Quadrature Formulas with Multiple Nodes for Fourier Coefficients

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Abstract

Gaussian quadrature formulas with multiple nodes and their optimal extensions for computing the Fourier coefficients, in expansions of functions with respect to a given system of orthogonal polynomials, are considered. A numerically stable construction of these quadratures is proposed. Error bounds for these quadrature formulas are derived. We present a survey of recent results on this topic.

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1 Introduction

Let $\{P_k\}_{k=0}^{\infty}$ be a system of orthonormal polynomials on $[a,b]$ with respect to a weight function ω (integrable, non-negative function on $[a,b]$ that vanishes only at isolated points). The approximation of f by the partial sums $S_n(f)$ of its series expansions $f(x)=\sum_{k=0}^{\infty} a_k(f)P_k(x)$ with respect to a given system of orthonormal polynomials $\{P_k\}_{k=0}^{\infty}$ is a classical way of recovery of f. The numerical computation of the coefficients $a_k(f)$,

$$a_k(f)=\int_a^b \omega(t)P_k(t)f(t)\,dt,$$

requires the use of a quadrature formula. Evidently, an application of the n-point Gaussian quadrature formula with respect to the weight ω will give the exact result for all polynomials of degree at most $2n-k-1$, $k<2n-1$.

Following Bojanov and Petrova [1] and using the same notation, we consider quadrature formulas of the type

$$\int_a^b \omega(t)P_k(t)f(t)\,dt \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j), \quad a<x_1<\cdots<x_n<b, \qquad (1)$$

where ν_j are given natural numbers (multiplicities) and $P_k(t)$ is a monic polynomial of degree k.

In [1], for the sake of convenience, Bojanov and Petrova defined the formula (1) to be Gaussian, if it has maximal algebraic degree of precision ADP.

Let

$$\pi_n(\mathbb{R}) := \left\{ P(t) \ : \ P(t) = \sum_{k=0}^{n} d_k t^k, \ d_k \in \mathbb{R} \right\}$$

represents the space of all polynomials in one variable of degree at most n. Bojanov and Petrova [1, Section 2] discuss general remarks concerning Gaussian quadrature formulas with multiple nodes, since the study of formulas of type (1) for Fourier coefficients can be reduced to the study of standard multiple node quadratures. We repeat the following theorem established by Ghizzetti and Ossicini [2].

Theorem 1.1 *For any given set of odd multiplicities $\nu_1, \ldots, \nu_n$ ($\nu_j = 2s_j + 1, \ s_j \in \mathbb{N}_0, \ j = 1, \ldots, n$), there exists a unique quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \quad a \le x_1 < \ldots < x_n \le b,$$

of ADP $= \nu_1 + \ldots + \nu_n + n - 1$, *which is the well known Chakalov-Popoviciu quadrature formula. The nodes $x_1, \ldots, x_n$ of this quadrature are determined uniquely by the orthogonality property*

$$\int_a^b \omega(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} Q(t)\, dt = 0, \quad \forall Q \in \pi_{n-1}(\mathbb{R}).$$

The corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is known in the classical literature as σ-orthogonal polynomial, with $\sigma = \sigma_n = (s_1, \ldots, s_n)$, where n indicates the size of the array.

Bojanov and Petrova [1] describe the connection between quadratures with multiple nodes and formulas of type (1). For the system of nodes $\mathbf{x} := (x_1, \ldots, x_n)$ with corresponding multiplicities $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, they define the polynomials

$$\Lambda(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m), \quad \Lambda_j(t; \mathbf{x}) := \frac{\Lambda(t; \mathbf{x})}{t - x_j}, \quad \Lambda^{\bar{\nu}}(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m)^{\nu_m},$$

set $x_j^{\nu_j} := (x_j, \ldots, x_j)$ [x_j repeats ν_j times], $j = 1, \ldots, n$, denote by $g[x_1, \ldots, x_m]$ the divided difference of g at the points $x_1, \ldots, x_m$, and state and prove the following important theorem which reveals the relation between the standard quadratures and the quadratures for Fourier coefficients.

Theorem 1.2 *For any given sets of multiplicities* $\bar{\mu} := (\mu_1, \ldots, \mu_k)$ *and* $\bar{\nu} := (\nu_1, \ldots, \nu_n)$*, and nodes* $y_1 < \cdots < y_k$*,* $x_1 < \cdots < x_n$*, there exists a quadrature formula of the form*

$$\int_a^b \omega(t)\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\,dt \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji} f^{(i)}(x_j), \tag{2}$$

with ADP $= N$ *if and only if there exists a quadrature formula of the form*

$$\int_a^b \omega(t)f(t)\,dt \approx \sum_{m=1}^{k}\sum_{\lambda=0}^{\mu_m-1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} a_{ji} f^{(i)}(x_j), \tag{3}$$

which has degree of precision $N + \mu_1 + \cdots + \mu_k$*. In the case* $y_m = x_j$ *for some* m *and* j*, the corresponding terms in both sums combine in one term of the form*

$$\sum_{\lambda=0}^{\mu_m+\nu_j-1} d_{m\lambda} f^{(\lambda)}(y_m).$$

2 Main Results

Let us suppose that the coefficients a_{ji} $(j = 1, \ldots, n;\ i = 0, \ldots, \nu_j - 1)$ in (3) are known. By acting as in the first part of the proof of Theorem 2.1 in [1] we can determine the coefficients c_{ji} $(j = 1, \ldots, n;\ i = 0, \ldots, \nu_j - 1)$ in (2). Namely, applying (3) to the polynomial $\Lambda^{\bar{\mu}}(\cdot;\mathbf{y})f$, where $f \in \pi_N(\mathbb{R})$, the first sum in (3) vanishes and we can obtain (see [1, Eq. (2.4)])

$$\int_a^b \omega(t)\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\,dt = \sum_{j=1}^{n}\left(\sum_{i=0}^{\nu_j-1} a_{ji}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\right]^{(i)}\Big|_{t=x_j}\right) = \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji} f^{(i)}(x_j),$$

where

$$c_{ji} = \sum_{s=i}^{\nu_j-1} a_{js}\binom{s}{i}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})\right]^{(s-i)}\Big|_{t=x_j} \quad (j = 1, 2, \ldots, n;\ i = 0, 1, \ldots, \nu_j - 1). \tag{4}$$

In [4], for a Chakalov-Popoviciu quadrature formula of type

$$\int_a^b \omega(t)f(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} a_{\nu i} f^{(i)}(x_\nu), \tag{5}$$

where $a \le x_1 < x_2 < \cdots < x_n \le b$, it was studied its extension to the interpolatory quadrature formula

$$\int_a^b \omega(t)f(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} b_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} c_{\mu j}^* f^{(j)}(x_\mu^*), \tag{6}$$

where x_ν are the same nodes as in (5), and the new nodes x_μ^* and new weights $b_{\nu i}, c_{\mu j}^*$ are chosen to maximize the degree of precision of (6), which is greater than or equal to

$$\sum_{\nu=1}^{n}(2s_\nu+1)+\sum_{\mu=1}^{m}(2s_\mu^*+1)+m-1=2\left(\sum_{\nu=1}^{n}s_\nu+\sum_{\mu=1}^{m}s_\mu^*\right)+n+2m-1.$$

The interpolatory quadrature formula (6) has in general ADP$=\sum_{\nu=1}^{n}(2s_\nu+1)+\sum_{\mu=1}^{m}(2s_\mu^*+1)-1$ which is higher than the ADP of the quadrature formula (5), i. e. $\sum_{\nu=1}^{n}(2s_\nu+1)+n-1$, if

$$2\sum_{\mu=1}^{m}s_\mu^*+m>n.$$

If there exist unique quadrature formulas (5), (6), then Theorem 1.2 implies that there exist unique quadratures for calculating the integrals

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,dt\approx\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1}\widehat{a}_{\nu i}f^{(i)}(x_\nu), \tag{7}$$

and

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,dt\approx\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1}\widehat{b}_{\nu i}f^{(i)}(x_\nu)+\sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*}\widehat{c}_{\mu j}^*f^{(j)}(x_\mu^*), \tag{8}$$

which represent the Fourier coefficients if the given σ-orthogonal polynomial $\pi_{n,\sigma}$ coincides to the corresponding ordinary orthogonal polynomial P_n with respect to the weight function ω, i.e., $\pi_{n,\sigma}(t)\equiv P_n(t)$ on $[a,b]$. Then, the error in (7) can be estimated by the well known method of computing the absolute value of the difference of the quadrature sums in (8) and (7).

Using the above presented method (see (7), (8)) for the case $\omega(t)=1/\sqrt{1-t^2}$, $t\in[-1,1]$, we have proved in [5] the following statement.

Theorem 2.1 *Let $n,s\in\mathbb{N}$ and $\omega(t)=1/\sqrt{1-t^2}$, $t\in[-1,1]$. Then, there exists a unique quadrature formula with multiple nodes for calculating the corresponding Fourier-Chebyshev coefficients $a_n(f)=\int_{-1}^{1}f(t)T_n(t)/\sqrt{1-t^2}\,dt$,*

$$\int_{-1}^{1}\frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,dt\approx\sum_{\nu=1}^{n}\sum_{i=0}^{2s-1}\widehat{A}_{i,\nu}f^{(i)}(\tau_\nu), \tag{9}$$

with ADP $=2sn+n-1$, *as well as its Kronrod extension*

$$\int_{-1}^{1}\frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,dt\approx\sum_{\nu=1}^{n}\sum_{i=0}^{2s-1}\widehat{B}_{i,\nu}f^{(i)}(\tau_\nu)+\sum_{j=1}^{n+1}\widehat{C}_jf(\hat{\tau}_j),$$

with ADP $=2sn+2n+1$.

In the special case when $s=1$ the quadrature formula (9) becomes the well known Micchelli-Rivlin quadrature formula (cf. [3]).

3 Conclusion

A numerically stable construction of the quadrature formulas with multiple nodes for Fourier coefficients that is proposed in [4], [5] enables us their calculation as well as estimation of its error. A part of those results is presented here.

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