the cases of the paramagnetics and ferromagnetics we have $(\mu\text{-}\mu_0)>0$ and the magnetic field is acting in the same direction of the elastic field. It means that the magnetic field is in corelation with the elasic field and they together gravity to return the plate in the equilibrium position. For diamagnetic materials we have $(\mu\text{-}\mu_0)<0$ and the magnetic field is acting in opposite direction of the direction of the elastic field.

Using relation (17) and Kirchoff hypothesis, on the base of a sign of $\nabla_1^2 w$, we have to conclude that in the case of the plate clamped on only one side we have opposite situation.

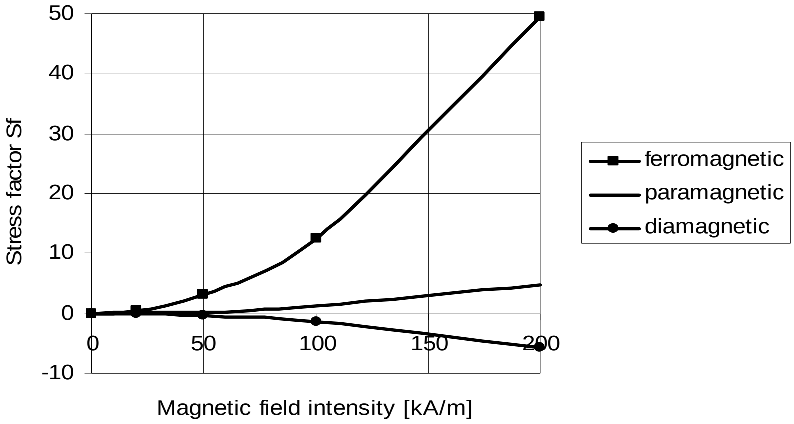

Figure 2. Stress factor as a function of magnetic intensity

On the base on the relation (17) stress factor, defined as

$$S_f = \frac{T_{33}^{+} - T_{33}^{-}}{\nabla_1^2 w}, \tag{21}$$

is presented on diagram from figure 2. As can be noticed, considered problem is interesting only for ferromagnetic plates and very strong magnetic fields.

In the reference [7] the hypothesis of the magnetoelasticity for real conductors were formed and the appropriate theory was developed. On that theory, for the case of the plate placed transversal to magnetic field we have the opposite conclusions. As that theory is not in agreement with Kirchoff hypothesis in the paper [12] modified hypothesis were defined. Than, the correct result can be involved and for a real conductors.

Presented theoretical analysis has the assumption that the plate is subjected to hardly uniform and hardly transversal magnetic field. Obtained theoretical results are valid only for very thin plates. In the case of thick plate we have the boundary effect, and we have to involve in consideration finite element method.

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