Error Estimates for Some Product Gauss Rules

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Abstract

Some integrals I^m over m-dimensional regions can be approximated by cubature formulas G_n^m constructed by the product of Gauss quadrature rules G_n. Using corresponding Gauss-Kronrod rules K_{2n+1} or corresponding generalized averaged Gauss rules $\widehat{G}_{2n+1}$ instead of G_n, we construct cubature formulas K_{2n+1}^m and $\widehat{G}_{2n+1}^m$. In order to estimate the error $|I^m - G_n^m|$ we use the differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$.

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1 Introduction

Consider the quadrature formula (q.f.) of the form

$$I(f) = \int_{\mathbb{R}} f(t) d\mu(t) \approx Q_n(f) = \sum_{k=1}^{n} \omega_k f(t_k).$$

The unique optimal interpolatory q.f. with n nodes and (algebraic) degree of exactnes $2n-1$ is Gauss q.f. G_n. The nodes of G_n are the eigenvalues and the weights are proportional to the squares of the first components of the corresponding eigenvectors of tridiagonal symmetric Jacobi matrix with diagonal elements $\alpha_0, ..., \alpha_{n-1}$ and subdiagonal elements $\sqrt{\beta_1}, ..., \sqrt{\beta_{n-1}}$, where αs and βs are coefficient of the three-term recurrence relation, satisfied by the monic orthogonal polynomials.

In order to (economically) estimate the error $|I - G_n|$ we can use the differences $|K_{2n+1} - G_n|$ and $|\widehat{G}_{2n+1} - G_n|$. K_{2n+1} is corresponding Gauss-Kronrod q.f. with degree of exactness $3n + 1$, and $\widehat{G}_{2n+1}$ is corresponding generalized averaged Gauss q.f. with degree of exactness $2n + 2$, both with $2n + 1$ nodes (n nodes of G_n form a subset). K_{2n+1} has higher degree of exactness, but $\widehat{G}_{2n+1}$ exists in some situations when K_{2n+1} does not and its numerical construction is simpler – Spalević in [2] proposed effective numerical procedure for constructing $\widehat{G}_{2n+1}$, where tridiagonal symmetric matrix has diagonal elements $\alpha_0, ..., \alpha_{n-1}; \alpha_n; \alpha_{n-1}, ..., \alpha_0$ and subdiagonal elements $\sqrt{\beta_1}, ..., \sqrt{\beta_{n-1}}; \sqrt{\beta_n}; \sqrt{\beta_{n+1}}; \sqrt{\beta_{n-1}}, ..., \sqrt{\beta_1}$.

Some integrals $I^m = \int_{\Omega^m} f(\boldsymbol{x})\omega(\boldsymbol{x})d\boldsymbol{x}$, $\omega(\boldsymbol{x}) \geq 0$, $\boldsymbol{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \geq 2$, over m-dimensional regions Ω^m, can be approximated by cubature formulas (c.f.) G_n^m constructed by the product of q.f. G_n. In order to estimate the error $|I^m - G_n^m|$ we first extend G_n^m to K_{2n+1}^m and $\widehat{G}_{2n+1}^m$, and than use the differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$, where K_{2n+1}^m denotes c.f. constructed by the product of corresponding q.f. K_{2n+1}, and $\widehat{G}_{2n+1}^m$ denotes c.f. constructed by the product of corresponding q.f. $\widehat{G}_{2n+1}$.

2 Main Results

In all considered cases we first introduce G_n^m constructed by the product of G_n (according to [1]). K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ can be introduced analogously, using corresponding K_{2n+1} and $\widehat{G}_{2n+1}$ instead od G_n. In all examples we first solve I^m analytically, and than show results for $|I^m - G_n^m|$, $|I^m - K_{2n+1}^m|$, $|K_{2n+1}^m - G_n^m|$, $|I^m - \widehat{G}_{2n+1}^m|$, $|\widehat{G}_{2n+1}^m - G_n^m|$, for different values of n. All results are calculated with 40 significant decimal digits.

Cube: $C^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid -1 \leq x_l \leq 1,\ l = 1, ..., m\}$. Integral of each variable x_l, $l = 1, ..., m$, can be approximated by n-point Gauss q.f. G_n with Legendre weight function $\omega(t) = 1$ on $[-1, 1]$,

$$\int_{-1}^{1} \varphi(t)dt \approx \sum_{k=1}^{n} \omega_k \varphi(t_k),$$

which leads to c.f.

$$I^m \approx G_n^m = \sum_{k_1,...,k_m=1}^{n} \omega_{k_1} \cdots \omega_{k_m} \cdot f(t_{k_1}, ..., t_{k_m}).$$

G_n^m has n^m, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(2n+1)^m$ nodes. Selected results are shown in table 1.

Simplex: $T^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_l \geq 0,\ l = 1, ..., m,\ x_1 + \cdots + x_m \leq 1\}$. Approximating integral of each variable x_l, $l = 1, ..., m$, by n-point Gauss q.f. G_n with Jacobi weight function $\omega(t) = (1-t)^{m-l}$, $l = 1, ..., m$, on $[0, 1]$,

$$\int_{0}^{1} (1-t)^{m-l} \varphi(t)dt \approx \sum_{k=1}^{n} \omega_{k,m-l} \varphi(t_{k,m-l}), \quad l = 1, ..., m,$$

we get c.f.

$$I^m \approx G_n^m = \sum_{k_1,...,k_m=1}^{n} \omega_{k_1,m-1} \cdots \omega_{k_m,0} \cdot f(\Pi(k_1), ..., \Pi(k_1, ..., k_m)),$$

$I^2 = \int_{-1}^{1}\int_{-1}^{1}\cos(x_1+x_2)dx_1dx_2 = (2\sin 1)^2 \approx 2.832...$					
n	$\lvert I^2 - G_n^2\rvert$	$\lvert I^2 - K_{2n+1}^2\rvert$	$\lvert K_{2n+1}^2 - G_n^2\rvert$	$\lvert I^2 - \widehat{G}_{2n+1}^2\rvert$	$\lvert \widehat{G}_{2n+1}^2 - G_n^2\rvert$
2	2.391e-02	2.979e-07	2.391e-02	2.979e-07	2.391e-02
4	9.455e-07	3.794e-16	9.455e-07	1.086e-13	9.455e-07
6	5.095e-12	8.249e-26	5.095e-12	4.534e-20	5.095e-12
$I^7 = \int_{-1}^{1}\cdots\int_{-1}^{1}\cos(x_1+\cdots+x_7)dx_1\cdots dx_7 = (2\sin 1)^7 \approx 38.237...$					
n	$\lvert I^7 - G_n^7\rvert$	$\lvert I^7 - K_{2n+1}^7\rvert$	$\lvert K_{2n+1}^7 - G_n^7\rvert$	$\lvert I^7 - \widehat{G}_{2n+1}^7\rvert$	$\lvert \widehat{G}_{2n+1}^7 - G_n^7\rvert$
2	1.118	1.408e-05	1.118	1.408e-05	1.118
4	4.468e-05	1.792e-14	4.468e-05	5.131e-12	4.468e-05

Table 1: Selected results for integrals over m-dimensional cube.

$I^3 = \int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\frac{dx_1dx_2dx_3}{(1+x_1+x_2+x_3)^3} = \frac{8\ln 2-5}{16} \approx 0.034...$					
n	$\lvert I^3 - G_n^3\rvert$	$\lvert I^3 - K_{2n+1}^3\rvert$	$\lvert K_{2n+1}^3 - G_n^3\rvert$	$\lvert I^3 - \widehat{G}_{2n+1}^3\rvert$	$\lvert \widehat{G}_{2n+1}^3 - G_n^3\rvert$
2	1.237e-04	1.353e-08	1.237e-04	6.196e-08	1.237e-04
4	1.285e-07	2.513e-14	1.285e-07	7.961e-12	1.285e-07
6	1.167e-10	2.024e-18	1.167e-10	2.337e-15	1.167e-10
$I^4 = \int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\int_0^{1-x_1-x_2-x_3}\frac{dx_1dx_2dx_3dx_4}{(1+x_1+x_2+x_3+x_4)^4} = \frac{24\ln 2-16}{144} \approx 0.004...$					
n	$\lvert I^4 - G_n^4\rvert$	$\lvert I^4 - K_{2n+1}^4\rvert$	$\lvert K_{2n+1}^4 - G_n^4\rvert$	$\lvert I^4 - \widehat{G}_{2n+1}^4\rvert$	$\lvert \widehat{G}_{2n+1}^4 - G_n^4\rvert$
2	1.959e-05	1.131e-09	1.959e-05	1.179e-08	1.960e-05
4	2.111e-08	-	-	1.661e-12	2.111e-08
6	1.937e-11	-	-	5.015e-16	1.937e-11

Table 2: Selected results for integrals over m-dimensional simplex.

$$\Pi(k_1) = t_{k_1,m-1},$$
$$\Pi(k_1,...,k_l) = (1-t_{k_1,m-1})\cdots(1-t_{k_{l-1},m-l+1})t_{k_l,m-l}, \quad l = 2,...,m.$$

G_n^m has n^m, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(2n+1)^m$ nodes.

Selected results are shown in table 2. In the cases of I^4, $n = 4, 6$, q.f. K_{2n+1} doesn't exist and c.f. K_{2n+1}^4 can't be constructed.

Sphere: $S^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_1^2 + \cdots + x_m^2 = r^2\}$. If we introduce spherical coordinates $r, \varphi_1, ..., \varphi_{m-1}$, than replace integral of variable φ_{m-1} by $(2n)$-point rectangle formula and approximate integral of each variable φ_{m-l-2}, $l = 0, ..., m-3$, by n-point Gauss q.f. G_n with Gegenbauer weight function $\omega(t) = (1-t^2)^{l/2}$, $l = 0, ..., m-3$, on $[-1, 1]$,

$$\int_{-1}^{1}(1-t^2)^{l/2}\varphi(t)dt \approx \sum_{k=1}^{n}\omega_{k,l}\varphi(t_{k,l}), \quad l = 0, ..., m-3,$$

$S^3 : x_1^2 + x_2^2 + x_3^2 = 1, \quad I^3 = \int_{S^3} e^{x_1} d\boldsymbol{x} = 2\pi(e - 1/e) \approx 14.768...$					
n	$\lvert I^3 - G_n^3\rvert$	$\lvert I^3 - K_{2n+1}^3\rvert$	$\lvert K_{2n+1}^3 - G_n^3\rvert$	$\lvert I^3 - \widehat{G}_{2n+1}^3\rvert$	$\lvert\widehat{G}_{2n+1}^3 - G_n^3\rvert$
2	4.842e-02	5.748e-07	4.842e-02	5.748e-07	4.842e-02
4	1.854e-06	7.429e-16	1.854e-06	2.123e-13	1.854e-06
6	9.855e-12	1.583e-25	9.855e-12	8.746e-20	9.855e-12

Table 3: Selected results for integrals over m-dimensional sphere.

we get c.f.

$$I^m \approx G_n^m = r^{m-1}\frac{\pi}{n} \sum_{k=1}^{2n} \sum_{k_1,...,k_{m-2}=1}^{n} \omega_{k_1,m-3} \cdots \omega_{k_{m-2},0} \cdot F(r, \varphi_{1,k_1}, ..., \varphi_{m-2,k_{m-2}}, \frac{\pi}{n}k),$$

$$F(r, \varphi_1, ..., \varphi_{m-1}) = f(r\cos\varphi_1, ..., r\sin\varphi_1 \cdots \sin\varphi_{m-1}),$$
$$\varphi_{m-l-2,k} = \arccos t_{k,l}, \quad l = 0, ..., m-3.$$

G_n^m has $2n^{m-1}$, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $2(2n+1)^{m-1}$ nodes.

Selected results are shown in table 3.

Ball: $B^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_1^2 + \cdots + x_m^2 \le 1\}$. I^m can be approximated by linear combination of n integrals over m-dimensional spheres of different radii,

$$I^m \approx \sum_{i=1}^{n} B_i \int_{S_i^m} f(\boldsymbol{x}) d\boldsymbol{x}, \quad S_i^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_1^2 + \cdots + x_m^2 = r_i^2\}.$$

If m is even, than $r_i^2 = \tau_i$, $2B_i r_i^{m-1} = \lambda_i$, $i = 1, ..., n$, where τ_i and λ_i are nodes and weights of Gauss q.f.

$$\int_0^1 t^{m/2-1}\varphi(t)dt \approx \sum_{i=1}^{n} \lambda_i \varphi(\tau_i).$$

If m is odd, than $r_i = \tau_i$, $B_i r_i^{m-1} = \lambda_i$, $i = 1, ..., n$, where τ_i and λ_i are nodes and weights of Gauss q.f.

$$\int_{-1}^1 t^{m-1}\varphi(t)dt \approx \sum_{\substack{i=-n \\ i\neq 0}}^{n} \lambda_i \varphi(\tau_i).$$

c.f. takes the form

$$I^m \approx G_n^m = \frac{\pi}{2n} \sum_{i=1}^{n} B_i r_i^{m-1} \sum_{k=1}^{4n} \sum_{k_1,...,k_{m-2}=1}^{2n} \omega_{k_1,m-3} \cdots \omega_{k_{m-2},0} \cdot F(r_i, \varphi_{1,k_1}, ..., \varphi_{m-2,k_{m-2}}, \frac{\pi}{2n}k).$$

$B^4: x_1^2+\cdots+x_4^2 \le 1, \quad I^4=\int_{B^4}\sqrt{(x_2^2+x_3^2+x_4^2)^{17}}d\boldsymbol{x}=\frac{524288}{4849845}\pi \approx 0.339...$					
n	$\lvert I^4-G_n^4\rvert$	$\lvert I^4-K_{2n+1}^4\rvert$	$\lvert K_{2n+1}^4-G_n^4\rvert$	$\lvert I^4-\widehat{G}_{2n+1}^4\rvert$	$\lvert \widehat{G}_{2n+1}^4-G_n^4\rvert$
2	1.084e-01	7.329e-06	1.084e-01	6.606e-05	1.084e-01
4	9.084e-05	9.728e-13	9.084e-05	4.984e-11	9.084e-05
6	4.369e-10	3.459e-16	4.369e-10	1.409e-14	4.369e-10

Table 4: Selected results for integrals over m-dimensional ball.

G_n^m has $(2n)^m$, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(4n+2)(4n+1)^{m-1}$ nodes.

Selected results are shown in table 4.

3 Conclusion

As expected, with the increase of n precision of all three c.f. G_n^m, K_{2n+1}^m, $\widehat{G}_{2n+1}^m$ increases. Also expected, both K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have better accuracy than G_n^m, while K_{2n+1}^m has better (or the same) accuracy than $\widehat{G}_{2n+1}^m$.

Both differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$ give very good estimates of error $|I^m - G_n^m|$. $\widehat{G}_{2n+1}^m$ exists in some situations when K_{2n+1}^m does not, and it's numerical construction is simpler than the construction of K_{2n+1}^m (since the construction of $\widehat{G}_{2n+1}$ is simpler than the construction of K_{2n+1}). So, $\widehat{G}_{2n+1}^m$ might be a better choice than K_{2n+1}^m for estimating error of G_n^m.

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