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NUMERICAL COMPUTATIONS OF ELASTIC TORSION USING THE FINITE VOLUME METHOD

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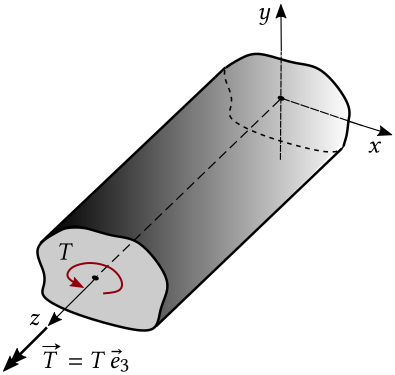
Cylindrical bar with arbitrary cross-section subjected to end loading moment.



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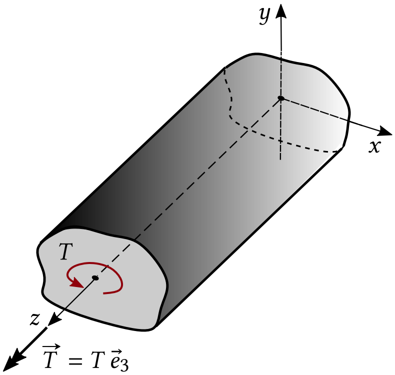

Saint-Venant principle: seeking for the solution that satisfies the same resultant loading (not the pointwise traction conditions at the ends).

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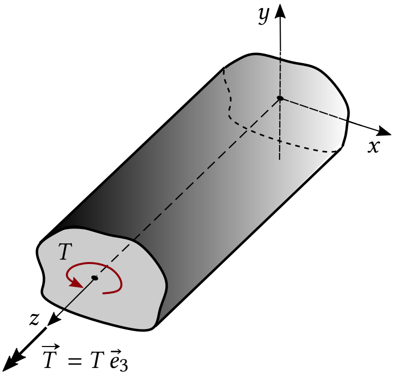

Reasonable assumptions:

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Physical Considerations: Formulation of the Problem

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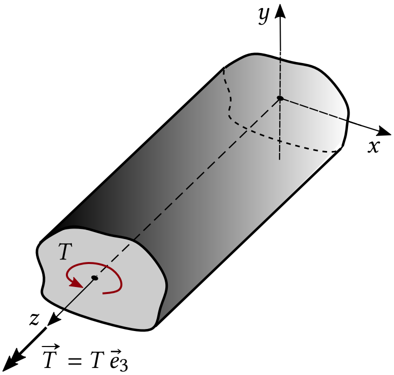

Reasonable assumptions:

- Each cross-section in (x, y)-plane rotates as rigid body around the central axis
- Amount of rotation is linear function of axial coordinate z (small deformation theory)

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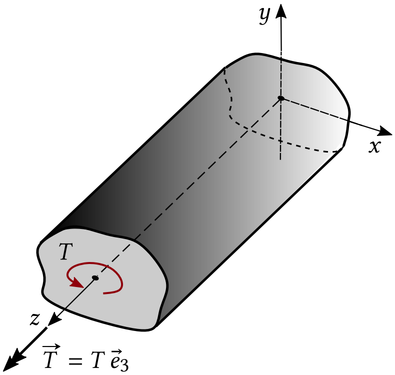

Reasonable assumptions:

- Each cross-section in (x, y)-plane rotates as rigid body around the central axis
- Amount of rotation is linear function of axial coordinate z (small deformation theory)
- Circular cross-sections remain plane after deformation
- Plane cross-sections do not remain plane after deformation - warping displacement

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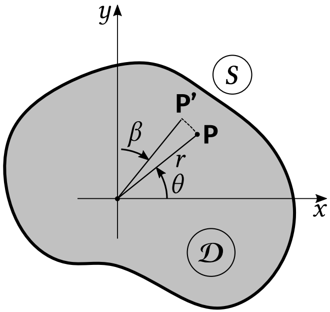

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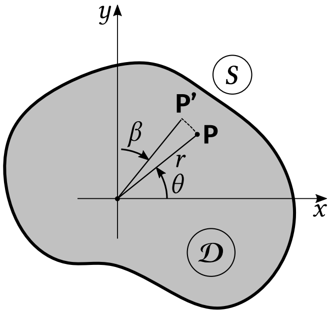
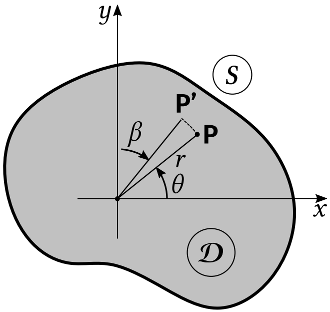


- Starting point: "known" displacement field

$$u = -\alpha z y, \quad v = \alpha x z, \quad w = \alpha w^*(x, y)$$

$\alpha = \mathrm{const}$ - angle of twist per unit length



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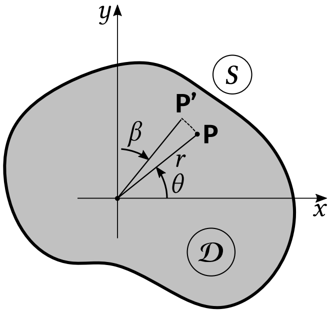

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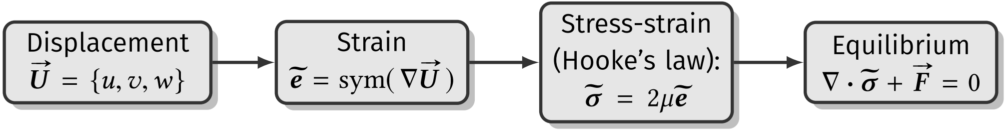

Stress field:

$$\sigma_{ii} = 0, \ \tau_{yx} = \tau_{xy} = 0$$

$$\tau_{zx} = \tau_{xz} = \mu\alpha\left(\frac{\partial w^*}{\partial x} - y\right)$$

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Various Formulations of Mathematical Model

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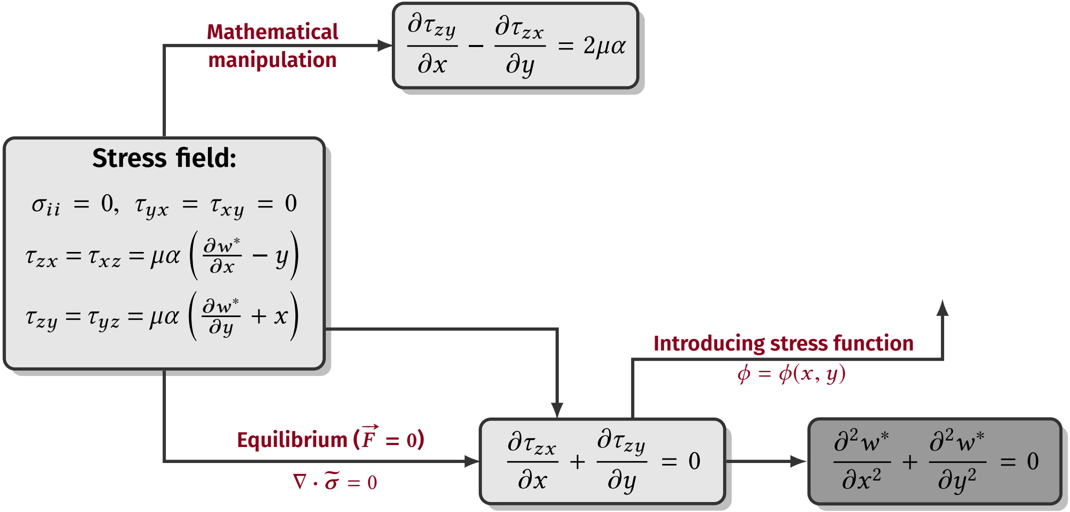

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Introducing stress function

$$\phi = \phi(x, y)$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \ \tau_{zy} = -\frac{\partial \phi}{\partial x}$$

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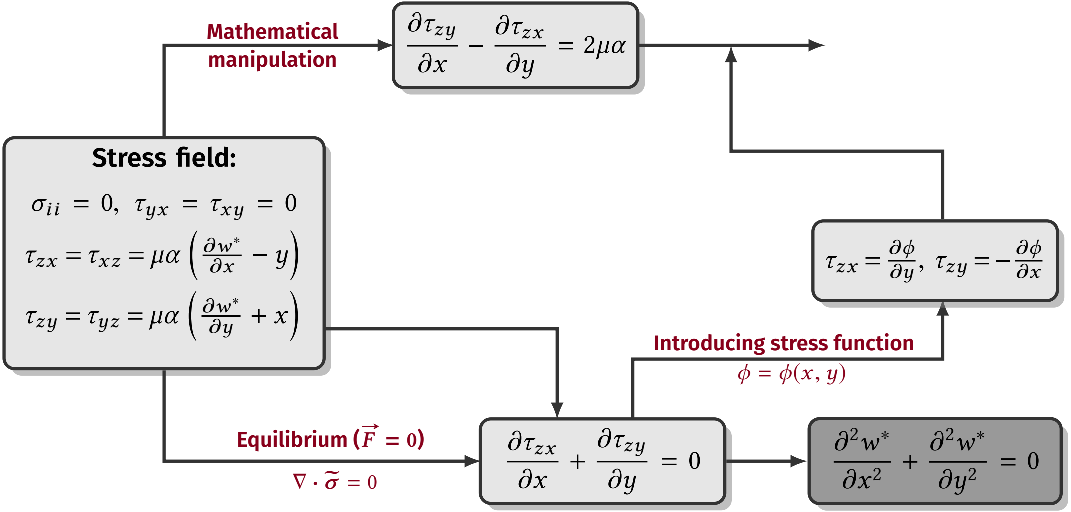

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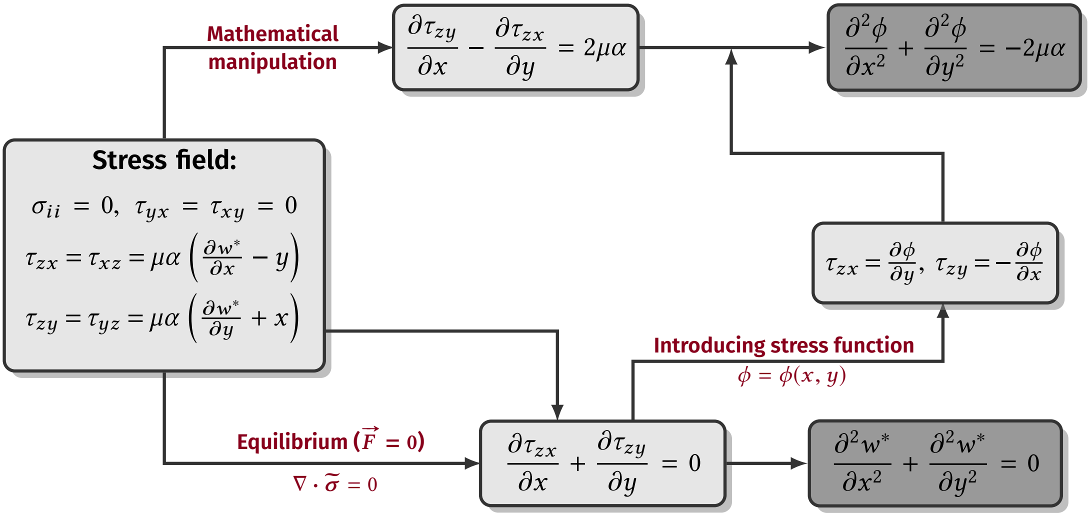

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$$\frac{\partial \tau_{zy}}{\partial x} - \frac{\partial \tau_{zx}}{\partial y} = 2\mu\alpha$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\alpha$$

Traction

$$\vec{t}_n = \vec{n} \cdot \tilde{\sigma}$$

Equilibrium ($\vec{F} = 0$)

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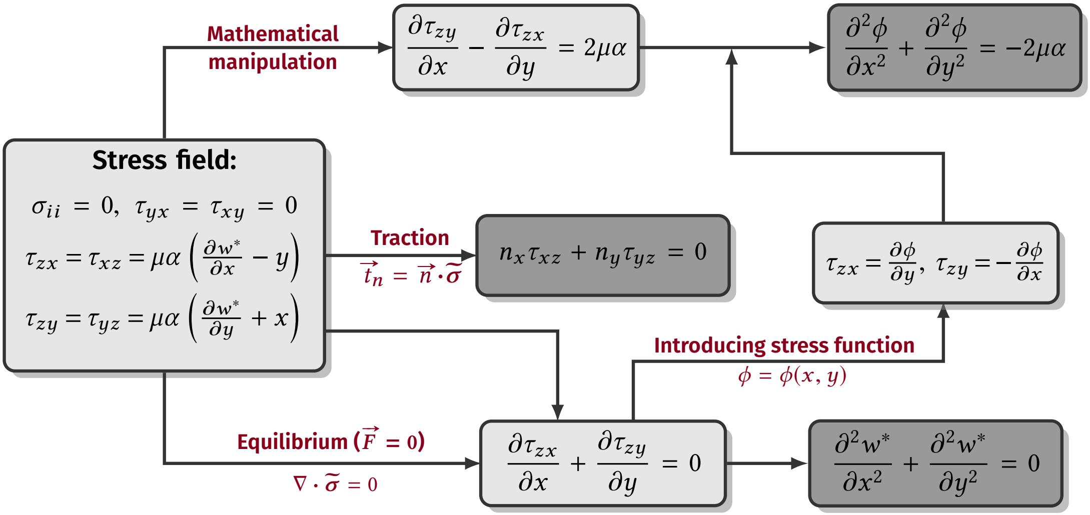


Final Formulation, With Simple Scaling: $\tau_{zx}/(\mu\alpha) = \tau_x$, $\tau_{zy}/(\mu\alpha) = \tau_y$

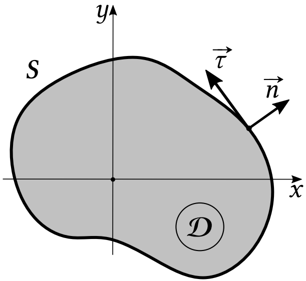
Elliptic type

Laplace's PDE:

$$\nabla^2 w^* = 0$$

Poisson's PDE:

$$\nabla^2 \phi^* = -2$$



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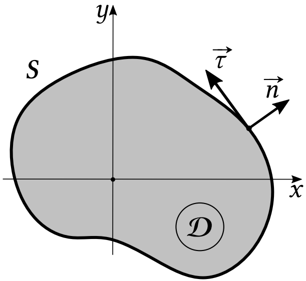

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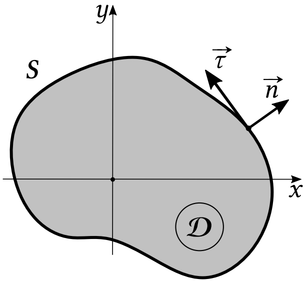

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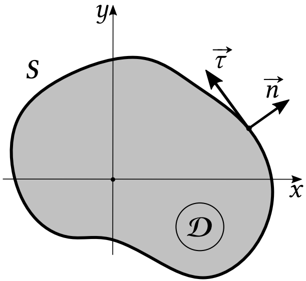

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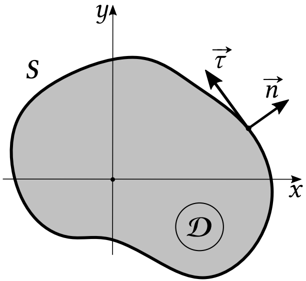

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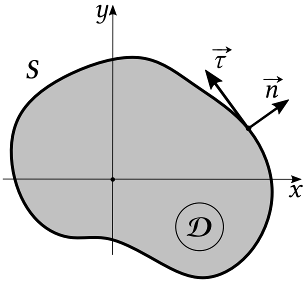

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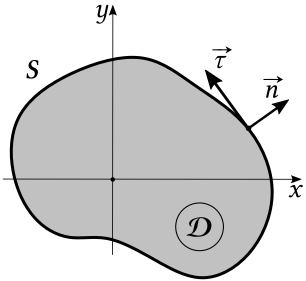

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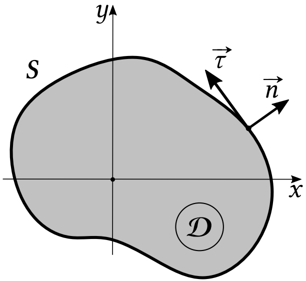

Simple-connected domain

Physical condition on S: zero traction

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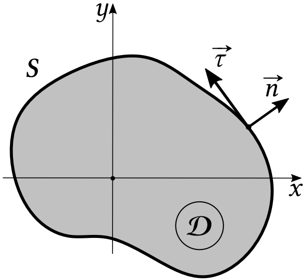

Simple-connected domain

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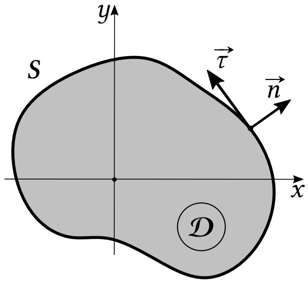

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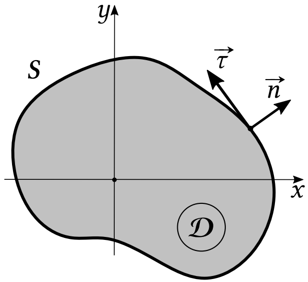

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On S: $\vec{\tau} \cdot \vec{n} = 0$ ↓

$$\frac{dw^*}{dn} = \frac{1}{2}\frac{d}{ds}(x^2 + y^2)$$

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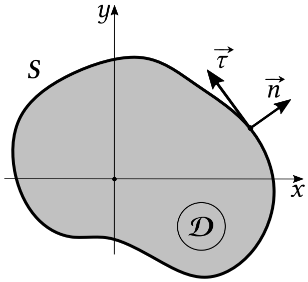

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Final Formulation, With Simple Scaling: $\tau_{zx}/(\mu\alpha) = \tau_x$, $\tau_{zy}/(\mu\alpha) = \tau_y$

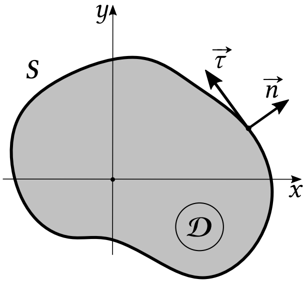
Elliptic type

Laplace's PDE:

$$\nabla^2 w^* = 0$$

Poisson's PDE:

$$\nabla^2 \phi^* = -2$$



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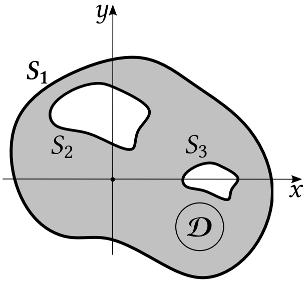

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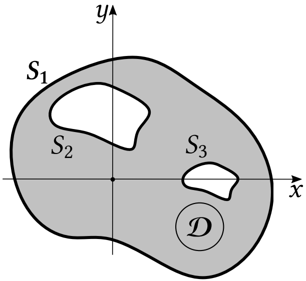

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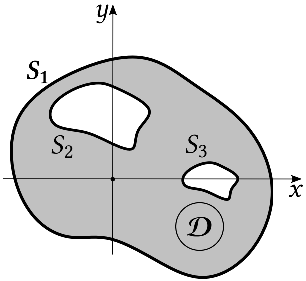

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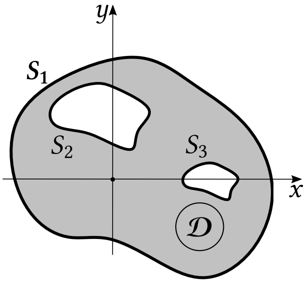

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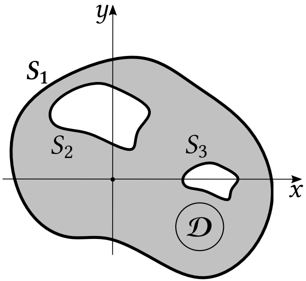

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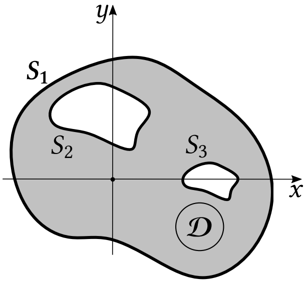

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Finite Volume Method (FVM): Numerical Method for Solving of PDE's

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- Pure steady-state diffusion, without source term

$$\oiint_A \vec{n} \cdot (\Gamma\,\nabla\varphi)\,\mathrm{d}A = 0 \quad \longrightarrow \quad \text{for } \Gamma = \text{const} \quad \longrightarrow \quad \oiint_A \vec{n} \cdot \nabla\varphi\,\mathrm{d}A = 0 \qquad (\text{PDE: } \nabla^2\varphi = 0)$$

FVM: General Principle of Discretization of Computational Domain

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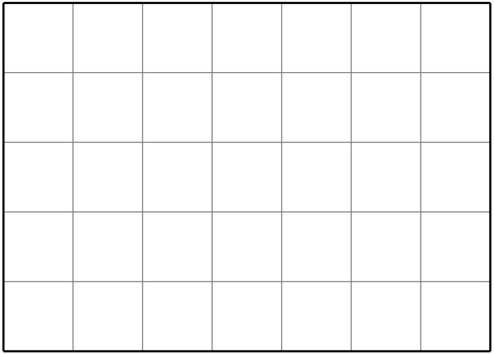

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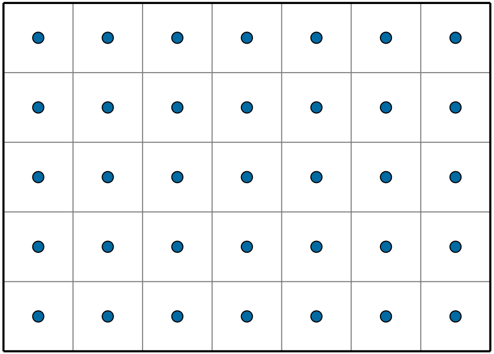

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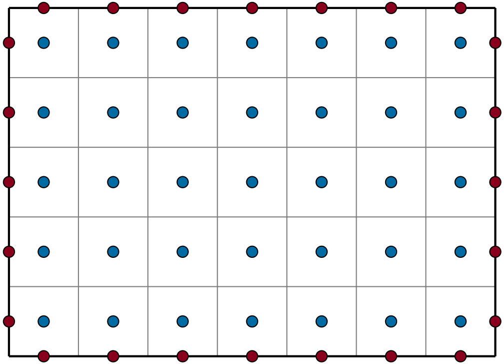

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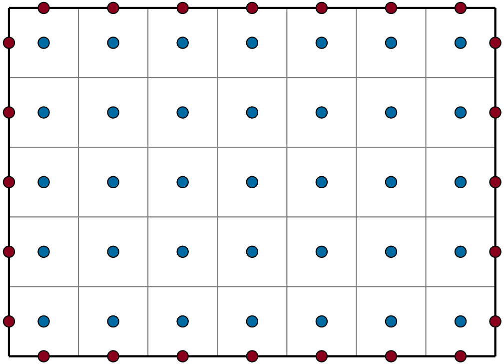

- Boundary conditions
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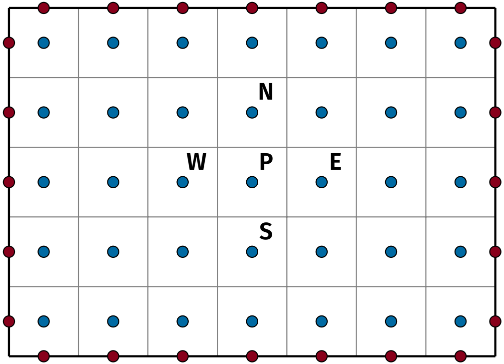

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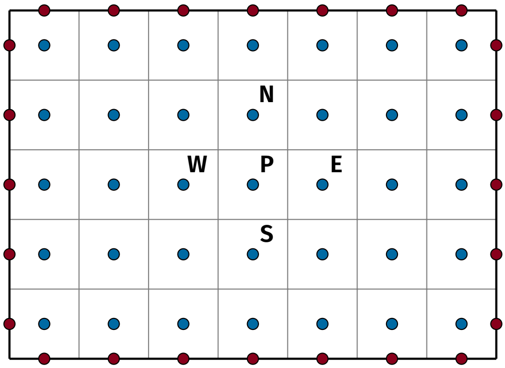

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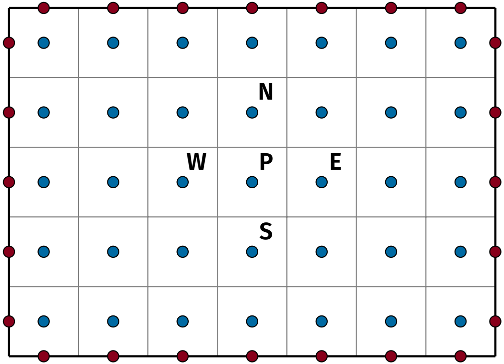

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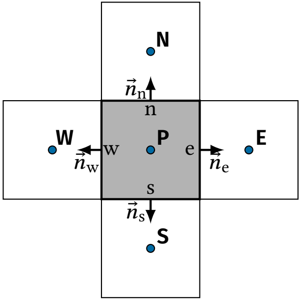

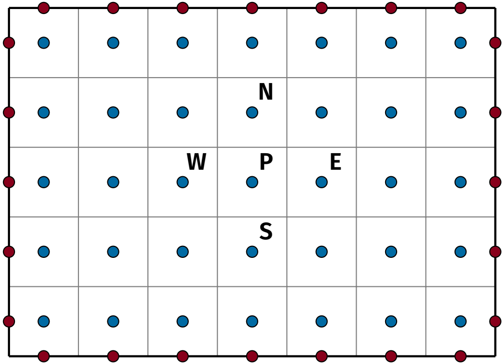

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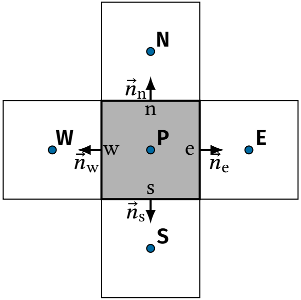

$$\oiint_A \vec{n} \cdot \nabla\varphi \, \mathrm{d}A = \sum_k \iint_{A_k} \vec{n} \cdot \nabla\varphi \, \mathrm{d}A, \quad k = \mathrm{e, w, n, s}$$

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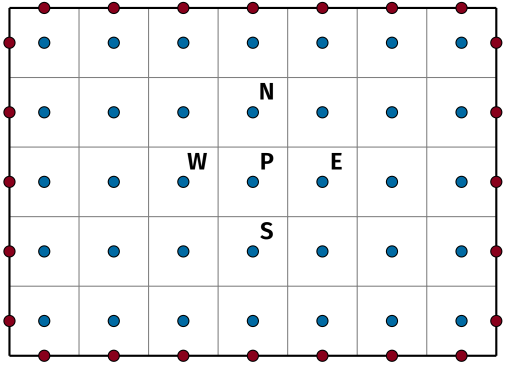

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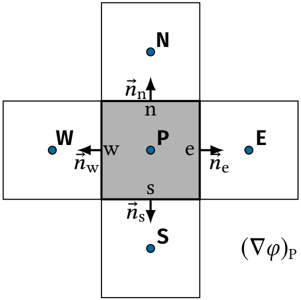

$$(\nabla\varphi)_P = \frac{1}{V_p}\sum_f \vec{n}_f \varphi_f A_f$$

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- **Open-source philosophy:** use existing codes and suite them for your needs

```
const volVectorField& C = mesh.C();
const surfaceVectorField& Sf = mesh.Sf();
volVectorField coord = -C.component(vector::Y)*ei + C.
     component(vector::X)*ej;

fvScalarMatrix WEqn
(
  fvm::ddt(W) - fvm::laplacian(k, W)
);
WEqn.relax();
WEqn.solve();

tau = fvc::grad(W) + coord;
phi = fvc::interpolate(tau) & Sf;
magTau =mag(tau);
```

```
const fvPatch& boundaryPatch = patch();
const vectorField& Cf = boundaryPatch.Cf();
// Vektori normala na granicnoj povrsi
const vectorField& Sf = boundaryPatch.Sf();
// Povrsine pojedinacnih povrsi
const scalarField& magSf = boundaryPatch.magSf();
scalarField& field = this->refGrad();
forAll(Cf, faceI)
{
  const scalar x = Cf[faceI].x();
  const scalar y = Cf[faceI].y();
  const scalar nx = Sf[faceI].x()/magSf[faceI];
  const scalar ny = Sf[faceI].y()/magSf[faceI];
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```

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- Modification of the solver `laplacianFoam`, create `torsionWarpingFoam`, with additional implementation for boundary condition

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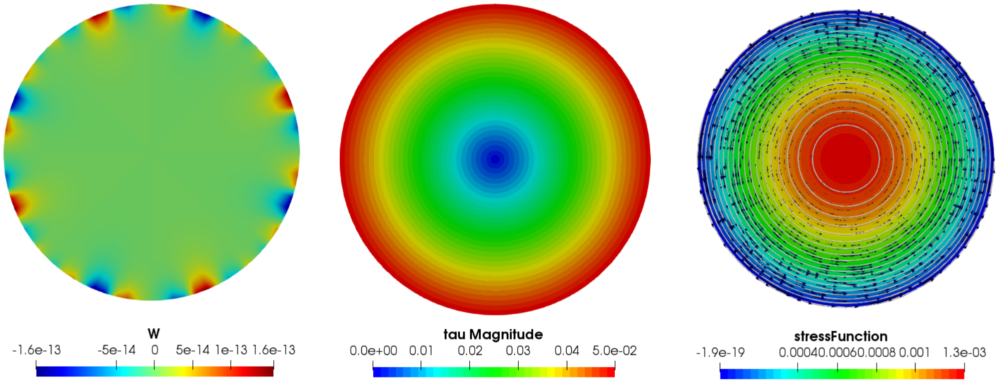
Simple Cases, Known Analytical Solution - Circle And Square Domain

- **Circle:** distribution of warping function, stress vectors, and stress function



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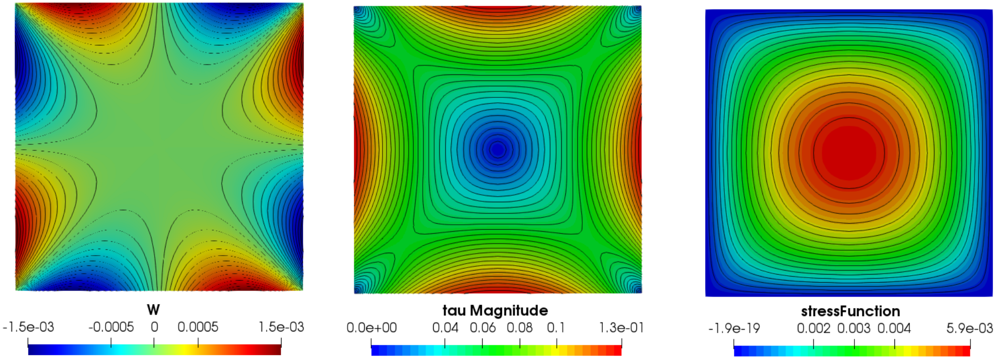

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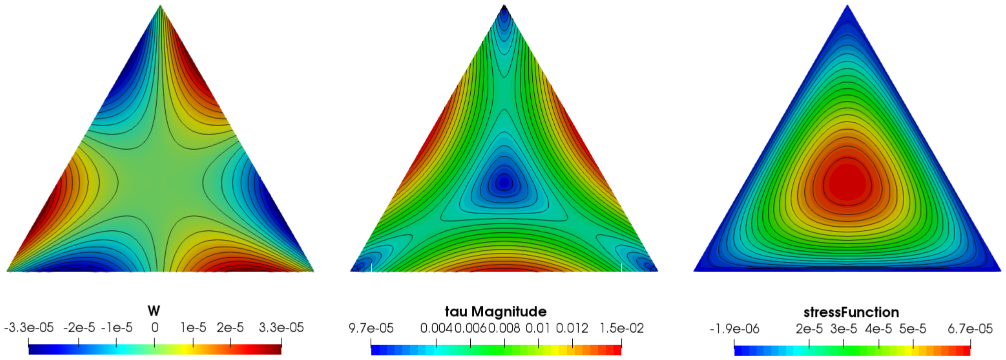
Simple Cases, Known Analytical Solution - Equiliteral Triangle

- **Equiliteral triangle:** distribution of warping function, stress vectors, and stress function



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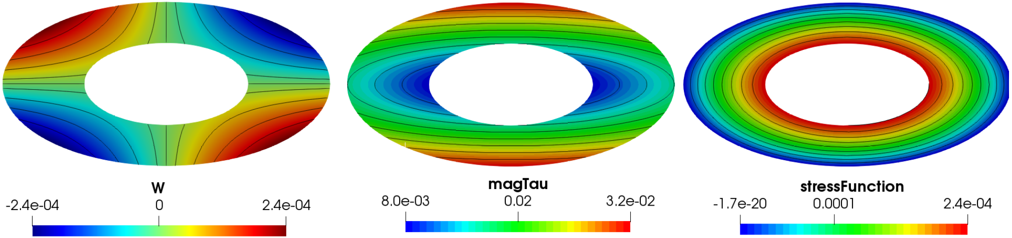
Multiply-connected Domains: Hollow Ellipse (Known Analytical Solution)

- Geometrical characteristics: $a/b = 2$, $k = 0.5$ (outer ellipse $a = 4\,\mathrm{cm}$, $b = 2\,\mathrm{cm}$)



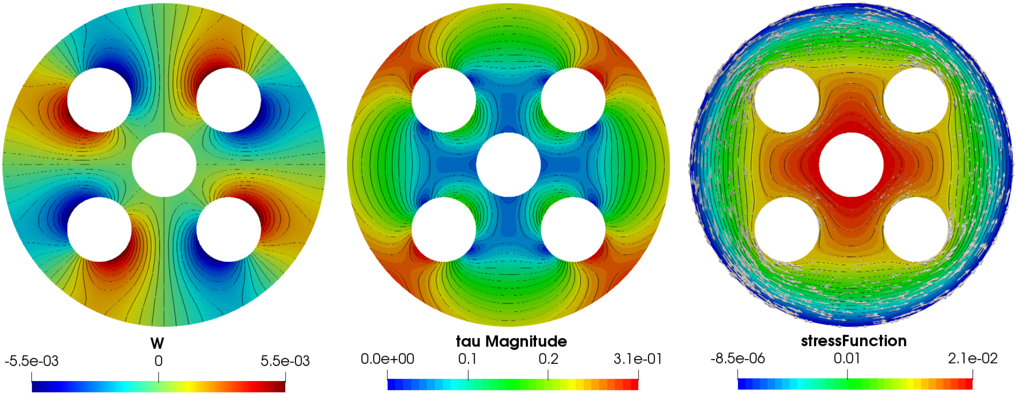
- Analytical solution for torque (torsional constant): $\dfrac{T}{\mu\alpha} = \dfrac{a^3 b^3 \pi}{a^2 + b^2}(1 - k^4) = 75.39822\,\mathrm{cm}^4$

- Numerical solution: based on solved distribution for w^* and numerical integration:

$$\frac{T}{\mu\alpha} = \iint_{\mathcal{D}} \underbrace{\left(x^2 + y^2 + x\frac{\partial w^*}{\partial y} - y\frac{\partial w^*}{\partial x}\right)}_{F} dxdy \;\longrightarrow\; [\text{fvc::domainIntegrate(F)}] \;\longrightarrow\; 75.38238\,\mathrm{cm}^4$$

Multiply-connected Domains: Playing Around With Some Random Shapes

- Domain symmetry - symmetry in solution (well known and expected)



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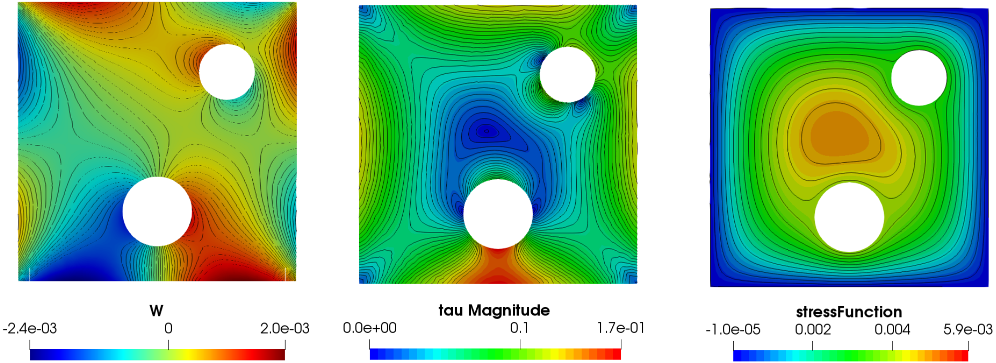
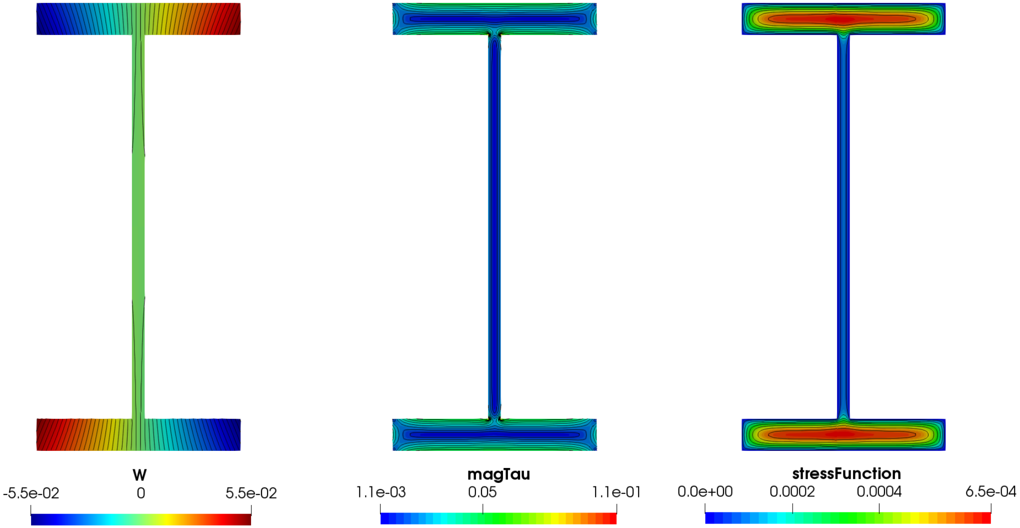


Practical, Engineering Applications (Thin-walled Cross-sections)



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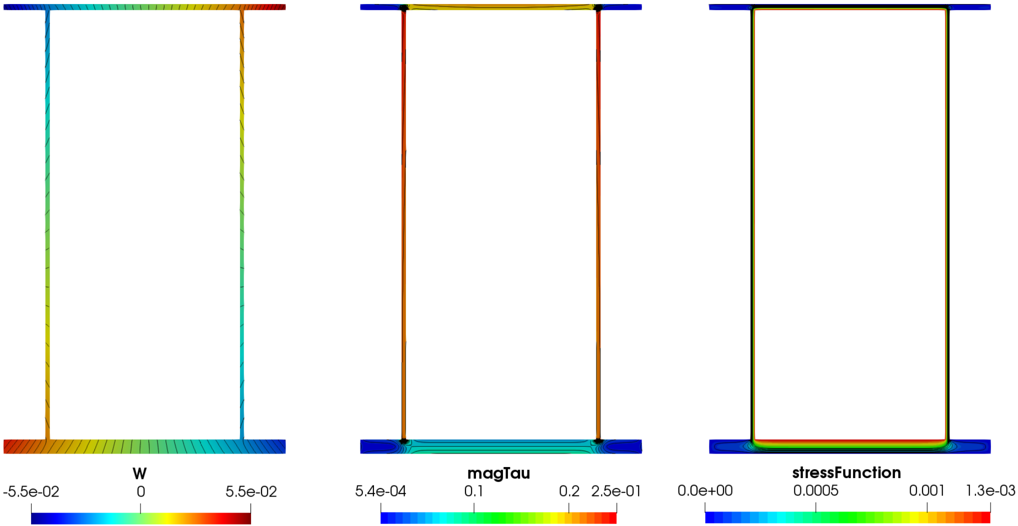

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SOLID MECHANICS – FINITE VOLUME SOLVERS

https://github.com/wyldckat/solidMechanics