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# ON MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES WITH CHEBYSHEV WEIGHTS: THE CASES S =1,2*

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**Abstract.** We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at $\pm 1$, when the weight $\omega$ is Chebyshev weight function of the first and third kind. It is shown that the modulus of the kernel attains its maximum on the real axis $\forall n \geq n_0,\ n_0 = n_0(\rho, s)$ in the case $s = 1$. Analogous results can be performed in the case $s = 2$.

## 1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$(1.1)\quad \int_{-1}^{1} f(t)\omega(t)dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f)\ (n \in \mathbb{N};\ s \in \mathbb{N}_0),$$

where $\omega$ is nonnegative and integrable function on interval $(-1, 1)$, which is exact for all algebraic polynomials of degree at most $2(s + 1)n - 1$. The nodes $\tau_\nu$ in (1.1) must be zeros of the $s$-orthogonal polynomials with respect to the weight function $\omega(t)$. The $s$-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k \omega(t)dt = 0\,, \qquad k = 0, 1, \ldots, n - 1.$$

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Numerically stable methods for constructing nodes $\tau_\nu$ and coefficients $A_{i,\nu}$ can be found in [1, 4, 6]. For more details on quadrature formulae with multiple nodes see [2] and [3].

Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and let $D$ be its interior. If integrand $f$ is analytic on $D$ and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see, for instance, [5] and reference therein)

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_\Gamma K_{n,s}(z) f(z)\,dz. \tag{1.2}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\rho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}\,, \qquad z \notin [-1,1],$$

where

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\omega(t)\,dt.$$

The modulus of the kernel is symmetric with respect to real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function in (1.1) is even the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|$ (see [5, Lemma 2.1.]).

The integral representation (1.2) leads to the error estimate

$$|R_{n,s}| \le \frac{l(\Gamma)}{2\pi} \left( \max_{z\in\Gamma} K_{n,s}(z) \right) \left( \max_{z\in\Gamma} (f(z) \right),$$

where $l(\Gamma)$ denotes the length of the contour $\Gamma$. First maximum depends only on the quadrature rule (i.e., on $\omega$) and not on $f$.

## 2. The Maximum Modulus of the Kernel on Confocal Ellipses

In this section we take as contour $\Gamma$ an ellipse $\mathcal{E}_\rho$ with foci at points $\pm 1$ and a sum of semiaxes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \; z = \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1} e^{-i\theta}\right), \; 0 \le \theta \le 2\pi \right\}.$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1,1]$, while with increasing $\rho$ it becomes more and more circle-like.

We study the magnitude of $|K_{n,s}(z)|$ on the contour $\mathcal{E}_\rho$ for the generalized Chebyshev weight functions of first and third kind, respectively, (cf. [5])

$$\omega_1(t) = (1-t^2)^{-1/2} \quad \text{and} \quad \omega_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}.$$

**2.1. The weight function** $\omega_1(t) = (1-t^2)^{-1/2}$. Explicit representation of the kernel $K^{(1)}_{n,s}(z)$ on the ellipse $\mathcal{E}_\rho$ for the weight function $\omega_1(t)$ was given by Milovanović and Spalević in [5], as well

$$\left|K^{(1)}_{n,s}(z)\right| = \frac{2^{1-s}\pi}{\rho^n} \frac{\left|Z^{(1)}_{n,s}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n}+\cos 2n\theta)^{1/2+s}}\ , \ z \in \mathcal{E}_\rho, \tag{2.1}$$

where

$$a_j = a_j(\rho) = \frac{1}{2}\left(\rho^j + \rho^{-j}\right), \quad j \in \mathbb{N}, \tag{2.2}$$

and

$$Z^{(1)}_{n,s}\left(\rho e^{i\theta}\right) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} \left(\rho e^{i\theta}\right)^{-2nk}. \tag{2.3}$$

The weight function $\omega_1(t)$ is even, so we can take $\theta \in [0, \pi/2]$.

Using the representation (2.1) Milovanović and Spalević stated the following conjecture:

**Conjecture 2.1.** *For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that*

$$\max_{z\in\mathcal{E}_\rho}\left|K^{(1)}_{n,s}(z)\right| = K^{(1)}_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1})\right)$$

*for each $n \ge n_0$.*

**Theorem 2.1.** *Conjecture* 2.1 *holds for $s = 1$.*

*Proof.* Because (2.1) it is sufficiently to prove

$$\begin{aligned}(9+6\rho^{-2n}\cos 2n\theta+\rho^{-4n})&(a_2-1)(a_{2n}+1)^3\\ &\le (9+6\rho^{-2n}+\rho^{-4n})(a_2-\cos 2\theta)(a_{2n}+\cos 2n\theta)^3,\end{aligned} \tag{2.4}$$

for sufficiently large $n$ ($n \ge n_0(\rho)$) and $\theta \in (0, \pi/2]$, where $a_j$ are given by (2.2). Introducing half-angles, this is equivalent to

$$\begin{aligned}&[(3+\rho^{-2n})^2 - 12\rho^{-2n}\sin^2 n\theta](a_2-1)(a_{2n}+1)^3\\ &\quad\le (3+\rho^{-2n})^2[(a_2-1)+2\sin^2\theta][(a_{2n}+1)^3 - 6a_{2n}^2\sin^2 n\theta\\ &\qquad -12a_{2n}\sin^2 n\theta\cos^2 n\theta - 6\sin^2 n\theta + 12\sin^4 n\theta - 8\sin^6 n\theta].\end{aligned}$$

Now, it is sufficiently to prove

$$
\begin{aligned}
(2.5) \qquad & (a_{2n}+1)^3 - \frac{\sin^2 n\theta}{\sin^2 n\theta}(a_2-1)(3a_{2n}^2+6a_{2n}\cos^2 n\theta \\
& +3-6\sin^2 n\theta+4\sin^4 n\theta)-2\sin^2 n\theta(3a_{2n}^2 \\
& +6a_{2n}\cos^2 n\theta+3-6\sin^2 n\theta+4\sin^4 n\theta)\geq 0,
\end{aligned}
$$

if $n \geq n_0(\rho)$ and $\theta \in (0,\pi/2]$. Since

$$\left|\frac{\sin n\theta}{\sin\theta}\right| \leq n, \ (a_2-1)>0,$$

and

$$(\forall n\in\mathbb{N}) \quad 3a_{2n}^2+6a_{2n}\cos^2 n\theta+3-6\sin^2 n\theta+4\sin^4 n\theta \geq 0,$$

the left-hand side of (2.5) is larger or equal to

$$(a_{2n}+1)^3 - n^2(a_2-1)(3a_{2n}^2+6a_{2n}+7)-2(3a_{2n}^2+6a_{2n}+7) := F(n).$$

Using (2.2) we get

$$
\begin{aligned}
F(n) = \frac{1}{8}\big[&\rho^{6n}-(3An^2+6)\rho^{4n}-(12An^2+33)\rho^{2n} \\
&-(34An^2+116)-(12An^2+33)\rho^{-2n}-(3An^2+6)\rho^{-4n}+\rho^{-6n}\big],
\end{aligned}
$$

where $A=(a_2-1)=(\rho-\rho^{-1})^2=$ const. Since $F(n)$ is continuous on $\mathbb{R}$ and $\lim_{n\to+\infty}=+\infty$, it follows that $F(n)>0$, for all $n>t$, where $t$ is the largest zero of $F(n)$. For $n_0$ we can take $[t]+1$. □

We can use the function $F(n)$ from the proof to estimate $n_0$. Numerical values of $[t]+1$ ($t$ is the largest zero of $F$) for some values of $\rho$ are presented in Table 1. The least possible values of $n_0$ are also presented. We can see that the least possible $n_0$ is estimated by $[t]+1$ very well.

Table 1

| $\rho$ | $[t]+1$ | the l.p. $n_0$ | $\rho$ | $[t]+1$ | the l.p. $n_0$ |
|---|---|---|---|---|---|
| 1.01 | 207 | 165 | 1.2 | 12 | 10 |
| 1.02 | 104 | 83 | 1.3 | 8 | 7 |
| 1.03 | 70 | 56 | 1.4 | 7 | 6 |
| 1.04 | 53 | 42 | 1.5 | 6 | 5 |
| 1.05 | 43 | 34 | 1.6 | 5 | 4 |
| 1.06 | 36 | 29 | 1.7 | 4 | 4 |
| 1.07 | 31 | 25 | 1.8 | 4 | 4 |
| 1.08 | 27 | 22 | 1.9 | 4 | 3 |
| 1.09 | 24 | 20 | 2 | 4 | 3 |
| 1.1 | 22 | 18 | 2.5 | 3 | 3 |

Analogous results can be derived in the case $s = 2$, in a similar way. But when $s$ increases the derivation becomes drastically complex.

**2.2. The weight function** $\omega_3(t) = (1+t)^{1/2+s}(1-t)^{-1/2}$. Explicit representation of the kernel $K_{n,s}^{(3)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the generalized Chebyshev weight function of third kind $\omega_3(t)$ was given by Milovanović and Spalević in [5], as well

$$\left|K_{n,s}^{(3)}(z)\right| = \frac{2^{1-s}\pi}{\rho^{n+1/2}} \frac{(a_1 + \cos\theta)\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos{(2n+1)\theta})^{1/2+s}}\ , \tag{2.6}$$

where

$$Z_{n,s}^{(3)}\left(\rho e^{i\theta}\right) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} \left(\rho e^{i\theta}\right)^{-(2n+1)k}. \tag{2.7}$$

Using representation (2.6) in [5] was been stated the following conjecture:

**Conjecture 2.** For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that

$$\max_{z\in\mathcal{E}_\rho} \left|K_{n,s}^{(3)}(z)\right| = K_{n,s}^{(3)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)$$

for each $n \geq n_0$.

**Theorem 2.2.** *The conjecture* 2 *holds for* $s = 1$.

**Proof.** Because (2.6) it is sufficiently to prove

$$\begin{aligned}(9 + 6\rho^{-2n-1}\cos{(2n+1)\theta} + \rho^{-4n-2})(a_2 - 1)(a_{2n+1} + 1)^3 \\ \leq (9 + 6\rho^{-2n-1} + \rho^{-4n-2})(a_2 - \cos 2\theta)(a_{2n+1} + \cos{(2n+1)\theta})^3,\end{aligned} \tag{2.8}$$

for enough large $n$ ($n \geq n_0(\rho)$) and $\theta \in (0, \pi]$, where $a_j$ are given by (2.2). Introducing the new variable $k$ with $n = (2k-1)/2$ inequality (2.8) becomes inequality (2.4), which holds $\forall k,\ k > t$, where $t$ is the largest zero of the function $F(k)$ from the proof of Theorem 2.1. Furthermore, we can conclude that inequality (2.8) holds for every $n$, such that $n > (2t-1)/2$. For $n_0$ we can take $[(2t-1)/2] + 1$. □

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