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OPTIMUM DESIGN OF OPEN SECTION THIN-WALLED STRUCTURAL ELEMENTS ACCORDING TO STRESS CONSTRAINT

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Abstract. *The purpose of this work is to present a possible approach to the mass minimization of structural thin-walled open section beams of the proposed shapes (I, Z and U-beam), submitted to the stress constraints and multiple load cases. Lagrange's multipliers method has been used to determine the optimization parameters. The area of the cross-section is used as the objective function, while the stress constraint is introduced and used as the constraint function. Numerical examples are presented to verify the analytically obtained values.*
Key words: *Optimization, thin-walled beams, optimal dimensions, stress constraints, saved mass.*

1. INTRODUCTION

There were a large number of research studies on the behaviour of thin-walled structures. The investigations of the behaviour of thin-walled members with open cross-sections have been carried out extensively since the early works of Timoshenko [13], Vlasov [14], Kollbruner and Hajdin [6], Murray [9], Rhodes et al. [10]. In recent years, there have emerged many studies devoted to the optimization of thin-walled cross-sections. The issue of solving various optimization problems has been discussed in various works and monographs. First of all, Gajewski and Życzkowski [5] provided a review of optimal designing of thin-walled structures; Magnucki and Monczak [7] presented a variational and parametrical optimization of open cross-section of a thin-walled beam subject to bending; Tian and Lu [12] optimized the cold-formed open-channel. There have been many studies dealing with optimization problems, treating the cases where geometric configurations of structures are specified and only the dimensions of structural members and the areas of their cross-sections are determined in order to attain the minimum structural weight or cost [8]. Afterwards, a series of studies have appeared where the optimization problem of various cross-sections, such as the triangular cross-section [11], *I*-section [3, 4], *U*-section [2] or *Z*-section beams [1] is solved by means of the Lagrange multiplier method. The idea of this paper is to develop an approach to the optimization of thin-walled *I*, *Z* and *U* cross-section beams.

2. SUBJECT OF RESEARCHING

Two basic parts can be found in this paper. In the first part, the mathematical model is created and the equations which define the problem are derived. In the second part the obtained system of equations, which defines the optimal relation between the parts of the considered thin-walled cross sections, is analytically solved. The open cross-sections (*I*, *Z* and *U*-beam) are considered as objects of optimization in the paper.

The *I* and *U*-sections of the considered cantilever beam (Fig. 1a, c) with principal centroidal axes X_i ($i = 1, 2$) have the axis of symmetry.

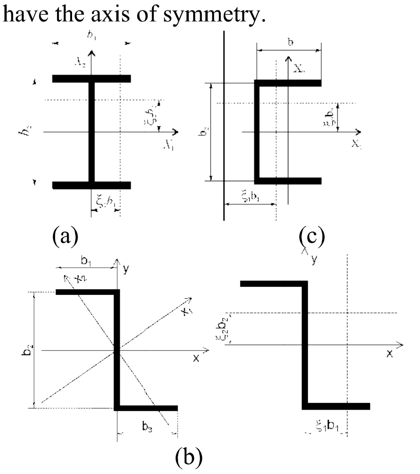

Fig. 1. Cross-section: a) I-beam, b) Z-beam, c) U-beam

The Z-cross-section (Fig. 2b) has the centre and not the axis of symmetry. It is assumed that its flanges have equal widths $b_1 = b_3$, and thicknesses $t_1 = t_3$, and that its web has the width b_2 and thickness t_2.The assumption is that the loads are applied in two longitudinal planes, parallel to the longitudinal centroidal axes at the distances $\xi_i\ b_i$ (i = 1, 2) (Fig. 1). In the case of the *I*-profile (Fig. 1a), the vertical longitudinal plane coincides with the shearing plane. In the *Z*-profile (Fig. 1b), the shearing centre corresponds with the centre of gravity, therefore, it can be said that the eccentricities $\xi_i\ b_i$ (i = 1,2) are defined with respect to the longitudinal planes, whereas in the *U*-profile (Fig. 1c) the distance in the case of vertical plane is measured from the shearing plane.

If loads are applied in such a way, they will cause the bending moments acting in the above defined two planes parallel to the longitudinal axis of the beam, with consequent effects of the constrained torsion occurring in the form of the bimoment *B*, causing the stresses [6].

The aim of the paper is to find the minimal cross-sectional area

$$A = A_{\min} \tag{1}$$

for the given loads and material and geometrical properties, while satisfying the constraints. In the considered problem the cross-sectional area will be treated as an objective function [1, 3]. Because $b_1 = b_3$, it is evident from the Fig. 1 that

$$A = A(b_1, b_2) = 2\ b_1\ t_1 + b_2\ t_2\ . \tag{2}$$

3. CONSTRAINTS

The cross-section of the considered *Z*-beam (Fig. 1b) with principal centroidal axes X_i (i = 1, 2) has the centre and not the axis of symmetry and, because of that, the expressions (3) for equivalent bending moments $\overline{M_x}$ and $\overline{M_y}$ taking into account the influence of the bending moments around centroidal axes *x* and *y*, denoted as M_x and M_y respectively, will be used [1],

$$\overline{M_x} = \frac{M_x - M_y\left(\dfrac{I_{xy}}{I_y}\right)}{1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}}\ , \overline{M_y} = \frac{M_y - M_x\left(\dfrac{I_{xy}}{I_x}\right)}{1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}} \tag{3}$$

where I_x, I_y are the moments of inertia of the cross-sectional area about the centroidal axes *x* and *y*, and I_{xy} is the product of inertia.

The normal stresses σ are caused by the bending moments M_{X1} and M_{X2} in the case of the *I* and *U*-section beam (σ_{X1} and σ_{X2}), i.e. $\overline{M_x}$ and $\overline{M_y}$ in the case of the *Z*-section beam ($\overline{\sigma_x}$ and $\overline{\sigma_y}$), and by the bimoment *B* that appears in the case of constrained torsion. The normal stresses caused by the bimoment will be denoted as σ_ω [6].

The bending moments are acting in planes that are parallel to the longitudinal axis (Fig.1) at the distances $\xi_i b_i$ (i=1,2). The bimoment B will occur as their consequence and it can be expressed as the function of the bending moments and the eccentricities of their planes $\xi_i b_i$ (i=1, 2) [6] for the *I* and *U*-section beam (4) and for the *Z*-section beam (5):

$$B = \xi_1\ b_1 M_{X1} + \xi_2 b_2 M_{X2} \tag{4}$$

$$B = \xi_1\ b_1 \overline{M_x} + \xi_2 b_2 \overline{M_y}\ . \tag{5}$$

If σ_0 stands for allowable stress, the constraint function can be written for the *I* and *U*-section beam (6) and for the *Z*-section beam (7):

$$\varphi = \varphi(\sigma) = \sigma_{X1\max} + \sigma_{X2\max} + \sigma_{\omega\max} \le \sigma_0 \tag{6}$$

$$\varphi = \varphi(\sigma) = \overline{\sigma_x}_{\max} + \overline{\sigma_y}_{\max} + \sigma_{\omega_{\max}} \le \sigma_0\ . \tag{7}$$

The maximal normal stresses [6] are defined in the forms

– for the *I* and *U*-section beam

$$\sigma_{Xi\max} = \frac{M_{Xi}}{W_{Xi}}\ (\mathrm{i} = 1,2),\ \sigma_{\omega\max} = \frac{B}{W_\omega} \tag{8}$$

where W_{Xi} (i = 1,2) are the section moduli for the principal axes for the *I* and *U*-section, and

– for the *Z*-section beam

$$\overline{\sigma}_{x\max} = \frac{\overline{M_x}}{W_x}, \overline{\sigma}_{y\max} = \frac{\overline{M_y}}{W_y}, \sigma_{\omega\max} = \frac{B}{W_\omega}\ . \tag{9}$$

For the considered cross-sections W_x and W_y are the section moduli for the longitidunal axes for the *Z*-section and W_ω is the sectorial section modulus.

After the introduction of Eq. (8) into Eq. (6), and Eq. (9) into Eq. (7), the constraint function becomes (10) for the *I* and *U*-section beam and (11) for the *Z*-section beam:

$$\varphi = \frac{M_{X1}}{W_{X1}} + \frac{M_{X2}}{W_{X2}} + \frac{B}{W_\omega} \le \sigma_0\ , \tag{10}$$

$$\varphi = \frac{\overline{M_x}}{W_x} + \frac{\overline{M_y}}{W_y} + \frac{B}{W_\omega} \le \sigma_0\ . \tag{11}$$

The constraint functions (10) and (11) are reduced to (12) for the *I*-section beam, (13) for the *Z*-section beam and (14) for the *U*-section beam. The expressions (13-15) represent the constraint functions that correspond to the given stress constraints.

$$\varphi = \varphi(b_1, b_2) = 6 M_{X1} \frac{1}{t_1 b_1 b_2\left(6 + \dfrac{t_2}{t_1}\dfrac{b_2}{b_1}\right)} + \\ + 3 M_{X2} \frac{1}{t_1 b_1^2} + 6B \frac{1}{t_1 b_1^2 b_2} \le \sigma_0, \tag{12}$$

$$\varphi=\varphi(b_1,b_2)=30M_x\frac{1}{t_1b_1b_2\left(3+2\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}+$$

$$+3M_y\cdot\frac{b_2}{b_1}\cdot\frac{9+\frac{t_2}{t_1}\frac{b_2}{b_1}}{t_1b_1b_2\left(3+2\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}+6B\frac{1+\frac{t_2}{t_1}\frac{b_2}{b_1}}{t_1b_1^2b_2\left(1+2\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}\le\sigma_0, \quad (13)$$

$$\varphi=\varphi(b_1,b_2)=6M_{X1}\frac{1}{t_1b_1b_2\left(6+\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}+$$

$$+3M_{X2}\frac{1+\frac{t_2}{t_1}\frac{b_2}{b_1}}{t_1b_1^2\left(1+2\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}+6B\frac{3+\frac{t_2}{t_1}\frac{b_2}{b_1}}{t_1b_1^2b_2\left(3+2\frac{t_2}{t_1}\frac{b_2}{b_1}\right)}\le\sigma_0. \quad (14)$$

4. RESULTS AND DISCUSSION

Optimization parameters have been determined by the Lagrange's multipliers method [1, 3, 5, 8]. Applying the Lagrange multiplier method to the vector which depends on two parameters b_i (i=1, 2)

$$\frac{\partial}{\partial b_i}\left[A(b_1,b_2)+\lambda\varphi(b_1,b_2)\right]=0,\quad i=1,2 \quad (15),$$

the system of equations will be obtained. After the elimination of the Lagrange multiplier λ, it will become

$$\frac{\partial A(b_1,b_2)}{\partial b_1}\cdot\frac{\partial\varphi(b_1,b_2)}{\partial b_2}=\frac{\partial A(b_1,b_2)}{\partial b_2}\cdot\frac{\partial\varphi(b_1,b_2)}{\partial b_1}. \quad (16)$$

4.1 Analytic solution

After introducing the expressions (4) and (5) for the bimoment into the equations (12), (13) and (14), the equation (16) can be reduced to an equation in the form (17) whose solutions give the optimal values of the ratio (18)

$$\sum_{k=0}^{n}c_kz^k=0, \quad (17)$$

where: $z=b_2/b_1$ (18) is the optimal ratio of the parts of the considered cross-section. The coefficients c_k are dependent on the ratio of the bending moments and on the eccentricities ξ_1 and ξ_2 of their planes.

The solutions are in the form of the fourth order for the considered *I*-section beam, the sixth order for the considered *Z*-section beam and the eighth order for the considered *U*-section beam.

4.2 Particular cases. Optimal values z = b₂ / b₁

In the general case, bending moments about both principal axes appear simultaneously with the bimoment. Depending on the ratio M_{X2}/M_{X1} for the *I* and *U*-secton and M_y / M_x for the *Z*-section beam, there are some particular cases to consider. The optimal ratios *z* (18) obtained from the equations (17) are calculated for M_{X2}/M_{X1} (M_y/M_x) = 0, 0.5, 1; ψ = 0.5, 0.75, 1 and for $0\le\xi_1\le 1$; $0\le\xi_2\le 1$. The highest and the lowest optimal values of *z* M_{X2}/M_{X1} (M_y / M_x)=0, 0.5, 1 and ψ = 0.5, 0.75 and 1.0, are shown in a shortened form in Table 1 for the considered cross-sections.

Table 1. Optimal z = b₂ /b₁ for ψ = 0.5; 0.75; 1

M_{X2}/M_{X1}		0	0.5	1
z	I-section	1.09 ≤ z ≤ 12	0.69 ≤z ≤1.72	0.51 ≤z≤ 1.59
	Z-section	1.25 ≤z ≤ 8.99	1.36 ≤z ≤ 2.74	1.26 ≤z≤ 2.49
	U-section	1.39 ≤z ≤ 12	0.93 ≤z ≤ 2.22	0.71 ≤z≤ 2.09

5. A NUMERICAL EXAMPLE. ANALYSIS OF RESULTS

This chapter will discuss some particular cases that occur depending on the loading case.

5.1 The loading cases

In this section the *I*, *U* and *Z*-section beams are fixed at one end and exposed to the concentrated bending moment M_{X1} (M_x) = 100 Nm; M_{X2} (M_y) = 0 at the free end of the beam in two ways as:

a) Loading case 1: $\xi_1=\xi_2=0$ and

b) Loading case 2: $\xi_1=0.5$, $\xi_2=0$.

The initial cross-sectional geometrical characteristics are calculated taking into account the initial dimensions of the *I*, *Z* and *U*-section beam. It is assumed that the considered section has the initial wall thicknesses: b_1 = 51.75 mm, b_2 = 92 mm, t_1 = 8 mm, t_2 = 6.5 mm. This serves as the **Initial model**, with the "Initial area" of the cross-section. Starting from the initial relation z_{initial} and for the initial cross-sectional geometrical characteristics t_1 and t_2 the optimal relation z_{optimal} is calculated defining the "Optimum area" of the cross-section.

5.2 Minimum mass determination

To illustrate the design optimization technique, we consider the weight minimization problem of clamped *I*, *U* and *Z*-section beams shown in Fig. 1.

The problem is discussed in two ways [14, 17]: 1) The optimum dimensions of the cross-sections $b_{1\text{optimum}}$ and $b_{2\text{optimum}}$ are arrived at by equalizing the "Initial" and the "Optimum area" ($A_{\text{initial}}=A_{\text{optimal}}$) and by using the calculated optimal relation *z*. This case represents the **Optimum model 1** (Table 2). 2) In the **Optimum model 2**, the optimal values $b_{1\text{optimum}}$ and $b_{2\text{optimum}}$ are obtained from the condition requiring that the stresses must be lower than the allowable stress. Using the optimum cross-sectional dimensions as the starting point, the optimum minimal cross-sectional area $A_{\min}$ is calculated for each loading case and the results that include the saved mass of the material are provided in Table 2.

Table 2 shows that greater saved mass was obtained for the *I*-section than for the channel and *Z*-sections. Also, for all loading cases, the level of stresses is reduced in the Optimum model 1, while the saved mass of the material is increased with regard to the initial stress limits in the Optimum model 2. The calculations have shown that the maximum saved material is obtained in the Loading case 1 and the minimum in the Loading case 2 for all three shapes of cross-sections.

This allows for the conclusion that if the distance of the loading plane from the shearing plane is increased, it is less necessary to perform the optimization of the cross-section.

Table 2. Optimum model 1: $z_{initial}$=1.78

Section	Loading case	$z_{optimum}$	$\sigma_{initial}$ [MPa]	$\sigma_{optimum1}$ [MPa]	$\sigma_{optimum\ 2}$ [MPa]	$A_{initial}=A_{optimum\ 1}$ [mm^2]	$A_{min}=A_{optimum\ 2}$ [mm^2]	Saved mass [%]
I-beam	1	7.39	2.02	1.58	2.02	1426	1260	11.64
	2	1.45	9.44	9.38	9.44		1423	0.22
Z-beam	1	5.58	13.4	7.5	13.4		1033	27.56
	2	1.84	15.9	12.7	15.9		1398	1.96
U-beam	1	7.38	2.21	1.71	2.21		1280	10.25
	2	1.84	8.43	8.42	8.43		1425	0.12

6. CONCLUSION

In this paper, one approach to the optimization of the thin-walled open section beams, loaded in a complex way, using the Lagrange multiplier method, is presented. Accepting the cross-sectional area for the objective function and stress constrains for the constrained functions, the regions of optimal values of dimensions of all considered cross-sections are defined. As the result of the calculation, the modified constrained functions are derived as the polynomials of the fourth, sixth and eighth order, depending on the shape of the profile. The obtained functions are subjected to the given constraints and the obtained solution results give the optimal values of the ratios of the parts of the considered cross-section.

Particular attention is paid to calculating the saved mass by means of the proposed analytical approach. The saved mass can also be calculated for different loading cases.

The aim of the paper is the optimization of thin-walled elements subjected to the complex loads. It can be concluded that the optimization approach considered in this paper gives the general results that can be effectively used for deriving the expressions recommendable for technical applications.

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