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QUADRATURE FORMULAE OF RADAU AND LOBATTO TYPE CONNECTED TO s-ORTHOGONAL POLYNOMIALS[1)]

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При помощи обобщенной проблемы Гаусса, используя квадратурные формулы Гизетти и Оссичини, автор строит квадратурные формулы Гаусса–Турана типов формул Радо и Лобатто. Используя функцию влияния, автор оценивает остаточные члены этих формул. Предложен метод использования этих квадратур для вычислений. Приведены результаты численных экспериментов. Библ. 18. Табл. 5.

1. INTRODUCTION

Quadrature formulae are generally obtained by substituting the integral $\int_a^b f(x)dx$ to be evaluated with the integral $\int_a^b \varphi(x)dx$, where $\varphi(x)$ is a function approximating $f(x)$, whose indefinite integral is expressible in elementary functions. The choice of $\varphi(x)$ is performed using interpolation methods. Thus the problem of approximating the integral of $f(x)$ is reduced to the approximation of $f(x)$ itself, that is to a problem not equivalent to the first one. Therefore, it is reasonable to think that it is possible to obtain quadrature formulae without using interpolation methods. In this way the theory of quadrature formulae is expounded in [1]. Our considerations will be based on the results which are expounded in [1], and because of that we will keep the notation from [1].

Let $w(x)$ be a weight function on the interval $[a, b]$, $-\infty \le a < b \le \infty$. A quadrature formula of the form

$$\int_a^b w(x)u(x)dx = \sum_{i=1}^{m}\sum_{h=0}^{2s} A_{hi}^G u^{(h)}(x_i) + R^G(u), \tag{1.1}$$

where $A_{hi}^G = A_{hi}^{(m,s)}$, $x_i = x_i^{(m,s)}$, which is exact for all algebraic polynomials of degree at most $2(s+1)m-1$, was considered firstly by Turán (see [2]), in the case when $w(x)dx = dx$ on $[-1, 1]$. The general case has been considered by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, etc. (see [3] and [1] for references).

The nodes x_i in (1.1) must be zeros of a (monic) polynomial $\pi_m(x)$ which minimizes the integral

$$F \equiv F(a_0, \ldots, a_{m-1}) = \int_a^b w(x)\pi_m(x)^{2s+2}dx,$$

where

$$\pi_m(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1x + a_0.$$

In order to minimize F we must have

$$\int_a^b w(x)\pi_m(x)^{2s+1}x^k dx = 0, \quad k = 0, 1, \ldots, m-1.$$

Such polynomials $\pi_m(x) = P_{s,m}(x)$, which satisfy this new type of orthogonality, *power orthogonality*, are known as s-orthogonal (or s-self associated) polynomials with respect to the measure $w(x)dx$.

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An iterative process for computing the coefficients of s-orthogonal polynomials in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [4].

Gautschi and Milovanović [3] gave a stable procedure for the numerical construction of s-orthogonal polynomials. In §3 of [3] a stable numerical procedure for calculating the coefficients A_{hi}^{G} in (1.1) was proposed. Some alternative methods were proposed by Stroud and Stancu [5] (see also [6]) and Milovanović and Spalević [7] (see also [8]).

Remark 1. A particularly interesting case is one of the Chebyshev weight $w(x) = (1-x^2)^{-1/2}$ with the corresponding s-orthogonal polynomials, i.e., Gauss–Turán quadrature formulas. In 1930, Bernstein [9] showed that the monic Chebyshev polynomial minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_m(x)|^{k+1}}{\sqrt{1-x^2}} dx, \; k \ge 0.$$

Thus, the Chebyshev polynomials are s-orthogonal on $[-1, 1]$ for each $s \ge 0$. Ossicini and Rosati found three other weights for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind (see [10]).

Considering the set of Jacobi polynomials, Ossicini and Rosati [11] showed that the only Jacobi polynomials which are s-orthogonal for a positive integer s are the Chebyshev polynomials of the first kind. Recently, Shi [12] (see also [13]) has proved that the Chebyshev weight is the only one (up to a linear transformation) having the property: for each fixed m, the solutions of the extremal problem

$$\int_{-1}^{1} \left(\prod_{i=1}^{m} (x - x_i) \right)^k w(x) dx = \min_{\pi(x) = x^m + \dots} \int_{-1}^{1} \pi(x)^k w(x) dx$$

for every even k are the same.

Gori and Micchelli [14] have introduced for each m a class of weight functions defined on $[-1, 1]$ for which explicit Gauss–Turán quadrature formulas of all orders can be found. In the other words, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, of the same degree, are independent of s.

2. EXISTENCE AND UNIQUENESS OF RADAU AND LOBATTO FORMULAE

Let

$$\int_{a}^{b} p(x)u(x)dx = \sum_{h=0}^{p} A_{h0}^{R} u^{(h)}(a) + \sum_{h=0}^{2s} \sum_{i=1}^{m} A_{hi}^{R} u^{(h)}(x_i) + R^{R}(u), \tag{2.1}$$

$-\infty < a < \infty$, $p \in N_0$, with

$$R^{R}(u) = 0 \quad \forall u \in \mathcal{P}_{2(s+1)m+p},$$

be the generalized Gauss-Turán quadrature formula of Radau type.

Let

$$\int_{a}^{b} p(x)u(x)dx = \sum_{h=0}^{p} A_{h0}^{L} u^{(h)}(a) + \sum_{h=0}^{2s} \sum_{i=1}^{m} A_{hi}^{L} u^{(h)}(x_i) + \sum_{h=0}^{q} A_{h,m+1}^{L} u^{(h)}(b) + R^{L}(u), \tag{2.2}$$

$-\infty < a < b < \infty$, $q, p \in N_0$, with

$$R^{L}(u) = 0 \quad \forall u \in \mathcal{P}_{2(s+1)m+p+q+1},$$

be the generalized Gauss–Turán quadrature formula of Lobatto type.

With $\mathcal{P}_k$ we denoted the set of all polynomials of degree at most k, $k \in N_0$.

By using the results, which have been given by Ghizzetti and Ossicini [1], it is easy to prove existence and uniqueness of the formulas (2.1) and (2.2).

Define the generalized Gauss problem (see [1, pp. 41–43]).

Let us consider the elementary quadrature formula

$$\int_a^b p(x)u(x)dx = \sum_{h=0}^{n-1}\sum_{i=1}^{m} A_{hi}u^{(h)}(x_i) + R(u), \quad E(u) = 0 \Rightarrow R(u) = 0,$$

where E is the linear differential operator of order n. The question is whether, having fixed integers p_i (with $0 \le p_i \le n-1$, $(\exists k = 1, 2, \dots, m)$ $1 \le p_k$), it is possible to make use of the arbitrary nature of these parameters to drop from the formula the values $u^{(h)}(x_i)$ of the derivatives of order higher than $n - p_i - 1$, $i = 1, 2, \dots, m$, that is whether there can exist a formula of the type

$$\int_a^b p(x)u(x)dx = \sum_{i=1}^{m}\sum_{h=0}^{n-p_i-1} A_{hi}u^{(h)}(x_i) + R(u), \quad E(u) = 0 \longrightarrow R(u) = 0. \tag{2.3}$$

The answer give the following theorem, which the proof can be done in the similar way as one of Theorem 2.5.I in [1] (see also the Problem 2 in [1, p. 45]).

Theorem 1. *Given the nodes* $x_1, \dots, x_m$, *which satisfy*

$$a \le x_1 < \dots < x_m \le b, \tag{2.4}$$

the linear differential operator E of order n and nonnegative integers $0 \le p_i \le n-1$, $(\exists k = 1, 2, \dots, m)$ $1 \le p_k$, *consider the homogeneous differential problem*

$$E(u) = 0; \quad u^{(h)}(x_i) = 0, \quad h = 0, 1, \dots, n - p_i - 1, \quad i = 1, 2, \dots, m. \tag{2.5}$$

If this problem has no non-trivial solutions (*whence* $n \le mn - \sum_{i=1}^{m} p_i$) *it is possible to write a quadrature formula of the type* (2.3) *in form a family of* $mn - \sum_{i=1}^{m} p_i - m$ *parameters. If on the other hand the problem* (2.5) *has q linearly independent solutions* $U_j(x)$ ($j = 1, 2, \dots, q$, *with* $n - nm + \sum_{i=1}^{m} p_i \le q \le p_i$ $(\forall i = 1, 2, \dots, m)$; $1 \le q$) *then the formula* (2.3) *may apply only if the q conditions*

$$\int_a^b p(x)U_j(x)dx = 0, \quad j = 1, 2, \dots, q,$$

are satisfied; if so it is possible to write a quadrature formula of the type (2.3) *in form a family of* $mn - \sum_{i=1}^{m} p_i - n + q$ *parameters.*

Consider the formula (2.1), with the conditions (2.4) for x_i, $i = 1, 2, \dots, m$, for which is $R^{R}(u) = 0$ $\forall u \in \mathcal{P}_{2m(s+1)+p}$.

Let $n = 2m(s+1) + p + 1$. By virtue of the theorem 1 we must consider the boundary problem

$$d^n u/dx^n = 0,$$

with

$$u^{(h)}(a) = 0, \quad h = 0, 1, \dots, p, \quad u^{(h)}(x_i) = 0, \quad h = 0, 1, \dots, 2s, \quad i = 1, 2, \dots, m,$$

and its non trivial solutions which are

$$x^k(x-a)^{p+1}\prod_{i=1}^{m}(x-x_i)^{2s+1}, \quad k = 0, 1, \dots, m-1.$$

Therefore, (2.1) is possible if and only if there are satisfied

$$\int_a^b (x-a)^{p+1}p(x)x^k\prod_{i=1}^{m}(x-x_i)^{2s+1}dx = 0, \quad k = 0, 1, \dots m-1,$$

and this shows that the nodes x_i must coincide with the zeros of the polynomial $P_{s,m}(x)$ of the s-orthogonal system relative to the measure $(x-a)^{p+1}p(x)dx$.

With such a choice of the nodes the formula (2.1) is unique since, with the notations of the theorem 1, we have that $mn - \sum_{i=1}^{m} p_i - n + q$ in our case is

$$m(2s+1)+p+1-n+m = 0.$$

Consider the formula (2.2), with the conditions (2.4) for x_i, $i = 1, 2, \ldots, m$, for which is $R^{\mathrm{L}}(u) = 0$ $\forall u \in \mathcal{P}_{2m(s+1)+p+q+1}$.

Let $n = 2m(s+1)+p+q+2$. By virtue of the theorem 1 we must consider the boundary problem

$$d^n u/dx^n = 0,$$

with

$$u^{(h)}(a) = 0, \quad h = 0, 1, \ldots, p, \quad u^{(h)}(x_i) = 0, \quad h = 0, 1, \ldots, 2s, \quad i = 1, 2, \ldots, m,$$

$$u^{(h)}(b) = 0, \quad h = 0, 1, \ldots, q,$$

and its non trivial solutions which are

$$x^k(x-a)^{p+1}(b-x)^{q+1}\prod_{i=1}^{m}(x-x_i)^{2s+1}, \quad k = 0, 1, \ldots, m-1.$$

Therefore, (2.2) is possible if and only if there are satisfied

$$\int_a^b (x-a)^{p+1}(b-x)^{q+1}p(x)x^k\prod_{i=1}^{m}(x-x_i)^{2s+1}dx = 0, \quad k = 0, 1, \ldots, m-1,$$

and this shows that the nodes x_i must coincide with the zeros of the polynomial $P_{s,m}(x)$ of the s-orthogonal system relative to the measure

$$(x-a)^{p+1}(b-x)^{q+1}p(x)dx.$$

With such a choice of the nodes the formula (2.2) is unique since, with the notations of the theorem 1, we have that $mn - \sum_{i=1}^{m} p_i - n + q$ in our case is

$$(m+2)n - p_0 - \sum_{i=1}^{m} p_i - p_{m+1} - n + m = mn + 2n - (n-p-1) - m(n-2s-1) -$$
$$-(n-q-1) - n + m = 2sm + 2m + p + q + 2 - n = 0$$

3. THE BOUNDING FORMULAS OF THE REMAINDER IN (2.1)

Concerning the assumptions on $p(x)$, $u(x)$ for the validity of (2.1) we have the following theorem:

Theorem 2. *Formula* (2.1) *is valid under the following hypotheses*:

$$p(x) \in L[a, b], \quad u(x) \in AC^{n-1}[a, b],$$

if b is finite,

$$x^n p(x) \in L[a, \infty), \quad u(x) \in AC_{\mathrm{loc}}^{n-1}[a, \infty), \quad u^{(n)}(x)\int_x^{\infty}\xi^{n-1}p(\xi)d\xi \in L[0, \infty).$$

The proof is the same as one of the theorem 4.13.I. in [1, pp. 132–133] and will be omitted.

Let p be even, without loss of generality.

(a). Let $p \le 2s$. We have that $n = 2m(s+1) + p + 1$ is odd. For the remainder in (2.1) we have (cf. [1]):

$$R^{\mathrm{R}}(u) \equiv R(u) = \int_a^b \Phi(x) u^{(n)}(x) dx, \tag{3.1}$$

where the influence function $\Phi(x)$ is expressed by

$$\Phi(x) = \varphi_{i+1}(x) \text{ for } x_i < x < x_{i+1}, \quad i = 0, 1, \dots, m; \quad x_0 = a, \quad x_{m+1} = b, \tag{3.2}$$

and the functions $\varphi_i(x)$, integrals of the differential equation $\varphi^{(n)}(x) = -p(x)$, are given by the formulae

$$\varphi_i(x) = -\int_a^x p(\xi) \frac{(x-\xi)^{n-1}}{(n-1)!} d\xi + \sum_{h=0}^{2s} \sum_{j=1}^{i-1} (-1)^h A_{hj} \frac{(x-x_j)^{n-h-1}}{(n-h-1)!} + \sum_{h=0}^{p} (-1)^h A_{h0} \frac{(x-a)^{n-h-1}}{(n-h-1)!}, \tag{3.3}$$

where $i = 1, 2, \dots, m+1$, and $A_{hj} = A_{hj}^{\mathrm{R}}$.

For $\varphi_{m+1}(x)$ we have

$$\varphi_{m+1}(x) = \int_x^b p(\xi) \frac{(x-\xi)^{n-1}}{(n-1)!} d\xi. \tag{3.4}$$

From (3.2), (3.3) it follows, differentiating k times (with $0 \le k \le n-1$):

$$\Phi^{(k)}(x) = \varphi_i^{(k)}(x) \text{ for } x \in (x_{i-1}, x_i), \quad i = 1, 2, \dots, m+1, \tag{3.5}$$

where

$$\varphi_i^{(k)}(x) = -\int_a^x p(\xi) \frac{(x-\xi)^{n-k-1}}{(n-k-1)!} d\xi + \sum_{h=0}^{M} \sum_{j=0}^{i-1} (-1)^h A_{hj} \frac{(x-x_j)^{n-h-k-1}}{(n-h-k-1)!} + \sum_{h=0}^{N} (-1)^h A_{h0} \frac{(x-a)^{n-h-k-1}}{(n-h-k-1)!}, \tag{3.6}$$

and $M := 2s$, $N := p$, for $0 \le k \le n-2s-2$; $M := n-k-1$, $N := p$, for $n-2s-1 \le k \le n-p-2$; $M := n-k-1$, $N := n-k-1$, for $n-p-1 \le k \le n-1$.

(b). If $p > 2s$ then (3.6) holds with $M := 2s$, $N := p$, for $0 \le k \le n-p-2$; $M := 2s$, $N := n-k-1$, for $n-p-1 \le k \le n-2s-2$; $M := n-k-1$, $N := n-k-1$, for $n-2s-1 \le k \le n-1$.

In both cases ((a) and (b)) we have

$$\varphi_{m+1}^{(k)}(x) = \int_x^b p(\xi) \frac{(x-\xi)^{n-k-1}}{(n-k-1)!} d\xi, \quad x \in (x_m, b). \tag{3.7}$$

Now, we can conclude that

$$\Phi^{(k)}(a) = 0, \quad k = 0, 1, \dots, n-p-2, \quad \Phi^{(k)}(b) = 0, \quad k = 0, 1, \dots, n-1, \tag{3.8}$$

and that the functions $\Phi(x)$, $\Phi'(x)$, …, $\Phi^{(n-2s-2)}(x)$ are continuous in $[a, b]$, while $\Phi^{(n-2s-1)}(x)$, …, $\Phi^{(n-1)}(x)$ have discontinuities of the first kind at the points $x_1, \dots, x_m$. From (3.7) we conclude

$$(-1)^k \Phi^{(k)}(x) > 0 \text{ for } x \in (x_m, b), \quad k = 0, 1, \dots, n-1, \tag{3.9}$$

and, particularly, $\Phi(x) > 0$ on (x_m, b).

We now prove the following theorem:

Theorem 3. *Under the hypothesis that the weight function $p(x)$ is not identically zero in any interval contained in $[a, b]$, the influence function $\Phi(x)$ defined by (3.2) (together with (3.3) and (3.4)) belongs to the class $C^{n-2s-2}[a, b]$, and it is positive inside such an interval.*

Proof. The first part of the theorem has just been proved. Now, we will prove that the influence function is positive on (a, b).

(i). Let firstly $p < 2s$. Prove that $\Phi(x)$ is positive on $(a, x_m]$ and, therefore, on $(a, b) = (a, x_m] \cup (x_m, b)$.

Consider $\Phi^{(n-2s-2)}(x)$. We will prove that $\Phi^{(n-2s-2)}(x)$ has at most $2s+2$ zeros in each interval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, m$. In fact, should it have $2s+3$ of them, for the Rolle theorem, $\Phi^{(n-2s-1)}(x)$ would have at least $2s+2$ zeros inside $[x_{i-1}, x_i]$, $\Phi^{(n-2s)}(x)$ would have at least $2s+1$ zeros and so on, until we may conclude that $\Phi^{(n-1)}(x)$ would have at least two zeros inside $[x_{i-1}, x_i]$. But this is absurd since from (3.6) there follows that, for $x \in (x_{i-1}, x_i)$, we have

$$\Phi^{(n-1)}(x) = \varphi_i^{(n-1)}(x) = -\int_a^x p(\xi)d\xi + \sum_{j=0}^{i-1} A_{0j},$$

and this function is decreasing since p is even.

Particularly, $\Phi^{(n-2s-2)}(x) = \varphi_1^{(n-2s-2)}(x)$ in (a, x_1) and $\varphi_1 \in C^{n-1}[a, x_1]$. Let $\varphi_1^{(n-2s-2)}(x)$ have α ($\alpha \in N_0$) zeros in (a, x_1). Because of the conditions $\varphi_1^{(k)}(a) = 0, k = 0, 1, \ldots, n-p-2$, for the Rolle theorem, we have that $\varphi_1^{(n-2s-1)}(x)$ has at least α zeros in (a, x_1), etc., $\varphi_1^{(n-p-2)}(x)$ has at least α zeros in (a, x_1), $\varphi_1^{(n-p-1)}(x)$ has at least α zeros in (a, x_1). But, we can prove as above that $\Phi^{(n-p-1)}(x)$ has at most $p+1$ zeros in $[a, x_1]$. Therefore, we have $\alpha \le p+1$, i.e., $\Phi^{(n-2s-2)}(x)$ has at most $p+1$ zeros in (a, x_1). Therefore, $\Phi^{(n-2s-2)}(x)$ has at most $p+1+(m-1)(2s+2)$ zeros in $(a, b) = (a, x_1) \cup [x_1, x_m] \cup (x_m, b)$.

We may then show that $\Phi(x)$ does not vanish inside $[a, b]$ and therefore is positive, because it is such on (x_m, b). In fact, if $\Phi(x)$ should vanish at one point in (a, b), using (3.8) and applying Rolle theorem, we find that $\Phi^{(n-2s-2)}(x)$ would vanish at least $n-2s-1$ times, in contraposition with the preceding deduction, because $n-2s-1 \le p+1+(m-1)(2s+2)$ gives $1 \le 0$.

(ii). Let $p > 2s$. Prove that $\Phi(x)$ is positive on (a, b).

Consider $\Phi^{(n-2s-2)}(x)$. We can prove as above that $\Phi^{(n-2s-2)}(x)$ has at most $2s+2$ zeros in each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$. Therefore, $\Phi^{(n-2s-2)}(x)$ has at most $m(2s+2)$ zeros inside $[a, b]$.

We may then show that $\Phi(x)$ does not vanish inside $[a, b]$ and therefore is positive, because it is such on (x_m, b). In fact, if $\Phi(x)$ should vanish at one point in (a, b), using (3.8) and applying Rolle theorem, we find that $\Phi^{(n-p-1)}(x)$ would vanish at least $n-p$ times, $\Phi^{(n-p)}(x)$ would vanish at least $n-p$ times, etc., $\Phi^{(n-2s-2)}(x)$ would vanish at least $n-p$ times, in contraposition with the preceding deduction, because $n-p \le m(2s+2)$ gives $1 \le 0$.

(iii). Let $p = 2s$. Prove that $\Phi(x)$ is positive on (a, b).

Consider $\Phi^{(n-2s-2)}(x)$. We can prove as above that $\Phi^{(n-2s-2)}(x)$ has at most $2s+2$ zeros in each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$. Therefore, $\Phi^{(n-2s-2)}(x)$ has at most $m(2s+2)-1$ zeros inside $[a, b]$, since $\Phi^{(n-2s-2)}(a) = 0$.

We may then show that $\Phi(x)$ does not vanish inside $[a, b]$ and therefore is positive, because it is such on (x_m, b). In fact, if $\Phi(x)$ should vanish at one point in (a, b), using (3.8) and applying Rolle theorem, we find as above that $\Phi^{(n-2s-2)}(x)$ would vanish at least $n-2s-1$ times, in contraposition with the preceding deduction, because $n-2s-1 \le m(2s+2)-1$ gives $1 \le 0$.

Now, we can estimate the remainder in the formulas of the type (2.1), by using (3.1).

1^0. If $u(x) \in AC^{n-1}[a, b]$ and $a, b \in R$ we have

$$|R(u)| \le \max_{a \le x \le b} \Phi(x) V_{n-1} = \Phi(t_0) V_{n-1},$$

where V_{n-1} denotes the total variation of the function $u^{(n-1)}(x)$ absolutely continuous on the interval $[a, b]$. Because $\Phi'(x)$ vanish in exact one point of the interval (a, b) it holds $(\exists t_0 \in (a, b))$ $\max_{a \le x \le b} \Phi(x) = \Phi(t_0)$.

2^0. If $u^{(n)}(x)$ is bounded in $[a, b]$, i.e.,

$$M_n = \sup_{a \le x \le b} |u^{(n)}(x)|, \quad b \le \infty,$$

we have

$$|R(u)| \le M_n \int_a^b \Phi(x) dx.$$

3^0. If $u \in C^n[a, b]$, $b < \infty$, because $\Phi(x) > 0$ on (a, b) we may apply the mean value theorem and write

$$R(u) = u^{(n)}(\xi) \int_a^b \Phi(x) dx, \quad \xi \in (a, b).$$

4. THE BOUNDING FORMULAS OF THE REMAINDER IN (2.2)

Concerning the assumptions on $p(x)$, $u(x)$ for the validity of (2.2) we have the following theorem:

Theorem 4. *Formula* (2.2) *is valid under the following hypotheses*:

$$p(x) \in L[a, b], \quad u(x) \in AC^{n-1}[a, b].$$

The proof is the same as one of the theorem 4.13.I. in [1, pp. 132–133] and will be omitted.

For the following considerations in the formula (2.2) we will distinguish the cases:

$$1^0.\ p > q > 2s, \quad 2^0.\ p = q = 2s, \quad 3^0.\ p < q < 2s, \text{ etc.},$$

and the subcases with p (or q) is odd or even.

Let, for example, $p > q > 2s$, $p + q$ – even, without loss of generality. Then $n = 2m(s + 1) + p + q + 2$ is even. For the remainder in (2.2) we have (see [1]):

$$R^{\mathrm{L}}(u) \equiv R(u) = \int_a^b \Phi(x) u^{(n)}(x) dx, \tag{4.1}$$

where the influence function $\Phi(x)$ is expressed by

$$\Phi(x) = \varphi_{i+1}(x) \text{ for } x_i < x < x_{i+1}, \quad i = 0, 1, \dots, m; \quad x_0 = a, \quad x_{m+1} = b, \tag{4.2}$$

and the functions $\varphi_i(x)$, integrals of the differential equation $\varphi^{(n)}(x) = p(x)$, are given by the formulae

$$\varphi_i(x) = \int_a^x p(\xi)\frac{(x-\xi)^{n-1}}{(n-1)!}d\xi - \sum_{h=0}^{2s}\sum_{j=1}^{i-1}(-1)^h A_{hj}\frac{(x-x_j)^{n-h-1}}{(n-h-1)!} - \sum_{h=0}^{p}(-1)^h A_{h0}\frac{(x-a)^{n-h-1}}{(n-h-1)!}, \tag{4.3}$$

where $i = 1, 2, \dots, m + 1$, and $A_{hj} = A_{hj}^{\mathrm{L}}$.

For $\varphi_{m+1}(x)$ we have

$$\varphi_{m+1}(x) = -\int_x^b p(\xi)\frac{(x-\xi)^{n-1}}{(n-1)!}d\xi + \sum_{h=0}^{q}(-1)^h A_{h,m+1}\frac{(x-b)^{n-h-1}}{(n-h-1)!}. \tag{4.4}$$

From (4.2), (4.3) it follows, differentiating k times (with $0 \le k \le n - 1$):

$$\Phi^{(k)}(x) = \varphi_i^{(k)}(x) \text{ for } x \in (x_{i-1}, x_i), \quad i = 1, 2, \dots, m+1,$$

where

$$\varphi_i^{(k)}(x) = \int_a^x p(\xi)\frac{(x-\xi)^{n-k-1}}{(n-k-1)!}d\xi - \sum_{h=0}^{M}\sum_{j=1}^{i-1}(-1)^h A_{hj}\frac{(x-x_j)^{n-h-k-1}}{(n-h-k-1)!} - \sum_{h=0}^{N}(-1)^h A_{h0}\frac{(x-a)^{n-h-k-1}}{(n-h-k-1)!},$$

and $M := 2s$, $N := p$, for $0 \le k \le n - p - 2$; $M := 2s$, $N := n - k - 1$, for $n - p - 1 \le k \le n - 2s - 2$; $M := n - k - 1$, $N := n - k - 1$, for $n - 2s - 1 \le k \le n - 1$.

For the derivatives of $\varphi_{m+1}(x)$ ($x \in (x_m, b)$), we can use the following formulas:

$$\varphi_{m+1}^{(k)}(x) = -\int_x^b p(\xi)\frac{(x-\xi)^{n-k-1}}{(n-k-1)!}d\xi + \sum_{h=0}^{M}(-1)^h A_{h,m+1}\frac{(x-b)^{n-h-k-1}}{(n-h-k-1)!},$$

with $M := q$, for $0 \le k \le n - q - 2$; $M := n - k - 1$, for $n - q - 1 \le k \le n - 1$.

Now, we can conclude that

$$\Phi^{(k)}(a) = 0, \quad k = 0, 1, \dots, n-p-2, \quad \Phi^{(k)}(b) = 0, \quad k = 0, 1, \dots, n-q-2, \tag{4.5}$$

and that the functions $\Phi(x), \Phi'(x), \dots, \Phi^{(n-2s-2)}(x)$ are continuous in $[a, b]$, while $\Phi^{(n-2s-1)}(x), \dots, \Phi^{(n-1)}(x)$ have discontinuities of first kind at the points $x_1, \dots, x_m$.

If we put $u(x) = (x-a)^{p+1}(b-x)^{q+1}[P_{s,m}(x)]^{2s+2}$ in (4.1), then

$$(-1)^{q+1}n!\int_a^b \Phi(x)dx = \int_a^b p(x)(x-a)^{p+1}(b-x)^{q+1}[P_{s,m}(x)]^{2s+2}dx.$$

So, we obtain that

$$\int_a^b \Phi(x)dx \begin{cases} <0, & \text{if } q \text{ is even,} \\ >0, & \text{if } q \text{ is odd.} \end{cases}$$

Therefore, if $\Phi(x)$ does not vanish in (a, b) it holds a sign on this interval.

We now prove the following theorem:

Theorem 5. *Under the hypothesis that the weight function $p(x)$ is not identically zero in any interval contained in $[a, b]$, the influence function $\Phi(x)$ defined by* (4.2) (*together with* (4.3) *and* (4.4)) *belongs to the class $C^{n-2s-2}[a, b]$, and one holds a sign inside such an interval.*

Proof. The first part of the theorem has just been proved. Now, we will prove that the influence function holds a sign on (a, b). We will give the proof for two cases, without loss of generality.

(i) Let firstly $p > q > 2s$.

Consider $\Phi^{(n-2s-2)}(x)$. We can prove as above that $\Phi^{(n-2s-2)}(x)$ has at most $2s+2$ zeros in each interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, m$. Therefore, $\Phi^{(n-2s-2)}(x)$ has at most $(m+1)(2s+2)$ zeros inside $[a, b]$.

We may then show that $\Phi(x)$ does not vanish inside $[a, b]$. If $\Phi(x)$ should vanish at one point in (a, b), using (4.8) and applying Rolle theorem, we find that $\Phi^{(n-p-1)}(x)$ would vanish at least $n-p$ times, $\Phi^{(n-p)}(x)$ would vanish at least $n-p$ times, etc., $\Phi^{(n-q-1)}(x)$ would vanish at least $n-p$ times, $\Phi^{(n-q)}(x)$ would vanish at least $n-p-1$ times, $\Phi^{(n-q+1)}(x)$ would vanish at least $n-p-2$ times, etc., $\Phi^{(n-2s-2)}(x)$ would vanish at least $n-p-q+2s+1$ times, in contraposition with the preceding deduction, because $n-p-q+2s+1 \le$ $\le (m+1)(2s+2)$ gives $1 \le 0$.

(ii) Let $p < q < 2s$.

Prove that $\Phi(x)$ holds a sign on (a, b). Consider $\Phi^{(n-2s-2)}(x)$. We can prove as above that $\Phi^{(n-2s-2)}(x)$ has at most $2s+2$ zeros in each interval $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, m-1$, and at most $p+1$ zeros in (a, x_1), i.e., at most $q+1$ zeros in (x_m, b). Therefore, $\Phi^{(n-2s-2)}(x)$ has at most $p+q+2+(m-1)(2s+2)$ zeros inside $[a, b]$.

We may then show that $\Phi(x)$ does not vanish inside $[a, b]$. In fact, if $\Phi(x)$ should vanish at one point in (a, b), using (4.5) and applying Rolle theorem, we find that $\Phi^{(n-2s-2)}(x)$ would vanish at least $n-2s-1$ times, in contraposition with the preceding deduction, because $n-2s-1 \le p+t+(m-1)(2s+2)$ gives $1 \le 0$.

Remark 2. The simplest case $p = 2s = q$ has been analyzed in [15] (see also the Problem 13 in [1, p. 147]). Thus, in this paper we complete the results from [15] (see also [1]) for the quadrature formulas (2.1), (2.2).

Now, we can estimate the remainder in the formulas of the type (2.2), by using (4.1).

1^0. If $u(x) \in AC^{n-1}[a, b]$ we have

$$|R(u)| = \max_{a \le x \le b} |\Phi(x)| V_{n-1} = |\Phi(t_0)| V_{n-1},$$

where V_{n-1} denotes the total variation of the function $u^{(n-1)}(x)$ absolutely continuous on the interval $[a, b]$. Because $\Phi'(x)$ vanish in exact one point of the interval (a, b) it holds $(\exists t_0 \in (a, b))$ $\max_{a \le x \le b} \Phi(x) = \Phi(t_0)$.

2^0. If $u^{(n)}(x)$ is bounded in $[a, b]$, i.e.,

$$M_n = \sup_{a \le x \le b} |u^{(n)}(x)|,$$

we have

$$|R(u)| \le M_n \int_a^b |\Phi(x)| dx.$$

3^0. If $u \in C^n[a, b]$, because $\Phi(x)$ holds a sign on (a, b) we may apply the mean value theorem and write

$$R(u) = u^{(n)}(\xi)\int_a^b \Phi(x)dx, \quad \xi \in (a, b).$$

5. CALCULATION OF QUADRATURE RULES (2.1), (2.2)

We give two results, which give a connection between the generalized Gauss–Turán quadrature (1.1) and the corresponding formulas of Radau and Lobatto type (see for example [16]).

Lemma 1. *If we have the generalized Gauss–Turán quadrature of Lobatto type* (2.2), *with distinct real zeros* $x_i = x_i^{(m)}$, $i = 1, 2, \ldots, m$, *all contained in the open interval* (a, b), *there exists then a generalized Gauss–Turán quadrature formula* (1.1), *where* $w(x) = (b-x)^{q+1}(x-a)^{p+1}p(x)$, *the nodes* $x_i^{(m)}$ *are the zeros of s-orthogonal polynomial* $\pi_m(\cdot, w(x)dx)$, *while the weights* A_{hi}^{G} *are expressible in terms of those in* (2.2) *by*

$$A_{hi}^{\mathrm{G}} = \sum_{k=h}^{2s}\binom{k}{h}[(b-x)^{q+1}(x-a)^{p+1}]_{x=x_i}^{(k-h)}A_{ki}^{\mathrm{L}}, \quad h = 0, 1, \ldots, 2s, \quad i = 1, 2, \ldots, m.$$

Proof. Let $g(x) = (b-x)^{q+1}(x-a)^{p+1}\pi(x)$, $\pi \in \mathcal{P}_{2(s+1)m-1}$ and $x_i = x_i^{(m)}$. We have by (2.2)

$$\int_a^b p(x)g(x)dx = \sum_{i=1}^{m}\sum_{k=0}^{2s}[(b-x)^{q+1}(x-a)^{p+1}\pi(x)]_{x=x_i}^{(k)}A_{ki}^{\mathrm{L}},$$

and by (1.1)

$$\int_a^b w(x)\pi(x)dx = \sum_{i=1}^{m}\sum_{h=0}^{2s}A_{hi}^{\mathrm{G}}\pi^{(h)}(x_i).$$

So, we have that

$$\sum_{i=1}^{m}\sum_{k=0}^{2s}[(b-x)^{q+1}(x-a)^{p+1}\pi(x)]_{x=x_i}^{(k)}A_{ki}^{\mathrm{L}} = \sum_{i=1}^{m}\sum_{h=0}^{2s}A_{hi}^{\mathrm{G}}\pi^{(h)}(x_i).$$

Applying the Leibnitz formula for the k-th derivative in the second sum, we find

$$\sum_{k=0}^{2s}\Big[(b-x)^{q+1}(x-a)^{p+1}\pi(x)\Big]_{x=x_i}^{(k)}A_{ki}^{\mathrm{L}} = \sum_{k=0}^{2s}\Bigg[\sum_{h=0}^{k}\binom{k}{h}[(b-x)^{q+1}(x-a)^{p+1}]^{(k-h)}\pi^{(h)}(x)\Bigg]_{x=x_i}A_{ki}^{\mathrm{L}} =$$

$$= \sum_{h=0}^{2s}\left(\sum_{k=h}^{2s}\binom{k}{h}[(b-x)^{q+1}(x-a)^{p+1}]_{x=x_i}^{(k-h)}A_{ki}^{\mathrm{L}}\pi^{(h)}(x_i)\right) = \sum_{h=0}^{2s}A_{hi}^{\mathrm{G}}\pi^{(h)}(x_i),$$

where

$$A_{hi}^{\mathrm{G}} = \sum_{k=h}^{2s}\binom{k}{h}[(b-x)^{q+1}(x-a)^{p+1}]_{x=x_i}^{(k-h)}A_{ki}^{\mathrm{L}}, \quad h = 0, 1, \ldots, 2s, \quad i = 1, 2, \ldots, m.$$

Similarly we can prove:

Lemma 2. *If we have the generalized Gauss–Turán quadrature of Radau type* (2.1), *with distinct real zeros* $x_i = x_i^{(m)*}$, $i = 1, 2, \ldots, m$, *all contained in the open interval* (a, b), *there exists then a generalized Gauss–Turán quadrature formula* (1.1), *where* $w(x) = w^*(x) = (x-a)^{p+1}p(x)$, *the nodes* $x_i^{(m)*}$ *are the zeros* *of s-orthogonal polynomial* $\pi_m(\cdot, w^*(x)dx)$, *while the weights* A_{hi}^{G} *are expressible in terms of those in* (2.1) *by*

$$A_{hi}^{\mathrm{G}} = \sum_{k=h}^{2s} \binom{k}{h} \left[(x-a)^{p+1}\right]_{x=x_i}^{(k-h)} A_{ki}^{\mathrm{R}}, \quad h = 0, 1, \ldots, 2s, \quad i = 1, 2, \ldots, m.$$

Remark 3. Recently the interest in the nodes and the coefficients of the formulas (1.1), (2.1), (2.2) has been underlined by some papers, dealing with a method of approximation of a function by means of moment preserving splines (see [16–18]).

Example. As an example we consider the Chebyshev weight $p(x) = \left(\sqrt{1-x^2}\right)^{-1}$ on the interval $I = [a, b] = [-1, 1]$ in the Lobatto case. Therefore, we have

$$w(x) = (1-x)^{1/2+q}(1+x)^{p+1/2}.$$

In Table 1 the nodes x_i of the corresponding Gauss–Turán quadrature formulae (1.1), for $s = 2$, $m = 4$, are given.

In Table 2 the weights A_{hi}^{G} of the corresponding Gauss–Turán formulae are given. For these calculations we have used the methods from [3] and [7] (see also [8]).

For $p = 1$ ($=q$, without loss of generality) in Table 3 the weights A_{hi}^{L} of the corresponding Gauss–Turán formulae of Lobatto type (2.2) are given. Table 4 gives the corresponding coefficients A_{h0}^{L}, A_{h5}^{L} in the endpoints –1, 1. The numbers in parentheses denote decimal exponents. The programs were realized in the double precision arithmetics in FORTRAN 77.

In the cases when the derivatives of the integrand we can obtain relative simply, the quadratures (1.1), (2.1), (2.2) can be used for the approximate calculation of such integral. We give an example where it is preferable to use a formula of Gauss–Turán type.

Table 1

i	x_{2i-1}	x_{2i}
1	–8.41567404859432(–1)	–3.28168606801998(–1)
2	3.28168606801998(–1)	8.41567404859432(–1)

Table 2

i	h	A_{hi}^{G}	$A_{h+1,i}^{\mathrm{G}}$
1	0	7.5765832442755(–2)	6.2654888450913(–3)
1	2	3.6654414242426(–4)	9.3958203613400(–6)
1	4	1.4588085631151(–7)	
2	0	5.1328279010533(–1)	2.1311377827516(–2)
2	2	6.5184626213695(–3)	1.1042906602195(–4)
2	4	1.3566511670870(–5)	
3	0	5.1328279010533(–1)	–2.1311377827516(–2)
3	2	6.5184626213695(–3)	–1.1042906602195(–4)
3	4	1.3566511670870(–5)	
4	0	7.5765832442755(–2)	–6.2654888450913(–3)
4	2	3.6654414242426(–4)	–9.3958203613400(–6)
4	4	1.4588085631151(–7)	

Table 3

i	h	A_{hi}^{L}	$A_{h+1,i}^{L}$
1	0	6.5802713035484(–1)	8.4368409927532(–3)
1	2	2.6795494468975(–3)	3.1286893729615(–5)
1	4	1.7136968279902(–6)	
2	0	6.6823331015210(–1)	3.2209234340733(–3)
2	2	8.3649374736572(–3)	3.8429666602366(–5)
2	4	1.7038885475690(–5)	
3	0	6.6823331015210(–1)	–3.2209234340733(–3)
3	2	8.3649374736572(–3)	–3.8429666602366(–5)
3	4	1.7038885475690(–5)	
4	0	6.5802713035484(–1)	–8.4368409927532(–3)
4	2	2.6795494468975(–3)	–3.1286893729615(–5)
4	4	1.7136968279902(–6)	

Table 4

h	A_{h0}^{L}	A_{h5}^{L}
0	2.4453588628796(–1)	2.4453588628796(–1)
1	1.4578947060761(–3)	–1.4578947060761(–3)

Table 5

m	s	p (=q)	re
2	1	0	1.0(–09)
3	1	0	3.6(–15)
4	1	0	5.6(–16)
2	1	1	2.5(–12)
2	1	2	6.5(–15)
2	2	1	4.5(–16)

By using the quadrature formula (2.2) and our methods we have calculated the integral

$$J = \pi I_0(1) = \int_{-1}^{1} \frac{e^t}{\sqrt{1-t^2}}dt = 3.97746326050642\ldots,$$

for some m, s, p (= q). I_0 is the modified Bessel function.

In general case the number of evaluations of integrand for (2.2) is $m(2s + 1) + p + q + 2$, but here one is $m + 2$.

In Table 5 the relative errors re of these calculations are given.

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