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BEHAVIOUR, EXAMINATION AND STABILITY OF THE CONSTRAINED MECHANICAL SYSTEM DESCRIBED WITH NONLINEAR EQUATIONS

Mitra V. Vesović[1], Radoslav D. Radulović[1]

[1] Faculty of Mechanical Engineering
University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35
e-mail: mvesovic@mas.bg.ac.rs, rradulovic@mas.bg.ac.rs

Abstract:

The paper considers motion and stability of a holonomic mechanical system in the vertical plane of an arbitrary force field. Differential equations of motion are created for a given system on the basis of general theorems of dynamics. Insights into generalized coordinates, Lagrange's equations of the second kind, covariant and Hamilton's equations are presented. Additionally to numerical procedures in the paper, a review of the theoretical foundations is performed. Also, the conditions of static equilibrium are solved numerically and by applying the intersection of the two curves. The paper introduced kinetic as well as the potential energy of the system. The spatial arrangement of equilibrium positions and behavior of the potential energy in the environment of the equilibrium positions is shown. Finally, the stability of motion for analysis is approached through Lagrange - Dirichlet theorem. Moreover, special attention is paid to examining effects responses of the disturbed and undisturbed system. Nonlinear and linearized equations are obtained in order to check the stability of the system for disturbed and undisturbed motion using Hurwitz stability criterion. Various procedures are verificated by drawing the same conclusions.

Key words: holonomic system, applied mechanics, system stability, nonlinear systems, disturbed motion analysis

1. Introduction

Designing a complex system presents a challenge for the engineers from the point of view of good analysis of a mechanical system, which will predict the behavior of the plant in different fields of industries. Literature related to the issue of explicit analysis of analytical mechanics and system motion can be found in [1] - [4]; an overview of system stability problems is well explained in [5] - [8]. In [9] authors have expressed and emphasized issue of system performance analysis consisting of slide mechanisms, as well as the material point; this paper, using the example of a holonomic mechanical system with limited reactions of constraints, presents the procedure of creating the differential equations of motion based on the general theorems of dynamics. The objectives in the present paper are similar and include a review of different approaches to modelling a specific multi-body system with given pseudo – codes for simulation and graphical representation of problem solutions. This paper also provides a procedure for solving constrained motion of a holonomic mechanical system in a plane with respect to its energy during motion with specified and given initial positions.

2. Description of a system and mathematical modelling

A mechanical system, shown on the Figure 1, is composed of two material points M_1 and M_2, as well as the slider M_3 which are tied with light rigid rods to each other. The fixed plane O_{xy} coincides with the vertical plane of motion, where the axis O_y is directed vertically upwards. In the configuration space, the position of the mechanical system in relation to the fixed coordinate system O_{xy} is defined by the set of Lagrangian coordinates (q^1, q^2) where $q^1 = \varphi$ and $q^2 = \theta$ are absolute angles, shown in figure. The rods OM_1 and OM_2 are of equal length l, while the rod M_1M_3 is of length $2l$. The spring with stiffness c_1, whose length is, in the unstressed state, $l_{01} = \overline{O_1O}$, is tied to the material point M_2, while at the other end it is tied to a fixed wall. Spring with stiffness c_2, whose length in the unstressed state $l_{02} = \overline{O_2A}$, has one end tied to material point M_1, while at the other end it is connected to a fixed wall. Slider M_3, which moves horizontally, in the direction which coincides with the direction of the O_x axis, is connected by a damper, with a coefficient of proportionality β, while its second end is attached to a fixed wall. On material point M_2, which is located at the end of the rod M_1M_2 acts force $F = F_0 e^{-\alpha t}$. All necessary numerical data are given in the Table 1.

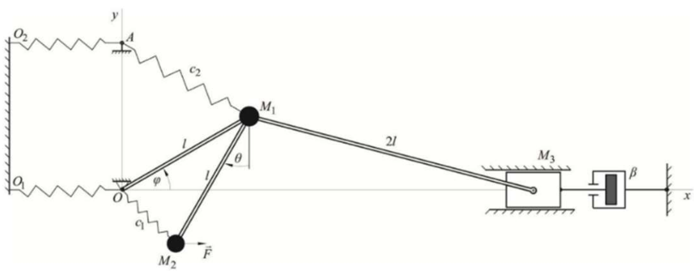


Fig. 1. Holonomic mechanical system

Parameters	Value	Name	Parameters	Value	Name
m_1	10 kg	Mass of material point M_1	β	121 N s / m	Coefficient of proportionality
m_2	12 kg	Mass of material point M_2	F_0	100 N	Static force
m_3	8 kg	Mass of slider - crank M_3	φ_0	4 π/13	Initial angle
l	1.25 m	length	θ_0	π / 7	Initial angle
c_1	162.5 N / m	Spring stiffness	$\dot{\varphi}_0$	0	Initial velocity
c_2	366.67 N / m	Spring stiffness	$\dot{\theta}_0$	0	Initial velocity
			α	0.7	Coefficient

Table 1. Numerical values

2.1 Constraints and Lagrange equations of the second kind

The state of a mechanical system of N material points M_ν ($\nu = 1,2, \dots N$), is determined in each moment t by the position and velocities of all its points in the inertial reference system (IRS). If a fixed Cartesian system is introduced into an IRS, the state of the system is determined by variable scalar quantities: coordinates x_ν, y_ν, z_ν and velocity projections $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$, which must satisfy the relations:

$$f_\mu(x_1, y_1, z_1, \dots, x_N, y_N, z_N, \dot{x}_1, \dot{y}_1, \dot{z}_1, \dots, \dot{x}_N, \dot{y}_N, \dot{z}_N; t) = 0, \quad \mu=1,2,\dots, m < 3N. \tag{1}$$

The motion of the considered system is limited by the following stationary holonomic constraints (1)-(5):

$$f^1 = x_2^2 + y_1^2 - l^2 = 0, \tag{2}$$
$$f^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0, \tag{3}$$
$$f^3 = y_3 = 0, \tag{4}$$
$$f^4 = (x_3 - x_1)^2 + (y_3 - y_1)^2 - 4l^2 = 0, \tag{5}$$

There are four geometric constrains (*p*=4), with zero differential equations (*q*=0) and $n = 2N - (p+q) = 2$, where *n* is the number of degrees of freedom, *N* is the number of material points. Instead of independent Cartesian coordinates, independent generalized coordinates are introduced, which also determine the position of the mechanical system. Independent generalized coordinates represent a minimum number of independent geometric parameters that can unambiguously describe the motion of the considered mechanical system in space. Selected geometric parameters will be marked as $q(t)$. Position of the mechanical system from the Figure 1 is defined by the set of Lagrange coordinates (q^1, q^2), where $q^1 = \varphi$, and $q^2 = \theta$ are the absolute angles. The coordinates of all points can be expressed via generalized coordinates:

$$\begin{aligned} &x_1 = l\,cos\varphi, \quad y_1 = l\,sin\varphi, \\ &x_2 = l\,cos\varphi - lsin\theta,\; y_2 = l \sin\varphi - l\cos\theta \\ &x_3 = l\left(cos\varphi + \sqrt{4 - sin^2\varphi}\right), \quad y_3 = 0 \end{aligned} \tag{6}$$

Squared values of the velocities of the material points are determined by:

$$\begin{aligned} &V_1^2 = l^2\dot{\varphi}^2, V_2^2 = l^2\,\dot{\varphi}^2 + 2l^2\, sin(\varphi+\theta)\,\dot{\varphi}\dot{\theta} + l^2\,\dot{\theta}^2, \\ &V_3^2 = l^2 sin^2\varphi\,(1 + \frac{cos\varphi}{\sqrt{4 - sin^2\varphi}})^2\,\dot{\varphi}^2 \end{aligned} \tag{7}$$

Kinetic energy of a mechanical system is calculated as the sum of the kinetic energies of each slider and the material point. This can be represented by the equation:

$$T = \frac{1}{2}(m_1 V_1^2 + m_2 V_2^2 + m_3 V_3^2) \tag{8}$$

By substituting the calculated velocities (7) into (8), the kinetic energy of the whole mechanical system can be calculated, which is presented in the equation (9). In addition to kinetic energy, one of the system's characteristics is its potential energy (10). When the total kinetic and potential energies of a mechanical system are calculated, it is possible to determine the coefficients of metric tensors $a_{\alpha\beta}$ and Christoffel symbols of the first kind $\Gamma_{\beta\gamma,\alpha}, \alpha, \beta, \gamma = 1, \dots, n$.

$$T = \frac{l^2}{2}\left[m_1 + m_2 + m_3 sin^2\varphi\left(1 + \frac{cos\varphi}{\sqrt{4 - sin^2\varphi}}\right)^2\right]\dot{\varphi}^2 + m_2 l^2 \sin(\varphi+\theta)\,\dot{\varphi}\dot{\theta} + \frac{m_2 l^2}{2} \tag{9}$$

$$\Pi = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + \frac{c_1}{2}(x_2^2 + y_2^2) + \frac{c_2}{2}(x_1^2 + (l - y_1^2)^2) \tag{10}$$

$$
\begin{aligned}
&T = \frac{1}{2}a_{11}\dot{\varphi}^2 + a_{12}\dot{\varphi}\dot{\theta} + \frac{1}{2}a_{22}\dot{\theta}^2\\
&a_{11} = l^2\left[m_1 + m_2 + m_3 sin^2\varphi\left(1 + \frac{cos\varphi}{\sqrt{4 - sin^2\varphi}}\right)^2\right]\\
&a_{12} = a_{21} = m_2 l^2\, sin(\varphi + \theta)\,, a_{22} = m_2 l^2
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\Gamma_{12,1} = \Gamma_{21,1} = \Gamma_{12,2} = \Gamma_{21,2} = \Gamma_{22,2} = 0, \Gamma_{22,1} = \Gamma_{11,2} = m_2 l^2\, cos(\varphi + \theta)\\
&\Gamma_{11,1} = -\frac{1}{(sin^2\varphi - 4)^2} m_3 l^2\, sin\varphi\left(cos\varphi + \sqrt{4 - sin^2\varphi}\right)^2 (-4\,cos\varphi + sin\varphi)^2\sqrt{4 - sin^2\varphi})
\end{aligned}
\tag{12}
$$

The last thing that needs to be determined in order to form Lagrangian equations of the second kind in covariant form are generalized forces Q_α. Generalized forces can be determined by the sum of potential and non-potential forces, so it is very easy to obtain Lagrange equations of the second kind in covariant form:

$$
a_{\alpha\beta}\ddot{q}^\beta + \Gamma_{\beta\gamma,\alpha}\dot{q}^\beta\dot{q}^\gamma = Q_\alpha, \quad \alpha,\beta,\gamma = 1,\dots,n,
\tag{13}
$$

so the graphical representation of the generalized coordinates $q^i = q^i(t), \ \ i = 1,2 \dots n$ is presented in Figure 2. Also, the change in the position of the slider and material points is shown in Figure 3. From the Figure 3 it is seen that the value of the coordinate y_3 does not change over time, because slider M_3 moves only in horizontal line. For verification accuracy of the results, the equations of relations (2-5) from the first point of this problem are analysed on the Figure 4.

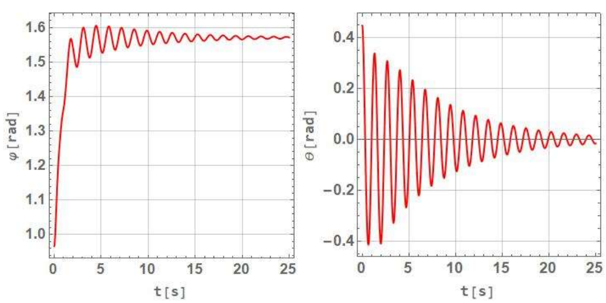
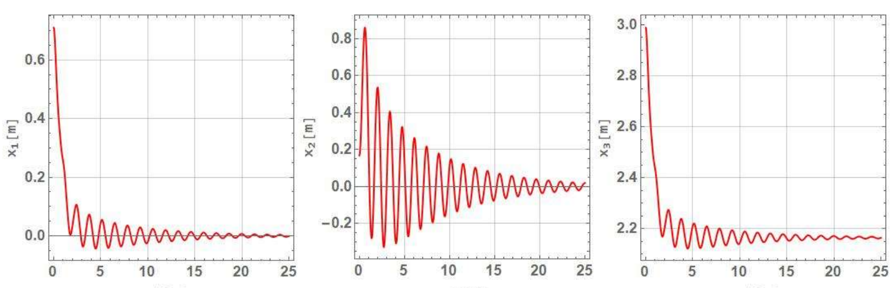


Fig. 2. Graph of generalized coordinates over time



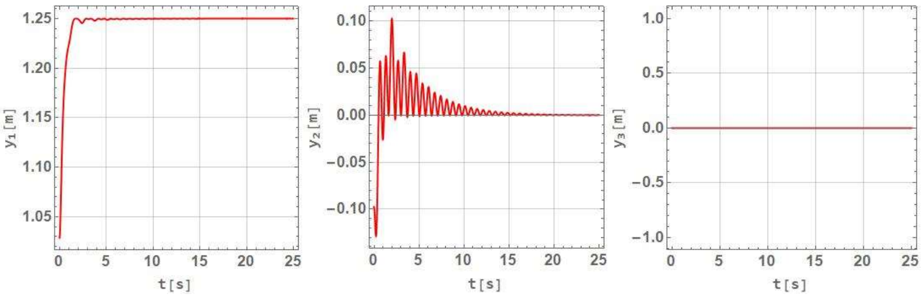


Fig. 3. Graph of coordinates over time

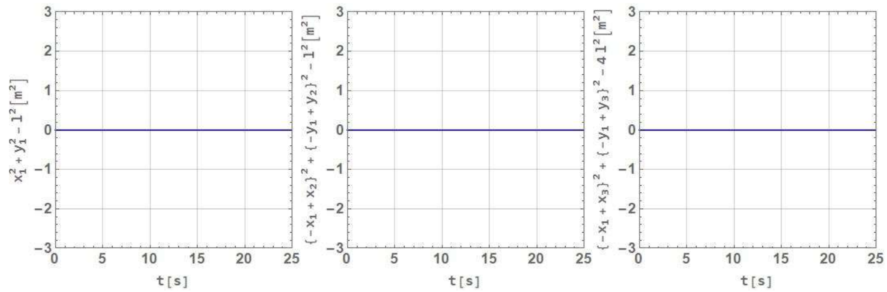


Fig. 4. Checking the constraints from (2) – (5)

3. Generalized momenta and Hamiltonian mechanics

Lagrange's variables $t; q^\alpha;\ \dot{q}_\alpha\ (\alpha = 1, \ldots, n)$ can easily be replaced by Canonical (Hamilton's) variables: $t; q^\alpha;\ p_\alpha\ (\alpha = 1, \ldots, n)$ which, also, completely characterize the state of the system. By introducing Hamilton's variables, Lagrange equations of the second kind, which make system of *n* second order differential equations for determining *n* functions $q^\alpha = q^\alpha(t)$, can be replaced by equivalent system of 2*n* first order differential equations for determining 2*n* functions of $q^\alpha = q^\alpha(t)\,, p_\alpha = p_\alpha(t)$. Hamiltonian function *H* is a function of generalized coordinates, generalized momenta and time i.e. $H = H(q^1, \ldots q^n; p_1, \ldots, p_n; t)$. Hence, differential equation of motion, under the action of nonconservative forces have the shape:

$$\dot{q}^\alpha = \frac{\partial H}{\partial p^\alpha}, \dot{p}_\alpha = -\frac{\partial H}{\partial p^\alpha} + \tilde{Q}_\alpha, \alpha = 1, \ldots, n \tag{14}$$

where $p_\alpha = \frac{\partial T}{\partial \dot{q}^\alpha},\quad \alpha = 1, \ldots, n$ are generalized momenta and $\tilde{Q}_\alpha$ represents nonpotential generalized forces. For the scleronomic system the Hamiltonian function *H*, has the form as given in (15) (because kinetic energy does not explicit depend on time):

$$H = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta + \Pi, \tag{15}$$

with $p_\varphi = \frac{\partial T}{\partial \dot{\varphi}}$ and $p_\theta = \frac{\partial T}{\partial \dot{\theta}}$, generalized momenta. For the proposed system, Hamiltonian is given as (16). All numerical research and calculations were done in the Wolfram Mathematica program. Change of the generalized coordinates and positions of material points are same as in Figures 2 – 4, because the same solutions are obtained by applying two different methods: Hamiltonian and Lagrange equations of the second kind in the covariant form. Generalized momenta are presented in the Fig. 5.

$$H = 826.82 - 147.10\,cos\theta - 303.24\,sin\varphi - 253.91\,sin(\theta + \varphi) + (p_\varphi^2\,(-1.28 + 0.32\,sin^2\varphi) + p_\theta^2\,(-2.43 + 0.03\,cos2\varphi + 0.05\,cos4\varphi - 0.43\,cos\varphi\,sin^2\varphi\,\sqrt{4 - sin^2\varphi}) + p_\varphi\,p_\theta\,(2.56 - 0.64\,sin^2\varphi)\,sin(\theta + \varphi))/(-65 + cos\,4\varphi - 24\,cos\,2(\theta + \varphi) + sin^2\varphi\,(-10 + 8\,sin^2\varphi) - 16\,cos\varphi\,\sqrt{4 - sin^2\varphi} - 12\,sin^2(\varphi + \theta) \qquad (16)$$

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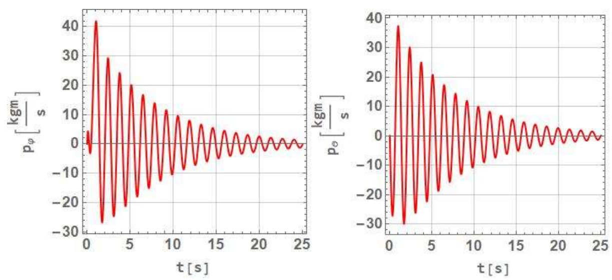

Fig. 5. Graph of generalized momenta over time

3. Stability of motion

The stability of the non – disturbed system was analysed in two different ways. First one is considering the conservative mechanical system, which is under the influence only of the conservative force:

$$0 = -\frac{\partial \Pi}{\partial q^\alpha} + \tilde{Q}_\alpha, \qquad \alpha = 1, \dots, n \qquad (17)$$

and because of the absence of non – conservative forces (17) becomes:

$$0 = -\frac{\partial \Pi}{\partial q^\alpha}, \qquad \alpha = 1, \dots, n \qquad (18)$$

Second method is uses Lagrange-Dirichlet theorem:

$$\Pi \approx \frac{1}{2} c_{\alpha\beta} q^\alpha q^\beta, \text{with: } c_{\alpha\beta} = \left(\frac{\partial^2 \Pi}{\partial q^\alpha\,\partial q^\beta}\right)_0, \alpha, \beta = 1, \dots, n. \qquad (19)$$

where $(\,)_0$ stands for the equilibrium position. The potential energy around the equilibrium position corresponds to the homogeneous quadratic form with constant coefficients $c_{\alpha\beta}$. For the force of constant intensity 100N acts on the material point M_2, the change of the potential energy and static equilibriums points were calculated. The both of the obtained constrains are same, so the both of the equilibrium points sets give the same angle positions. The results are also given in the following Table 2, numerically, and in the Figure 6 graphically (second case). Figure also shows the change of the potential energy with change of angles (first case). From both of the cases, there are four equilibrium positions (see Table 2).

Angle	Case 1	Case 2	Case 3	Case 4
φ_{st}	0.68803	1.45058	- 1.35581	- 2.44693
θ_{st}	- 2.91483	- 0.23257	- 2.52980	1.52262

Table 2. Static equilibriums

The resulting static equilibrium conditions are in form of the two equations:

$$-303.234\,cos\varphi - 253.906\,cos(\theta + \varphi) + 125\ sin\varphi = 0$$
$$125\,cos\theta - 253.906\,cos(\theta + \varphi) + 147.1\ sin\theta = 0 \qquad (20)$$

The equilibrium position is stable, only when the quadratic form (19) is positive definite, which can be confirmed using Silvester's criterion. According to Sylvester's criterion, for the Hermit matrix to be positively definite, it is necessary and sufficient for all its major minors to be positive [9].

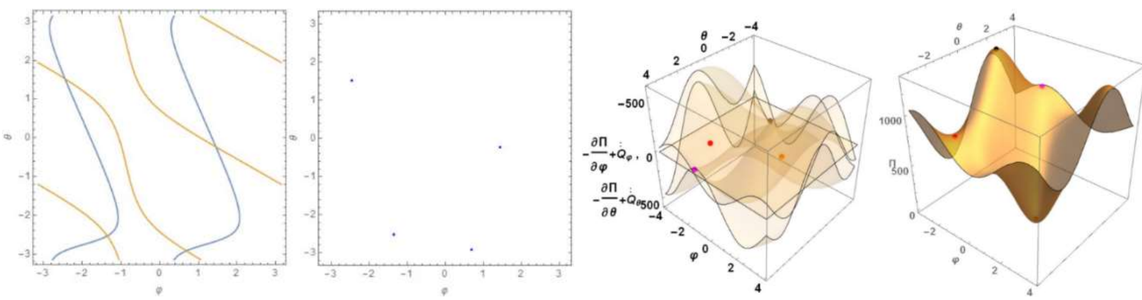

Fig. 6. Static equilibrium, positions of equilibrium points in space and Change of potential energy with change of the generalized coordinates

Lagrange Dirichlet theorem is not the only way for determining stability. In control theory one of the most popular is Hurwitz criterion, with analysis of the characteristic polynomial, which are given in Table 4.

Case	Case 1		Case 2		Case 3		Case 4	
$\boldsymbol{C} = \boldsymbol{c}_{\alpha\beta}$	-97.63	171.96	554.30	238.27	-97.63	171.96	-492.81	-202.66
	171.96	123.33	238.27	410.22	171.96	123.33	-202.67	-320.45
Postiion stability	unstable		stable		unstable		unstable	

Table 3. Static equilibriums

Case	**Characteristic polynomial**	**Stability**
Case 1	$54.3907 - 36.6394\,\lambda + 0.13725\lambda^2 - 5.5703\lambda^3 - \lambda^4$	unstable
Case 2	$-284.85 - 145.864\,\lambda - 36.54\lambda^2 - 6.6671\lambda^3 - \lambda^4$	stable
Case 3	$54.3907 - 36.6394\,\lambda + 0.13725\lambda^2 - 5.5703\lambda^3 - \lambda^4$	unstable
Case 4	$-257.085 + 19.3073\,\lambda + 59.192\lambda^2 - 1.12973\lambda^3 - \lambda^4$	unstable

Table 4. Hurwitz criteria and polynomial

A linear system has only one isolated equilibrium point and if it is stable all of the states will converge to this equilibrium. However, equations which describes this system are nonlinear and there is no sense in checking *system's* stability, only the *stability of the equilibriums*. For example system which has two equilibrium points is being considered. This system can initially be in equilibrium number one, which is unstable. When the disturbance is applied, it can happen that the second equilibrium attracts trajectory and the state can converge in the second equilibrium [10]. As it can be seen from Table 4 Case 1 and Case 3 have the same characteristic polynomial, so their graphs are the same (Figure 7).

The disturbed equations of motion are:

$$\bar{q}^{\alpha} = \bar{q}^{\alpha}(t; t_0, \bar{\boldsymbol{q}}_0, \dot{\bar{\boldsymbol{q}}}_0), \dot{\bar{q}}^{\alpha} = \dot{\bar{q}}^{\alpha}(t; t_0, \bar{\boldsymbol{q}}_0, \dot{\bar{\boldsymbol{q}}}_0), \quad \alpha = 1, \dots, n, \tag{21}$$

with: $\bar{\boldsymbol{q}}_0 = (\bar{q}_{01}, \bar{q}_{02}, \dots, \bar{q}_{0n})$, $\dot{\bar{\boldsymbol{q}}}_0 = (\dot{\bar{q}}_{01}, \dot{\bar{q}}_{02}, \dots, \dot{\bar{q}}_{0n})$, and $\bar{q}^{\alpha}(t) - q^{\alpha}(t) = \xi^{\alpha}(t)$, $\dot{\bar{q}}^{\alpha}(t) - \dot{q}^{\alpha} = \eta^{\alpha}(t)$ are disturbances.

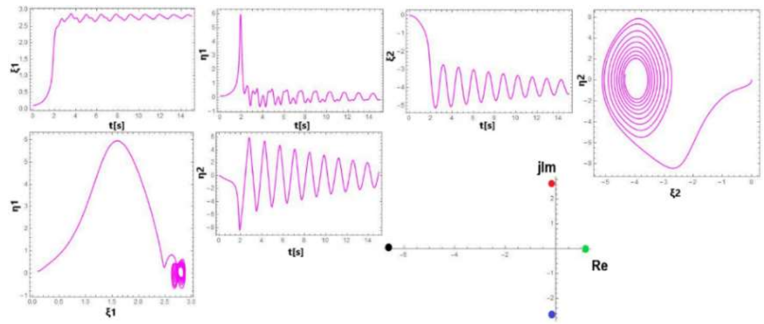

Fig. 7. Case 1 and 3: Disturbed motion of unstable equilibrium point and plane with polynomial solutions

Disturbances are taken to be: $\xi_1 = 0.1, \xi_2 = 0, \eta_1 = 0.1, \ \eta_2 = 0.$ There is root in right half of the plane so this two equilibriums are not stable. After linearization equations of disturbed motion in the vicinity of the positions of static equilibrium points for all cases were calculated. Linearized equations were calculated and they differ greatly from the nonlinearized (Figure 8).

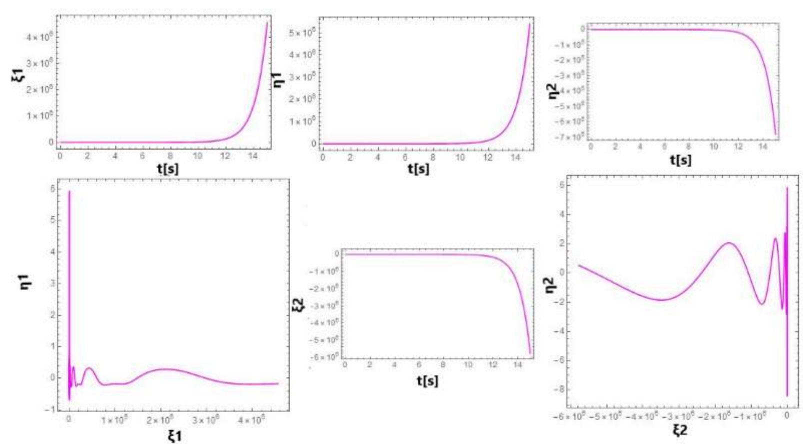

Fig. 8. Case 1 and 3: Disturbed motion of unstable equilibrium point with linearized equations

Many of the potential problems with modal testing only become apparent during the actual test. Frequently, it is not possible to predict such problems beforehand either because there is no analytical model or because the model of the structure is unrepresentative [11]. For holonomic system from this task solutions were relatively easily obtained. An example of the non-holonomic system was investigated in [12] and [13]. Many of the authors treated special classes of the systems, most frequently, linear systems in companion canonical form [14]. For stable equilibrium points norms of disturbed motion strives to zero. In both stability check give the same results. Only in case 2 norm of disturbed motion strives to zero – because it is only stable equilibrium point (Figures 9 - 10).

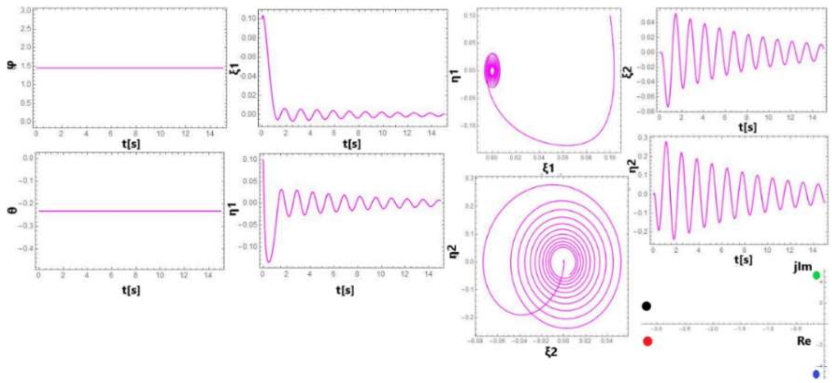

Fig. 9. Case 2: Disturbed motion of stable equilibrium point and plane with polynomial solutions

For the second, stable equilibrium point, case this motion can be seen in Figures 9 which represents both, disturbed motion in case of the nonlinear and linearized equations. The case 4 form Table 4 is analysed in Figures 10 and 11. Same conclusions are drawn – equilibrium is unstable and there are roots in right half of $Re - j\ Im$ plane. Linearized figures are similar as the nonlinear one in Case 2. Investigation of a linear time-invariant discrete-time plant were done.

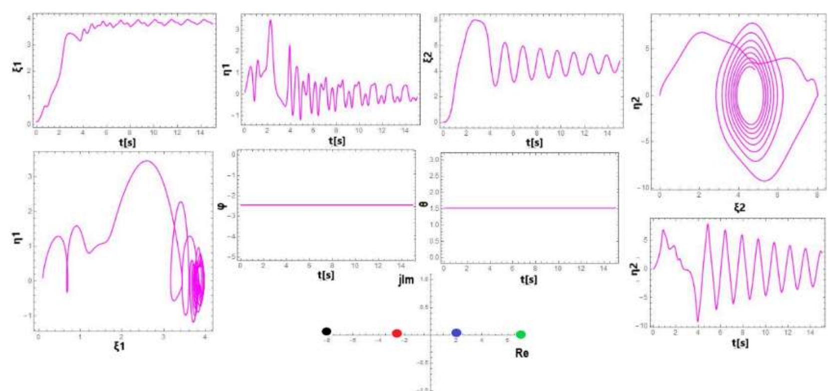

Fig. 10. Case 4: Disturbed motion of unstable equilibrium point and plane with polynomial solutions

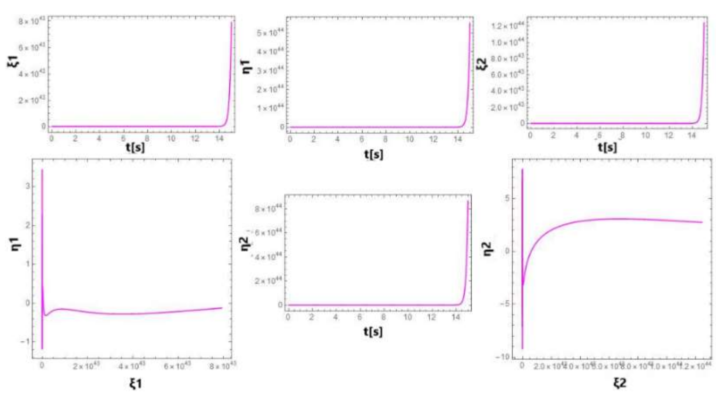

Fig. 11. Case 43: Disturbed motion of unstable equilibrium point with linearized equations

3. Conclusions

In this article, using a model of holonomic mechanical system, the motion at specified initial condition is analyzed with different approaches: using Lagrange equations of the second kind and

Hamiltonian mechanics. The change of system's coordinates, generalized coordinates, as well as the Hamiltonians momenta are obtained, so the system's motion has been confirmed. The stable and unstable equilibrium points are determined and found to be at the minimum of the potential energy. After concluding that only one equilibrium position is stable, a check was made by introducing disturbances into the system. Disturbed motion all of the equilibrium points with nonlinear and linearized equations is presented. From the real – imaginary $Re - j\ Im$ plane with roots of the characteristic polynomial the same judgements as from the graphs of system's motion are made.

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