Springer

Dear Author,

Here are the proofs of your article.

- You can submit your corrections **online**, via **e-mail** or by **fax**.
- For **online** submission please insert your corrections in the online correction form. Always indicate the line number to which the correction refers.
- You can also insert your corrections in the proof PDF and **email** the annotated PDF.
- For fax submission, please ensure that your corrections are clearly legible. Use a fine black pen and write the correction in the margin, not too close to the edge of the page.
- Remember to note the **journal title**, **article number**, and **your name** when sending your response via e-mail or fax.
- **Check** the metadata sheet to make sure that the header information, especially author names and the corresponding affiliations are correctly shown.
- **Check** the questions that may have arisen during copy editing and insert your answers/ corrections.
- **Check** that the text is complete and that all figures, tables and their legends are included. Also check the accuracy of special characters, equations, and electronic supplementary material if applicable. If necessary refer to the *Edited manuscript*.
- The publication of inaccurate data such as dosages and units can have serious consequences. Please take particular care that all such details are correct.
- Please **do not** make changes that involve only matters of style. We have generally introduced forms that follow the journal's style.
Substantial changes in content, e.g., new results, corrected values, title and authorship are not allowed without the approval of the responsible editor. In such a case, please contact the Editorial Office and return his/her consent together with the proof.
- If we do not receive your corrections **within 48 hours**, we will send you a reminder.
- Your article will be published **Online First** approximately one week after receipt of your corrected proofs. This is the **official first publication** citable with the DOI. **Further changes are, therefore, not possible.**
- The **printed version** will follow in a forthcoming issue.

Please note

After online publication, subscribers (personal/institutional) to this journal will have access to the complete article via the DOI using the URL: http://dx.doi.org/[DOI].
If you would like to know when your article has been published online, take advantage of our free alert service. For registration and further information go to: http://www.link.springer.com.

Due to the electronic nature of the procedure, the manuscript and the original figures will only be returned to you on special request. When you return your corrections, please inform us if you would like to have these documents returned.

Metadata of the article that will be visualized in OnlineFirst

Please note: Images will appear in color online but will be printed in black and white.

ArticleTitle	The brachistochronic motion of a wheeled vehicle	
Article Sub-Title		
Article CopyRight	Springer Science+Business Media Dordrecht (This will be the copyright line in the final PDF)	
Journal Name	Nonlinear Dynamics	
Corresponding Author	Family Name	**Šalinić**
	Particle	
	Given Name	**Slaviša**
	Suffix	
	Division	Faculty of Mechanical and Civil Engineering in Kraljevo
	Organization	University of Kragujevac
	Address	Dositejeva 19, 36000, Kraljevo, Serbia
	Phone	+381 36 383269
	Fax	
	Email	salinic.s@ptt.rs
	URL	
	ORCID	
Author	Family Name	**Radulović**
	Particle	
	Given Name	**Radoslav**
	Suffix	
	Division	Faculty of Mechanical Engineering
	Organization	University of Belgrade
	Address	Kraljice Marije 16, 11120, Belgrade 35, Serbia
	Phone	
	Fax	
	Email	
	URL	
	ORCID	
Author	Family Name	**Obradović**
	Particle	
	Given Name	**Aleksandar**
	Suffix	
	Division	Faculty of Mechanical Engineering
	Organization	University of Belgrade
	Address	Kraljice Marije 16, 11120, Belgrade 35, Serbia
	Phone	
	Fax	
	Email	

	URL	
	ORCID	
Author	Family Name	**Mitrović**
	Particle	
	Given Name	**Zoran**
	Suffix	
	Division	Faculty of Mechanical Engineering
	Organization	University of Belgrade
	Address	Kraljice Marije 16, 11120, Belgrade 35, Serbia
	Phone	
	Fax	
	Email	
	URL	
	ORCID	

Schedule	Received	12 June 2015
	Revised	
	Accepted	17 August 2016

Abstract	The paper considers the brachistochronic motion of a wheeled vehicle on a horizontal plane surface. The objective is to transfer the vehicle from the specified initial position with given initial kinetic energy to the specified terminal position in minimum time with conserved total mechanical energy of the vehicle. The problem is solved by applying Pontryagin's maximum principle and singular optimal control theory. The projection of the reaction force of the horizontal plane applied on the front vehicle wheels onto the axis of the front vehicle axle is taken for a control variable. The cases of unbounded and bounded value of this projection are considered. The shooting method is used to solve the two-point boundary value problem arising from Pontryagin's maximum principle and singular optimal control theory.

Keywords (separated by '-')	Brachistochronic motion - Nonholonomic system - Wheeled vehicle - Optimal control
Footnote Information	

Nonlinear Dyn
DOI 10.1007/s11071-016-3035-3

CrossMark

Author Proof

ORIGINAL PAPER

The brachistochronic motion of a wheeled vehicle

Radoslav Radulović · Aleksandar Obradović · Slaviša Šalinić · Zoran Mitrović

Received: 12 June 2015 / Accepted: 17 August 2016
© Springer Science+Business Media Dordrecht 2016

Abstract The paper considers the brachistochronic motion of a wheeled vehicle on a horizontal plane surface. The objective is to transfer the vehicle from the specified initial position with given initial kinetic energy to the specified terminal position in minimum time with conserved total mechanical energy of the vehicle. The problem is solved by applying Pontryagin's maximum principle and singular optimal control theory. The projection of the reaction force of the horizontal plane applied on the front vehicle wheels onto the axis of the front vehicle axle is taken for a control variable. The cases of unbounded and bounded value of this projection are considered. The shooting method is used to solve the two-point boundary value problem arising from Pontryagin's maximum principle and singular optimal control theory.

Keywords Brachistochronic motion · Nonholonomic system · Wheeled vehicle · Optimal control

R. Radulović, A. Obradović, Z. Mitrović
Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, Belgrade 35 11120, Serbia

S. Šalinić (✉)
Faculty of Mechanical and Civil Engineering in Kraljevo, University of Kragujevac, Dositejeva 19, Kraljevo 36000, Serbia
e-mail: salinic.s@ptt.rs

1 Introduction

The subject of this research paper is a wheeled vehicle shown in Fig. 1. The motion of the vehicle is observed with respect to the fixed reference frame $O\xi\eta\zeta$ whose coordinate plane $O\xi\eta$ coincides with the horizontal plane of the vehicle motion. The moving coordinate frame $Axyz$ is rigidly attached to the vehicle body, so that the coordinate plane Axy coincides with the plane $O\xi\eta$ where point A represents the mass center of the front vehicle axle.

The unit vectors of the axes x, y, and z are $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. The axis Ax passes through the mass center C of the vehicle body, and it is normal to the rear vehicle axle. Masses of the vehicle body and the front axle are M_1 and M_2, respectively. It is assumed that masses of the wheels and the rear axle are negligible. Let J_1 and J_2, respectively, be the moments of inertia of the vehicle body and the front axle about its central axes of inertia normal to the plane Axy, where $J_1 \gg J_2$. The vehicle configuration relative to the frame $O\xi\eta\zeta$ is defined by a set of Lagrangian coordinates (q_1, q_2, q_3, q_4), where $q_1 = \xi_B$ and $q_2 = \eta_B$ are the Cartesian coordinates of the vehicle point B, $q_3 = \varphi$ is the angle between the axes $O\xi$ and Ax, while $q_4 = \theta$ represents the angle between the axis Ay and the axis of the front axle. The point B coincides with the middle of the rear axle. Further analysis refers to the case when point A is prevented from moving in the direction of the front axle, whereas point B is prevented from moving in the direction of the rear axle.

uncorrected proof

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

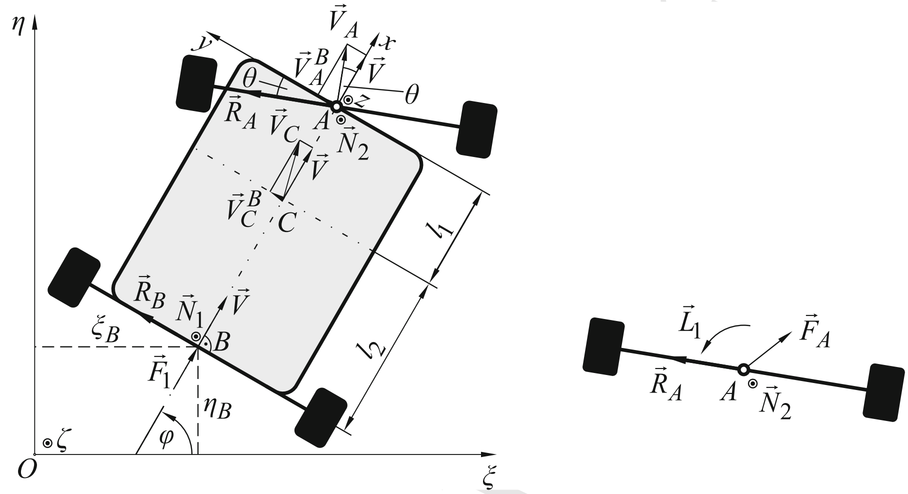


Fig. 1 A simplified model of the vehicle and its front axle

This means there is no side slipping of the front and rear wheels during the vehicle motion. Due to thus imposed restrictions on the vehicle motion, which can be written in the form of two ideal independent nonholonomic homogeneous constraints [1] :

$$-\dot{\xi}_B \sin\varphi + \dot{\eta}_B \cos\varphi = 0, \tag{1}$$

$$-\dot{\xi}_A \sin(\varphi+\theta) + \dot{\eta}_A \cos(\varphi+\theta) = 0, \tag{2}$$

horizontal reactions of the horizontal plane $O\xi\eta$ occur at the contact points between the wheels and the plane. The net reaction forces acting on the front and rear axles read, respectively:

$$\overrightarrow{R_A} = -R_A \sin\theta\, \overrightarrow{i} + R_A \cos\theta\, \overrightarrow{j}\,, \quad \overrightarrow{R_B} = R_B\, \overrightarrow{j}\,. \tag{3}$$

Note that as a consequence of the constraints imposed on the motion is that the velocity $\overrightarrow{V}$ of point B has the direction of the axis Ax, so the relations (1) and (2) can be also represented in the form:

$$\dot{\xi}_B = V\cos\varphi, \quad \dot{\eta}_B = V\sin\varphi, \quad \dot{\varphi} = \frac{V}{l}\tan\theta, \tag{4}$$

where $V = \overrightarrow{V}\cdot\overrightarrow{i}$ and $l = l_1 + l_2$ is the distance between the rear and front axles. As in [2] it is taken that, during the motion, the vehicle is acted on by a driving force $\overrightarrow{F}_1 = F_1(t)\,\overrightarrow{i}$ at point B of the rear axle, as well as by an internal turning torque $\overrightarrow{L}_1 = L_1(t)\,\overrightarrow{k}$ acted about the vertical axis Az. Further, differential equations of motion of the considered vehicle will be generated using general theorems of dynamics [3]:

$$\frac{\mathrm{d}\overrightarrow{K}}{\mathrm{d}t} = \overrightarrow{F}^s_R, \qquad \frac{\mathrm{d}\overrightarrow{L}_B}{\mathrm{d}t} + \overrightarrow{V}\times\overrightarrow{K} = \overrightarrow{M}^s_B, \tag{5}$$

where $\overrightarrow{K}$ is the linear momentum of the vehicle, $\overrightarrow{L}_B$ is the angular momentum about point B of the vehicle, $\overrightarrow{F}^s_R$ is the total external force, and $\overrightarrow{M}^s_B$ is the total moment of the external forces about point B. Note that in [2] the Hamel–Boltzmann and Maggi equations are used, while in [4] the Appell equations are chosen.

Projecting Eq. (5) on the axes of coordinate frame $Axyz$ yields:

$$M\left[\dot{V} - \left(l_2 + \frac{M_2}{M}l_1\right)\omega^2\right] = F_1 - R_A\sin\theta, \tag{6}$$

$$M\left[\omega V + \left(l_2 + \frac{M_2}{M}l_1\right)\dot{\omega}\right] = R_A\cos\theta + R_B, \tag{7}$$

$$0 = N_1 + N_2 - Mg, \tag{8}$$

$$J^*\dot{\omega} + J_2\ddot{\theta} + M\left(l_2 + \frac{M_2}{M}l_1\right)\omega V = R_A l\cos\theta, \tag{9}$$

$$0 = M_1 g l_2 + M_2 g l - N_2 l, \tag{10}$$

where $\omega = \dot{\varphi}$ is the vehicle body angular velocity, $M = M_1 + M_2$, $J = J_1 + J_2$, $J^* = M_1 l_2^2 + M_2 l^2 + J$ is the moment of inertia of the vehicle about the axis $B\zeta$, N_1 and N_2, respectively, are normal reactions of the horizontal plane on the rear and front axles, g is the gravity acceleration, and $\cos\theta = V/\sqrt{V^2 + l^2\omega^2}$ (see Fig. 1). Further, the differential equation of the front axle rotation about the axis Az reads:

uncorrected

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

Uncorrected Proof

$$J_2\left(\dot{\omega}+\ddot{\theta}\right)=L_1. \tag{11}$$

Now, based on Eqs. (6)–(11) it is possible to determine the reactions of nonholonomic constraints, as well as the driving force and the turning torque required to realize motion as follows:

$$R_A\left(t\right)=\frac{1}{l}\frac{\sqrt{V^2+l^2\omega^2}}{V}\left[J^*\dot{\omega}\right.$$
$$\left.+\left(Ml_2+M_2l_1\right)\omega V+J_2\ddot{\theta}\right], \tag{12}$$

$$R_B\left(t\right)=\frac{1}{l}\left[M_1l_1\omega V+\left(M_1l_1l_2-J\right)\dot{\omega}-J_2\ddot{\theta}\right], \tag{13}$$

$$F_1\left(t\right)=M\dot{V}+\frac{\omega}{V}\left(J^*\dot{\omega}+J_2\ddot{\theta}\right), \tag{14}$$

$$L_1\left(t\right)=J_2\left(\dot{\omega}+\ddot{\theta}\right), \tag{15}$$

as well as the reactions N_1 and N_2:

$$N_1=\frac{M_1l_1g}{l}, \tag{16}$$

$$N_2=\frac{M_1gl_2}{l}+M_2g. \tag{17}$$

From above equations it is obvious that the reactions N_1 and N_2 are constant during the vehicle motion.

The kinetic energy of the vehicle reads:

$$T=\frac{1}{2}\left(MV^2+J^*\omega^2+2J_2\omega\dot{\theta}+J_2\dot{\theta}^2\right). \tag{18}$$

Since the power of the active control forces equals zero, the law of conservation of total mechanical energy of the vehicle holds:

$$\frac{\mathrm{d}T}{\mathrm{d}t}=F_1V+L_1\dot{\theta}=0, \tag{19}$$

that is:

$$\Phi(V,\,\omega)\equiv MV^2+J^*\omega^2+2J_2\omega\dot{\theta}+J_2\dot{\theta}^2-2T_0=0, \tag{20}$$

where T_0 is the kinetic energy of the vehicle at the initial instant $t_0=0$.

Note that differential equations of the vehicle motion (6), (7), (9), and (11), as well as the reactions of nonholonomic constraints and the driving force and torque (12)–(15) are obtained in accordance with the constraints (1) and (2). Taking this into account and the Coulomb friction laws, necessary dynamic conditions for the realization of motion in accordance with the constraints (1) and (2) are that the magnitudes of interaction forces between the front and rear vehicle wheels and the horizontal plane of vehicle motion do not exceed the corresponding limit values of the Coulomb dry friction forces [2]. In accordance with aforesaid, the following inequalities [2]:

$$\mu_2(t)N_2=|R_A|<F_2^{\mathrm{fr}}=\mu N_2, \tag{21}$$

$$\mu_1(t)N_1=F_B=\sqrt{R_B^2+F_1^2}<F_1^{\mathrm{fr}}=\mu N_1, \tag{22}$$

must hold where F_1^{fr} and F_2^{fr} are the dry friction forces between the rear and front wheels and the horizontal plane, respectively; μ_1 and μ_2, respectively, are laws of change of minimum required value of the coefficients of sliding friction between the rear and front wheels and the plane of motion; μ is the coefficient of dry friction between the front and rear wheels and the plane of motion. Let us emphasize that in the conditions (22) it has been taken into account that the interaction force between the wheels on the rear axle and the horizontal plane is determined by the vector sum of the driving force $\overrightarrow{F}_1$ and the reaction of nonholonomic constraint $\overrightarrow{R}_B$ (see [2]). The corresponding dynamic conditions for the realization of motion in case of the absence of the driving forces $\overrightarrow{F}_1$ are derived in [4].

The objective of this paper is to analyze the brachistochronic motion of the described vehicle. In that regard, the paper is based on the ideas and approaches presented in references [5–7] . To the best of the authors' knowledge the results concerning the brachistochronic motion of this vehicle type have not been reported elsewhere before. For the problem of controlling the motion of this type of vehicle along a given trajectory see [8,9]. In the next section, the formulation of the brachistochrone problem for the considered vehicle is given.

2 Optimal control problem formulation

In order to define state equations that describe the motion of the considered system in the state space, it is necessary first to express $\dot{\omega}$ and $\dot{V}$ as a function of defined state quantities and their time derivatives. The relation $J_2<<J_1$ will be further employed to obtain a simpler form of the state equations. Namely, by omitting the third term in Eq. (12) and solving for $\dot{\omega}$, it is obtained:

$$\dot{\omega}=\frac{l}{J^*}\frac{V}{\sqrt{V^2+l^2\omega^2}}R_A-\frac{\omega}{J^*}\left(l_2M+l_1M_2\right)V. \tag{23}$$

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Further, omitting the terms $2J_2\omega\dot{\theta}$ and $J_2\dot{\theta}^2$ in Eq. (18) and performing the time derivative of such simplified expression of the kinetic energy yields:

$$\dot{V} = -\frac{l\omega}{M\sqrt{V^2+l^2\omega^2}} R_A + \frac{\omega^2}{M}\,(l_2 M + l_1 M_2)\,, \quad (24)$$

where Eq. (23) is taken into account. Now, in accordance with Eqs. (4), (23), and (24), the requested state equations read:

$$\dot{\xi}_B = V\cos\varphi,\;\; \dot{\eta}_B = V\sin\varphi,\;\; \dot{\varphi} = \omega, \quad (25)$$

$$\dot{\omega} = \frac{l}{J^*}\frac{V}{\sqrt{V^2+l^2\omega^2}}u - \frac{\omega}{J^*}\,(l_2 M + l_1 M_2)\,V, \quad (26)$$

$$\dot{V} = -\frac{l\omega}{M\sqrt{V^2+l^2\omega^2}}u + \frac{\omega^2}{M}\,(l_2 M + l_1 M_2)\,, \quad (27)$$

where the control variable u is defined as $u = R_A$. Note that the realization of the vehicle brachistochronic motion is achieved by both the active control force $\vec{F}_1$ and the active control torque $\vec{L}_1$.

Let the values of the state variables ξ_B, η_B, and φ and the initial kinetic energy T_0 of the vehicle be specified at the beginning of motion on the manifold:

$$t_0 = 0,\;\; \xi_B(t_0) = 0,\;\; \eta_B(t_0) = 0,\;\; \varphi(t_0) = 0, \quad (28)$$

$$MV^2(t_0) + J^*\omega^2(t_0) - 2T_0 = 0, \quad (29)$$

as well as the values of the state variables ξ_B, η_B, and φ at the terminal position of the vehicle on the manifold:

$$t = t_f,\;\; \xi_B(t_f) = a,\;\; \eta_B(t_f) = b,\;\; \varphi(t_f) = \varphi_f, \quad (30)$$

where t_f is free and represents the instant corresponding to the terminal state of the vehicle.

For the vehicle, whose motion is described by the differential equations (25)–(27), the problem of brachistochronic motion consists of determining the optimal control u and the state variables ξ_B, η_B, and φ, so that the vehicle moves from the initial state on the manifold (28), (29) to the terminal state on the manifold (30), with conserved total mechanical energy (20), in minimum time t_f. This can be expressed in terms of the condition that the functional

$$I = \int_{t_0}^{t_f} \mathrm{d}t, \quad (31)$$

on the interval $[t_0,\;\; t_f]$ has a minimum value.

Here, it should be pointed out that the brachistochrone problem and the minimum time optimal control problems (see e.g., [10–13]) are very similar. The difference between these two types of optimal control problems is that in the minimum time optimal control problems the request for the conservation of total mechanical energy of the controlled mechanical system is not imposed on control forces.

3 Optimal control in the case of unbounded reaction forces

In order to solve the posed problem by Pontryagin's maximum principle [14,15], the Hamiltonian (Pontryagin's function) is formed:

$$\begin{aligned} H &= \lambda_0 + \lambda_\xi V\cos\varphi + \lambda_\eta V\sin\varphi + \lambda_\varphi\omega \\ &\quad + \lambda_\omega\left[\frac{l}{J^*}\frac{V}{\sqrt{V^2+l^2\omega^2}}u - \frac{\omega}{J^*}\,(l_2M + l_1M_2)\,V\right] \\ &\quad + \lambda_V\left[-\frac{l\omega}{M\sqrt{V^2+l^2\omega^2}}u + \frac{\omega^2}{M}\,(l_2M + l_1M_2)\right], \end{aligned} \quad (32)$$

where $\lambda_0 = \text{const.} \le 0$, λ_ξ, λ_η, λ_φ, λ_ω, and λ_V are costates and where it can be taken that $\lambda_0 = -1$ (see [14]). For the needs of further considerations, a switching function H_1 is defined as follows:

$$H_1 = \frac{\partial H}{\partial u} = \frac{l}{\sqrt{V^2+l^2\omega^2}}\left(\lambda_\omega\frac{V}{J^*} - \lambda_V\frac{\omega}{M}\right). \quad (33)$$

Now, based on Eqs. (32) and (33), the Hamiltonian H can be written in the form:

$$H = H_0 + H_1 u, \quad (34)$$

where

$$\begin{aligned} H_0 &= -1 + \lambda_\xi V\cos\varphi + \lambda_\eta V\sin\varphi + \lambda_\varphi\omega \\ &\quad - \lambda_\omega\,(l_2M + l_1M_2)\,V\frac{\omega}{J^*} \\ &\quad + \lambda_V\,(l_2M + l_1M_2)\,\frac{\omega^2}{M}. \end{aligned} \quad (35)$$

Such a case, when the Hamiltonian is linear in the control u, is known as the singular control case [15], where the optimal control u cannot be explicitly determined from the necessary optimality condition:

$$H_1 = 0. \quad (36)$$

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

uncorrected proof

Author Proof

If the control u belongs to an open set, as is the case in this section, condition (36) represents the only requirement for determining an optimal control. For the case of bounded constraint reaction, which is the subject of analysis in the next section, an optimal control represents a combination of singular and bang-bang controls.

Taking into account the boundary conditions (28)–(30) and the fact that time does not explicitly appear in the state equations (25)–(27), this problem of optimal control can be solved by directly applying Theorem 3 (see [14]) that also involves the application of Theorem 1 (see [14]).

Based on Eq. (32), the corresponding costate equations [14,15] read:

$$\dot{\lambda}_\xi = 0, \qquad \dot{\lambda}_\eta = 0, \tag{37}$$

$$\dot{\lambda}_\varphi = \left(\lambda_\xi \sin\varphi - \lambda_\eta \cos\varphi\right) V, \tag{38}$$

$$\begin{aligned}\dot{\lambda}_\omega = -\lambda_\varphi + \lambda_\omega &\left[(l_2 M + l_1 M_2)\, V + \frac{l^3 \omega V}{\left(V^2 + l^2\omega^2\right)^{\frac{3}{2}}} u\right] \frac{1}{J^*} \\ &+ \lambda_V \left[\frac{l V^2}{\left(V^2 + l^2\omega^2\right)^{\frac{3}{2}}} u - 2\omega\,(l_2 M + l_1 M_2)\right] \frac{1}{M},\end{aligned} \tag{39}$$

$$\begin{aligned}\dot{\lambda}_V = -\lambda_\xi \cos\varphi - \lambda_\eta \sin\varphi &+ \lambda_\omega \left[l_2 M + l_1 M_2 - \frac{l^3\omega}{\left(V^2 + l^2\omega^2\right)^{\frac{3}{2}}} u\right] \frac{\omega}{J^*} \\ &- \lambda_V \frac{l V}{\left(V^2 + l^2\omega^2\right)^{\frac{3}{2}}} u \frac{\omega}{M},\end{aligned} \tag{40}$$

from where it follows that $\lambda_\xi = const.$ and $\lambda_\eta = const.$ Further, the boundary conditions (28)–(30) imply the transversality conditions [14] at the initial and terminal positions, respectively, as follows:

$$\lambda_\omega (0)\, M V (0) - \lambda_V (0)\, J^* \omega (0) = 0, \tag{41}$$

$$\lambda_\omega (t_f) = 0, \quad \lambda_V (t_f) = 0. \tag{42}$$

Note that the transversality conditions (41) and (42) also satisfy the optimality condition (36). Since t_f is free, in solving the system of Eqs. (25)–(27) and (37)–(40) the boundary and transversality conditions (28), (29), (30), (41), and (42) should be adjoined by the condition that the value of the Hamiltonian equals zero at any instant (see Theorem 1 [14]):

$$H(t) = 0, \tag{43}$$

which, taking into account Eq. (42), leads to the following condition at the terminal instant t_f:

$$\begin{aligned}&-1 + \lambda_\xi V(t_f) \cos\varphi(t_f) + \lambda_\eta V(t_f) \sin\varphi(t_f) \\ &\quad + \lambda_\varphi(t_f)\omega(t_f) = 0.\end{aligned} \tag{44}$$

Now, the procedure of determining the optimal control u consists of further differentiating the switching function H_1 with respect to time in accordance with Eqs. (25)–(27) and (37)–(40), as long as the control u appears explicitly. For this purpose, the Poisson bracket formalism is employed [17] as follows:

$$\dot{H}_1 = \{H_1, H\} = \{H_1, H_0\} + \{H_1, H_1\} u = 0. \tag{45}$$

Based on Eq. (36) and the fact that in the case of a singular control along an optimal state-space trajectory the following relation holds (see [17]):

$$\{H_1, H_1\} = 0, \tag{46}$$

it is obtained that:

$$\{H_1, H_0\} = 0. \tag{47}$$

Further differentiating Eq. (45) with respect to time yields:

$$\ddot{H}_1 = \{\{H_1, H_0\}, H_0\} + \{\{H_1, H_0\}, H_1\} u = 0. \tag{48}$$

Now, it is possible to determine the first-order singular control in the form:

$$u = -\frac{\{\{H_1, H_0\}, H_0\}}{\{\{H_1, H_0\}, H_1\}}. \tag{49}$$

If

$$\{\{H_1, H_0\}, H_1\} = 0, \tag{50}$$

it is necessary to continue the differentiating procedure of the expression (48). The Kelley necessary optimality condition (also known as the generalized Legendre–Clebsch condition) for singular control of the first order [15,18] can be written by means of the Poisson brackets [17] as follows:

$$\{\{H_1, H_0\}, H_1\} > 0. \tag{51}$$

uncorrected proof

Journal: 11071 MS: 3035 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

Now, in accordance with Eqs. (33) and (36), it is obtained that:

$$\lambda_\omega = \frac{J^*\omega}{MV}\lambda_V. \tag{52}$$

Based on Eqs. (20), (35), (36), (43), (47), and (52) one has that

$$\lambda_\varphi = \frac{J^*\omega}{2T_0}, \quad V = \frac{2T_0}{M}\left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right), \tag{53}$$

from where, based on the initial state of the system (28) and (29), the initial velocity of point B can be determined as:

$$V_0 = \frac{2T_0}{M}\lambda_\xi. \tag{54}$$

Finally, based on Eqs. (33), (35), (49), (52), and (53), the singular control of the first order takes the form:

$$\begin{aligned} u = \frac{\sqrt{V^2 + l^2\omega^2}}{l}\, &\left[2T_0\left(\lambda_\xi \sin\varphi - \lambda_\eta \cos\varphi\right)\right. \\ &\left. + \omega\left(l_2 M + l_1 M_2\right)\right], \end{aligned} \tag{55}$$

while the Kelley optimality condition (51) becomes:

$$\{\{H_1,\ H_0\},\ H_1\} = \frac{l^2}{J^* M\left(V^2 + l^2\omega^2\right)} > 0. \tag{56}$$

It can be deduced from Eq. (56) that the Kelley optimality condition is satisfied for $\forall t \in [t_0,\ t_f]$. For the needs of further considerations, the relation (47) can be written, in accordance with Eqs. (52) and (53), in the form:

$$\begin{aligned} \{H_1,\ H_0\} = \frac{l\omega}{2T_0 M\sqrt{V^2 + l^2\omega^2}}\, &\left[2T_0\left(\lambda_\xi \cos\varphi\right.\right. \\ &\left.\left. + \lambda_\eta \sin\varphi\right) - MV\right] = 0. \end{aligned} \tag{57}$$

Substituting the expressions (53) and (55) into the state Eqs. (25)–(27) yields:

$$\begin{aligned} \dot{\xi}_B &= \frac{2T_0}{M}\left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right)\cos\varphi, \\ \dot{\eta}_B &= \frac{2T_0}{M}\left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right)\sin\varphi, \end{aligned} \tag{58}$$

$$\begin{aligned} \dot{\varphi} &= \frac{2T_0}{J^*}\lambda_\varphi, \quad \dot{\omega} = \frac{4T_0^2}{MJ^*}\left(\lambda_\xi \cos\varphi\right. \\ &\quad \left. + \lambda_\eta \sin\varphi\right)\left(\lambda_\xi \sin\varphi - \lambda_\eta \cos\varphi\right), \end{aligned} \tag{59}$$

$$\dot{V} = \frac{4T_0^2}{MJ^*}\left(-\lambda_\xi \sin\varphi + \lambda_\eta \cos\varphi\right)\lambda_\varphi, \tag{60}$$

whereas, based on Eqs. (52), (53), and (55), the costate equations (37)–(40) become:

$$\dot{\lambda}_\xi = 0, \quad \dot{\lambda}_\eta = 0, \tag{61}$$

$$\dot{\lambda}_\varphi = \frac{2T_0}{M}\left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right)\left(\lambda_\xi \sin\varphi - \lambda_\eta \cos\varphi\right), \tag{62}$$

$$\dot{\lambda}_\omega = -\lambda_\varphi + \frac{2T_0}{M}\left(\lambda_\xi \sin\varphi - \lambda_\eta \cos\varphi\right), \tag{63}$$

$$\begin{aligned} \dot{\lambda}_V &= \frac{2T_0(-\lambda_\xi \sin\varphi + \lambda_\eta \cos\varphi)}{J^*(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi)}\lambda_\varphi\lambda_V \\ &\quad - \left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right), \end{aligned} \tag{64}$$

and the first integral (43) obtains the following explicit form:

$$-1 + \frac{2T_0}{J^*}\lambda_\varphi^2 + \frac{2T_0}{M}\left(\lambda_\xi \cos\varphi + \lambda_\eta \sin\varphi\right)^2 = 0. \tag{65}$$

Also, from Eq. (65), for the initial instant t_0 one has

$$\lambda_\varphi^2(t_0) = \frac{J^*}{2T_0}\left(1 - \frac{2T_0}{M}\lambda_\xi^2\right), \tag{66}$$

from where one can give a global estimation of the value of the costate variable λ_ξ

$$-\sqrt{\frac{M}{2T_0}} \le \lambda_\xi \le \sqrt{\frac{M}{2T_0}}, \tag{67}$$

whereas the estimation of the value of the costate variable λ_η can be given based on the value of angle φ_f and Eqs. (20) and (53) in the form:

$$-\sqrt{\frac{M}{2T_0}}\cot\frac{\varphi_f}{2} \le \lambda_\eta \le \sqrt{\frac{M}{2T_0}}\cot\frac{\varphi_f}{2}, \quad \forall\varphi_f \ne 0. \tag{68}$$

Solving the two-point boundary value problem determined by Eqs. (28), (29), (30), (58)–(60), (61)–(64), and (66) is based on the shooting method [19]. A three-parameter shooting consists of determining unknown values of the costate variables λ_ξ and λ_η, as well as a minimum required time t_f. The procedure of determining unknown parameters by shooting method consists of "shooting" the coordinates of the terminal state (30), in accordance with Eqs. (58)–(60) and (61)–(64), for the known initial state (28) and (29) as well as for (66). The application of the shooting method requires

uncorrected proof

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

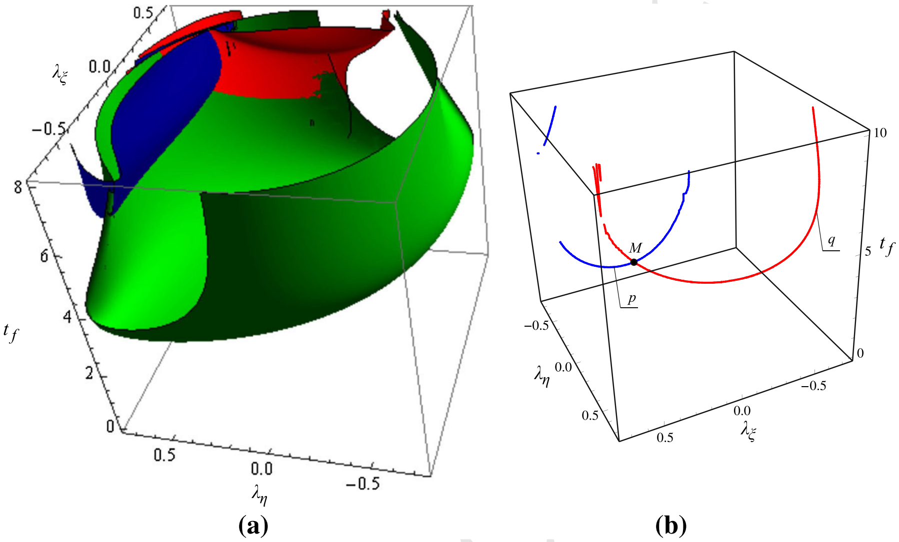


Fig. 2 **a** Crossing of the surfaces $\xi_B(t_\mathrm{f}) = f_\xi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f})$, $\eta_B(t_\mathrm{f}) = f_\eta(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f})$, and $\varphi(t_\mathrm{f}) = f_\varphi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f})$ for $\varphi_\mathrm{f} = \pi/2$, **b** crossing of the curves $p_\mathrm{f} = f_p(\lambda_\xi,\ t_\mathrm{f})$, $q_\mathrm{f} = f_q(\lambda_\xi,\ t_\mathrm{f})$ for $\varphi_\mathrm{f} = \pi/2$

the estimation of intervals containing the values of parameters to be determined. Based on estimates for the interval of values of the costate variables λ_ξ and λ_η, given by the inequalities (67) and (68), it can be stated that all solutions for the respective two-point boundary value problem are certainly found within the given intervals, thereby the global minimum time in the brachistochronic motion of the vehicle. For the case of multiple solutions of the maximum principle, global minimum is the solution that has minimum value of the time t_f.

In solving the two-point boundary value problem, the following relations can be established in a numerical form:

$$\xi_B(t_\mathrm{f}) = f_\xi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}),\quad \eta_B(t_\mathrm{f}) = f_\eta(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}),\quad \varphi(t_\mathrm{f}) = f_\varphi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}). \tag{69}$$

Each of the surfaces in Eq. (69) conforms to the fulfillment of one end condition on the manifold (30), respectively. The surfaces (69) can be graphically represented in the three-dimensional $\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}$-space of unknown parameters, where the solution to the two-point boundary value problem is found at the intersection of these surfaces.

The considered two-point boundary value problem is solved for the following values of the parameters:

$$T_0 = 1000\,\frac{\mathrm{kgm}^2}{\mathrm{s}^2},\quad \varphi(t_\mathrm{f}) = \frac{\pi}{2},\quad M_1 = 1000\,\mathrm{kg},\ M_2 = 110\,\mathrm{kg},\ J_1 = 1500\,\mathrm{kgm}^2,\quad J_2 = 30\,\mathrm{kgm}^2,\ l_1 = 0.75\,\mathrm{m},\ l_2 = 1.65\,\mathrm{m},\ a = 5\,\mathrm{m},\ b = 5\,\mathrm{m}. \tag{70}$$

Based on Eqs. (67), (68), and (70), the estimated values of the costate variables λ_ξ and λ_η read:

$$-0.74498 \le \lambda_\xi \le 0.74498,$$
$$-0.74498 \le \lambda_\eta \le 0.74498, \tag{71}$$

which is also graphically represented in Fig. 2. Finally, for the given values of the parameters, one has that $t_\mathrm{f} = 6.22571$ s, $\lambda_\xi = 0.51219$ s/m, and $\lambda_\eta = 0.51219$ s/m.

As above mentioned, global minimum of the time of the brachistochronic motion of the vehicle as well as the values of the costate variables λ_ξ and λ_η can be also determined based on the graphical representation of the solution of the system of nonlinear equations (69), as shown in Fig. 2a. It is evident from Fig. 2a that the solution to the considered two-point boundary value problem is unique, that is, the surfaces intersect at one point.

Note that the solution of the two-point boundary value problem considered may be determined in an another way. Namely, it is possible now to determine the intersections of the surfaces (69) as:

$$p_\mathrm{f} = f_\xi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}) \cap f_\varphi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}),\quad q_\mathrm{f} = f_\eta(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}) \cap f_\varphi(\lambda_\xi,\ \lambda_\eta,\ t_\mathrm{f}), \tag{72}$$

where p_f and q_f are the space curves represented by the following dependencies established in numerical form:

$$p_\mathrm{f} = f_p(\lambda_\xi,\ t_\mathrm{f}),\quad q_\mathrm{f} = f_q(\lambda_\xi,\ t_\mathrm{f}). \tag{73}$$

Journal: 11071 MS: 3035 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

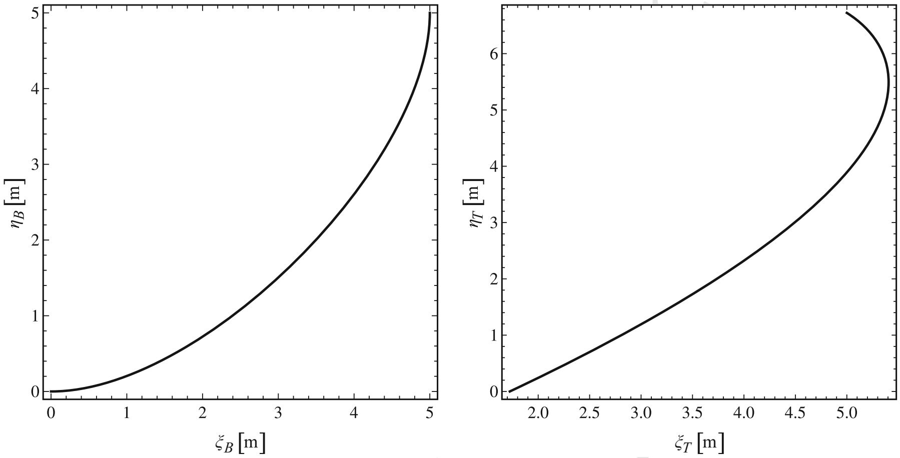


Fig. 3 Trajectories of the point B and the vehicle mass center T

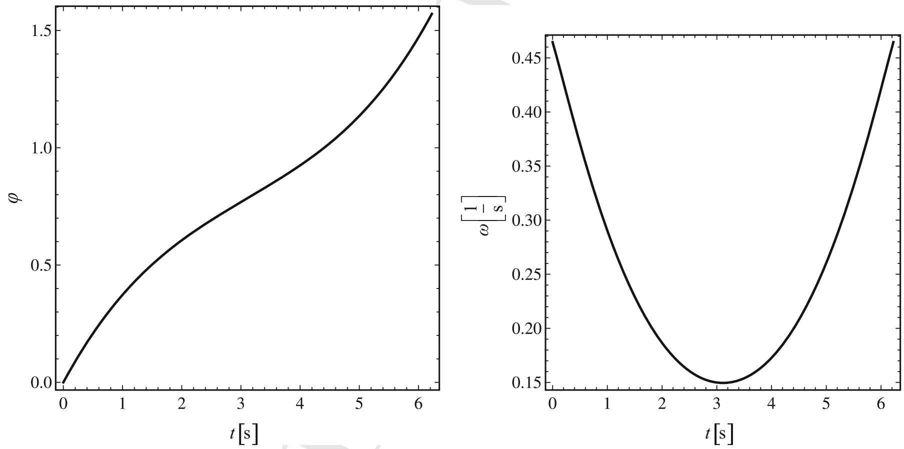


Fig. 4 Graphs of the angle φ and the angular velocity ω

Now, the solution of the two-point boundary value problem considered can be represented geometrically by the crossing points of the curves (73). This approach allows easy observation of the crossing points. The implementation of the method of crossing of curves (73) shown in Fig. 2b is realized by means of the built-in ContourPlot3D() function of the software package Mathematica. In Fig. 3 the trajectories of point B and the vehicle mass center are shown, while in Figs. 4, 5, 6, and 7 the changes versus time of the quantities φ, ω, θ, V, R_A, R_B, F_1, and L_1, respectively, are depicted.

Based on previous considerations, now it is possible to determine the laws of change of minimum required sliding friction coefficients μ_1 and μ_2 determined by the inequalities (21) and (22), as shown in Fig. 8. In accordance with Eq. (70), the normal reactions of the horizontal plane amount to $N_1 = 3065.6\,\mathrm{N}$ and $N_2 = 7823.5\,\mathrm{N}$.

The simultaneous fulfillment of conditions (21) and (22) leads to the conclusion (see Fig. 8) that minimum required value of the sliding friction coefficient is determined by $\mu_2(t_\mathrm{f}) = |R_A(t_\mathrm{f})|/N_2 = 0.14789$.

This shows that the realization of brachistochronic motion in accordance with the nonholonomic constraints (1) and (2) can be achieved only by the singular control if the sliding friction coefficient between the vehicle wheels and horizontal plane satisfies the inequality $\mu > 0.14789$.

Now, let us show, by using the obtained numerical solution of the problem, why it is justifiable to neglect particular terms in Eq. (18) in deriving state equations (25)–(27). Namely, the ratio of trade between the neglected part of kinetic energy, $T^* = (2J_2\omega\dot{\theta} + J_2\dot{\theta}^2)/2$, and used part of kinetic energy, $T = (MV^2 + J^*\omega^2)/2$, denoted by $\triangle T = |T^*/T| \cdot 100\,\%$, is shown in Fig. 9. By observing Fig. 9 it is noted that the maximum value of the quantity $\triangle T$ is lower than

Author Proof

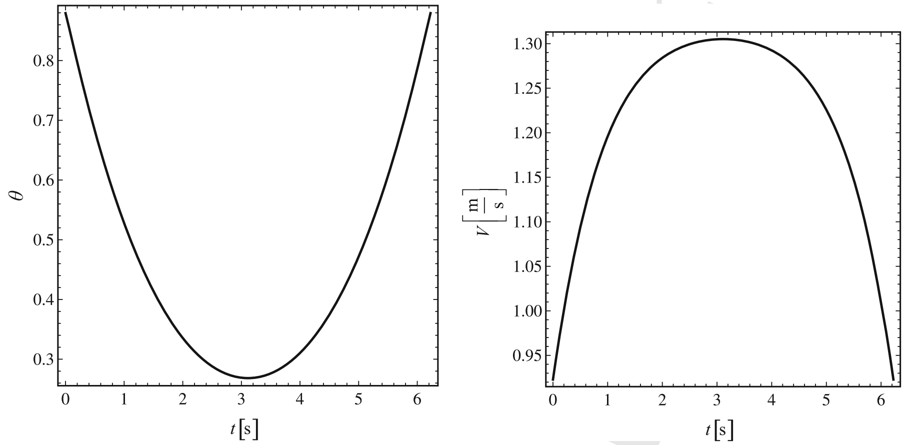


Fig. 5 Graphs of the angle θ and the speed V

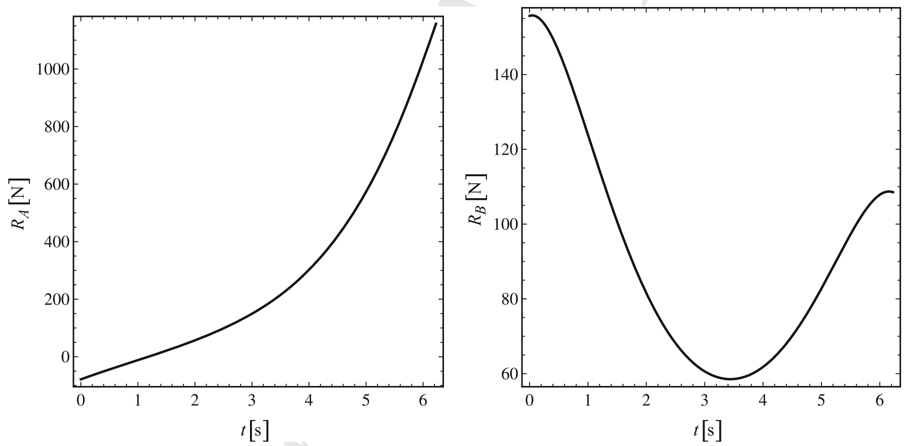


Fig. 6 Graphs of the reactions of constraints R_A and R_B

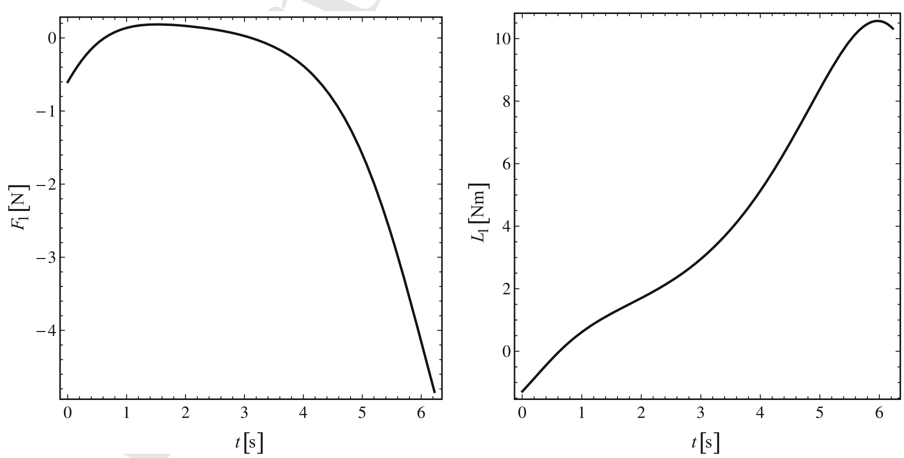


Fig. 7 Graphs of the driving force F_1 and the turning torque L_1

1 %, whereby the justification of neglecting mentioned terms in Eq. (18) is shown.

Also, the vehicle kinetic energy can be represented as $T = T_{\text{tr}} + T_{\text{rot}}$ where $T_{\text{tr}} = MV^2/2$ is the vehicle kinetic energy referring to translational motion of the vehicle body with velocity $\overrightarrow{V}$ and $T_{\text{rot}} = J^*\omega^2/2$ represents the vehicle kinetic energy referring to rotational motion of the vehicle body around axis $B\zeta$. By observing Fig. 10 it can be noted that there is a transfer between the energies T_{tr} and T_{rot} such that

Journal: **11071** MS: **3035** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2016/8/24** Pages: **15** Layout: **Medium**

Author Proof

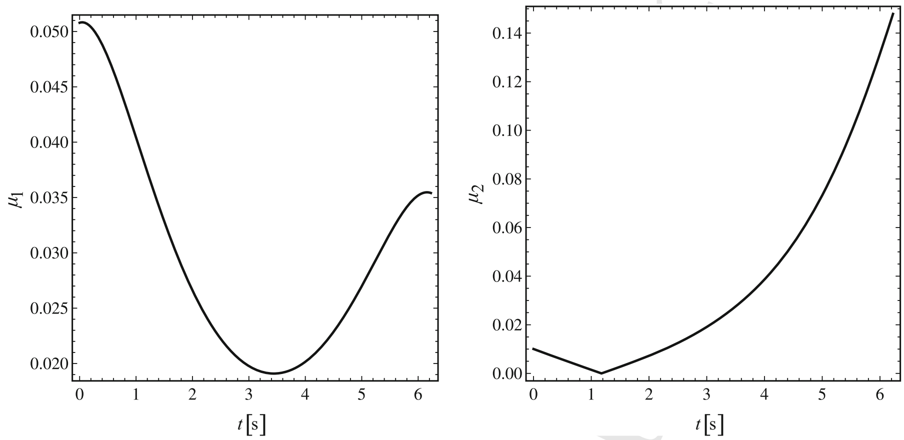


Fig. 8 Graphs of minimum required sliding friction coefficients μ_1 and μ_2

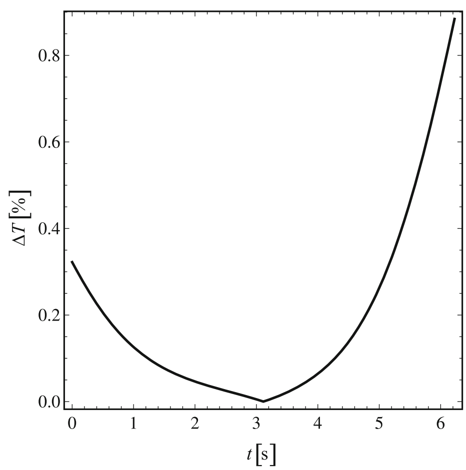


Fig. 9 The ratio $\triangle T$ versus time

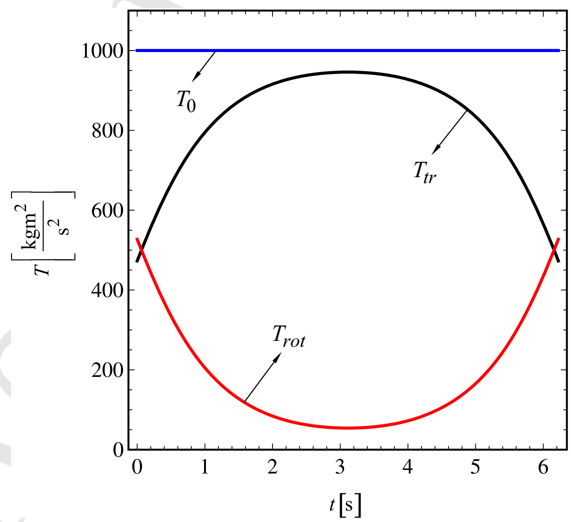


Fig. 10 The kinetic energies T_{tr} and T_{rot} versus time

$T(t) = \text{const.} = T_0$ holds. This is explained by the fact that, taking into account that the vehicle potential energy is constant, the conservation of the total mechanical energy of the system can be achieved only by mutual trade between the kinetic energies T_{tr} and T_{rot}.

4 Optimal control in the case of bounded reaction forces

In this section, the analysis of brachistochronic motion will be carried out for the specified value of the dry friction coefficient μ between the wheels and plane of motion that satisfies the following double inequality:

$$\mu_{1\,\max} < \mu < 0.14789, \tag{74}$$

where $\mu_{1\,\max}$ is the maximum value of the function $\mu_1(t)$ that expresses the law of change of minimum required value of the sliding friction coefficient between the wheels of the rear vehicle axle and the plane of motion on the interval $[t_0, t_f]$. In this case, due to the specified value for the coefficient μ, the restriction (21) must be imposed on the projection R_A. Note that if condition (74) is satisfied, then side slipping of the rear wheels does not occur. Based on the graphic of the function $R_A(t)$ shown in Fig. 6, the controller sequence is to be sing-max, i.e.,

$$u = \begin{cases} u_{\text{sing}}, & \text{if} \quad 0 \le t < t_1 \\ u_{\max} = \mu N_2, & \text{if} \quad t_1 \le t \le t_f \end{cases}, \tag{75}$$

where singular control u_{sing} is determined by the expression (55), whereas t_1 is the time instant corresponding to the discontinuity point of the function $u(t)$.

Author Proof

uncorrected proof

It should be pointed out that at the junction between singular and nonsingular subarcs of an optimal control, the necessary conditions for the optimality of junctions must be satisfied, as regulated by Theorem 1 from [20,21]. Namely, let $2q$ be the lowest order time derivative of the switching function H_1 which contains the control u explicitly and $u^{(r)}$ $(r \geq 0)$ be the lowest order derivative of the control u which is discontinuous at the junction point. Then, in accordance with Theorem 1 [20,21], the necessary condition for the junction between singular and nonsingular subarcs is expressed by the condition that the sum of $q+r$ is an odd integer. For our case, we have that $q = 1$ and $r = 0$ (see Eq. 75); consequently, it holds that $q+r = 1$, which means the necessary condition for the optimality of junction is satisfied.

As in the previous case, numerical procedure for solving the two-point boundary value problem, for different values of the sliding friction coefficient μ that satisfy the inequality (74), is based on shooting method. In this case, we have a five-parameter shooting that consists of defining the unknown costate variables λ_ξ and λ_η, time instants t_1 and t_f as well as the value of speed V_f corresponding to the time instant t_f. Numerical procedure for solving Cauchy's problem of the system of differential equations of the first kind by applying the Runge–Kutta method can be represented by the following step scheme:

- In the first step, for the time interval $[t_1,\ t_\text{f}]$ that corresponds to the nonsingular subarc of the control u, backward integration of the differential equations (25)–(27) and (37)–(40) is performed, with the initial conditions $\xi_B(t_\text{f}) = a$, $\eta_B(t_\text{f}) = b$, $\varphi(t_\text{f}) = \varphi_\text{f}$, $\omega(t_\text{f}) = \sqrt{\left(2T_0 - MV_\text{f}^2\right)/J^*}$, $V(t_\text{f}) = V_\text{f}$, $\lambda_\varphi(t_\text{f}) = \left[1 - V_\text{f}\left(\lambda_\xi \cos\varphi_\text{f} + \lambda_\eta \sin\varphi_\text{f}\right)\right]/\omega(t_\text{f})$, $\lambda_\omega(t_\text{f}) = 0$, and $\lambda_V(t_\text{f}) = 0$. Using the switching function (33) as well as its time derivative defined by Eq. (57), the following functional dependencies can be generated in the numerical form $H_1(t_1) = f_1(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f})$ and $\{H_1,\ H_0\}(t_1) = f_2(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f})$ corresponding to the time instant t_1.
- In the second step, for the time interval $[t_0, t_1]$ that corresponds to the singular subarc of the control u, the backward integration of the differential equations (58)–(60) and (61)–(64) is performed, with the initial conditions $\xi_B(t_1)$, $\eta_B(t_1)$, $\varphi(t_1)$, $\omega(t_1)$, $V(t_1)$, $\lambda_\varphi(t_1)$, $\lambda_\omega(t_1)$, and $\lambda_V(t_1)$ obtained in the previous step. Using the initial conditions (28) and (29), the following functional dependencies can be generated in the numerical form $\xi_B(0) = f_3(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f})$, $\eta_B(0) = f_4(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f})$, and $\varphi(0) = f_5(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f})$ corresponding to the time instant $t_0 = 0$.

Solving the system of nonlinear equations defined in the previous step, and using Eqs. (28), (29), (36), and (57), it is obtained:

$$f_i(t_\text{f},\ t_1,\ \lambda_\xi,\ \lambda_\eta,\ V_\text{f}) = 0,\ i = \overline{1, 5}, \tag{76}$$

from where the unknowns λ_ξ, λ_η, t_1, t_f, and V_f are determined.

The estimate of V_f can be given by observing the relation (20) at the instant t_f as follows:

$$-\sqrt{\frac{2T_0}{M}} \leq V_\text{f} \leq \sqrt{\frac{2T_0}{M}}, \tag{77}$$

where $t_1 \geq 0$ and $t_\text{f} \geq 0$. In this case, the estimate of the costate variables λ_ξ and λ_η cannot be explicitly given, but the values determined in the previous section can be taken for initial values. For the values of the parameters (70) as well as for different values of the coefficient μ chosen in accordance with the inequality (74), the values of the parameters λ_ξ, λ_η, t_1, t_f, and V_f were determined, as shown in Table 1. Figure 11 displays the graphs of the function $\eta_B(\xi_B)$ for different values of the coefficient μ. Figures 12 and 13 give graphical representation of the laws of change of the functions $\varphi(t)$, $\theta(t)$, $\omega(t)$, and $V(t)$ for differ-

Table 1 Values of λ_ξ, λ_η, t_1, t_f, V_f, and $\mu_{1\,\max}$ for various values of the coefficient μ

μ	λ_ξ (s/m)	λ_η (s/m)	t_1 (s)	t_f (s)	V_f (m/s)	$\mu_{1\,\max}$
0.13	0.51221	0.51218	5.86088	6.22571	0.93142	0.05210
0.11	0.51252	0.51187	5.40027	6.22581	0.96214	0.05870
0.09	0.51424	0.51011	4.84490	6.22668	1.01417	0.05320
0.08	0.51650	0.50778	4.50509	6.22813	1.04739	0.05143

Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

ent values of the coefficient μ. Figure 14 shows the driving force $F_1(t)$ and the turning torque $L_1(t)$ versus time, whereas Fig. 15 represents the graphs of optimal control $u(t)$ and $R_B(t)$ for different values of the coefficient μ.

Since the optimal control $u(t)$ has a discontinuity at the junction point of subarcs (see Fig. 15) it can be readily deduced that the conditions for the junction between singular and nonsingular subarcs on the optimal trajectory are satisfied. Also, from Fig. 16, where the function $\mu_1(t)$ for different values of μ is shown, as well as from the last column of Table 1 it can be deduced that at any time instant the condition (74) is satisfied, and accordingly the condition (13) as well. From numerically determined values displayed in Table 1, it is noticeable that the decreasing of the coefficient μ is accompanied by the decreasing of the singular subarc, that is, the time instant t_1 tends to zero. On the basis of previously defined numerical algorithm, the value of $\mu = \mu^*$can be determined, where $t_1 = 0$, in such a way that functional dependencies in the numerical form, $f_i(t_\mathrm{f},\, t_1,\, \lambda_\xi,\, \lambda_\eta,\, V_\mathrm{f},\, \mu^*) = 0,\ i = \overline{1, 5}$, will be adjoined by the functional dependence $f_6(t_\mathrm{f},\, t_1,\, \lambda_\xi,\, \lambda_\eta,\, V_\mathrm{f},\, \mu^*) = 0$ in the same way

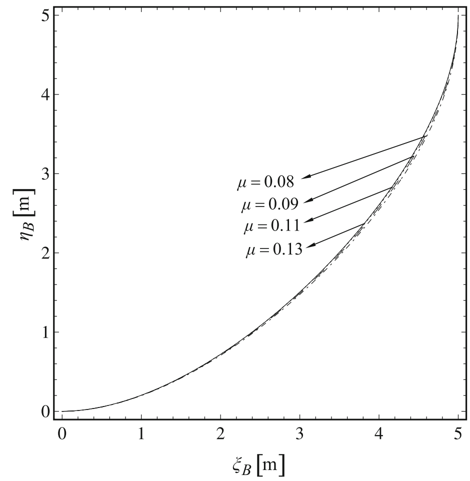


Fig. 11 Trajectories of the point B for different values of the coefficient μ

Fig. 12 Graphs of the angles φ and θ for different values of the coefficient μ

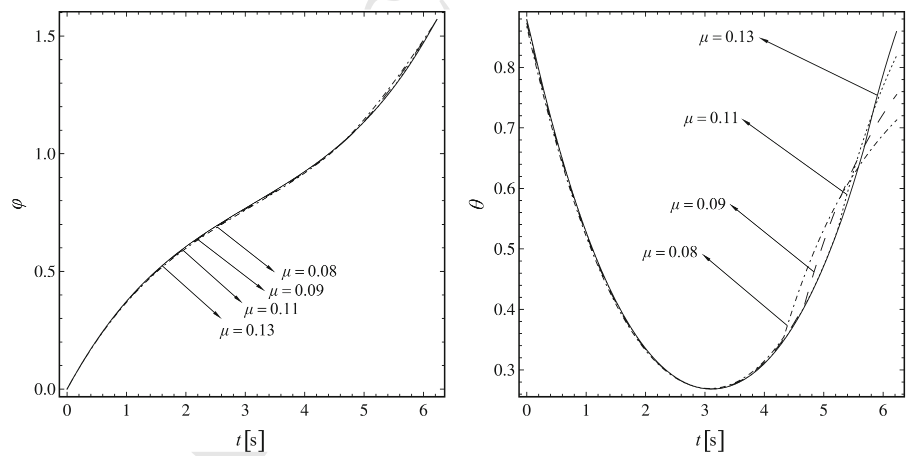
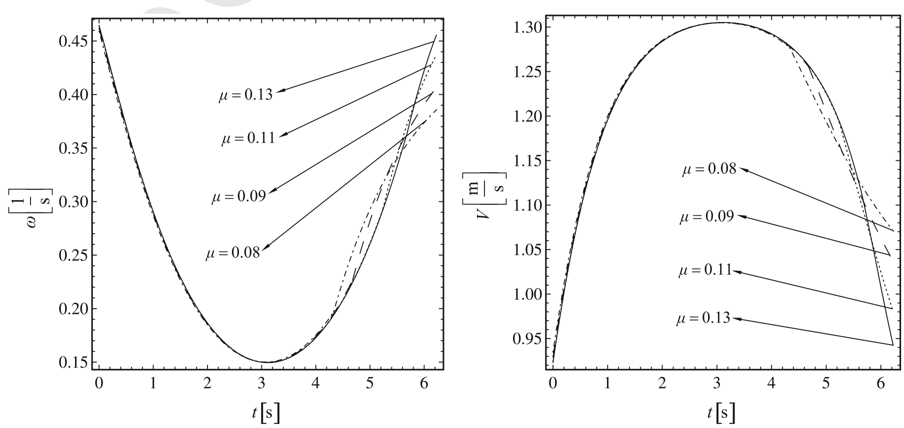


Fig. 13 Graphs of the angular velocity ω and the velocity V of point B of the vehicle for different values of the coefficient μ



Journal: 11071 MS: 3035 TYPESET DISK LE CP Disp.:2016/8/24 Pages: 15 Layout: Medium

Author Proof

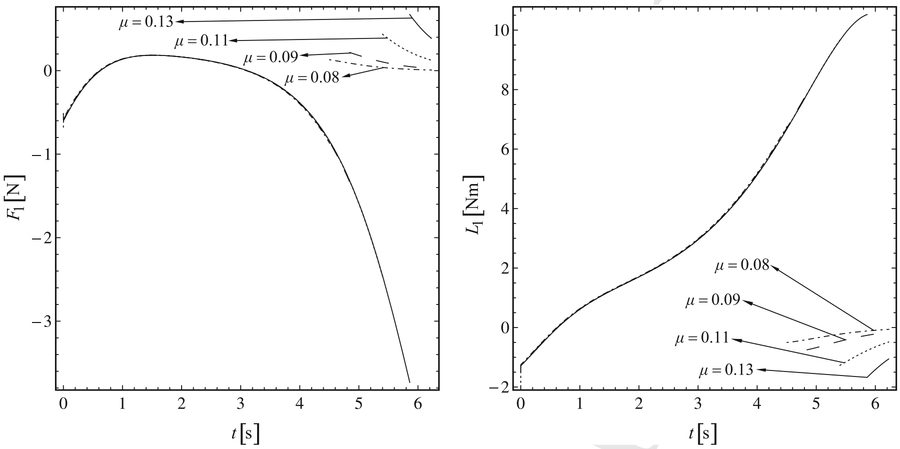


Fig. 14 Graphs of the driving force F_1 and the turning torque L_1 for different values of the coefficient μ

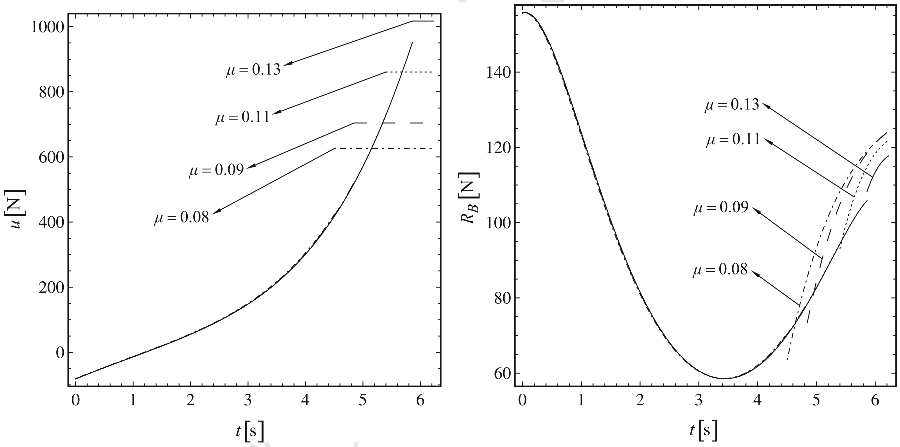


Fig. 15 Graphs of the optimal control u and the reaction force R_B for different values of the coefficient μ

as defined in the first step. This dependence expresses the fact that $t_1 = 0$ holds when μ takes the value μ^*. In accordance with that, the following values are obtained: $t_\mathrm{f} = 6.34533\,\mathrm{s}$, $\mu^* = 0.03465$, $\lambda_\xi = 0.68696\,\mathrm{s/m}$, $\lambda_\eta = 0.25953\,\mathrm{s/m}$, and $V_\mathrm{f} = 1.23776\,\mathrm{m/s}$. On the basis of numerical values of $\mu_{1\,\max}$ shown in Table 1, as well as a defined value of $\mu^* = 0.03465$, it can be deduced that μ^* is not in accordance with condition (74), thereby with a setup optimal control problem and necessary dynamic condition (22). Consequently, the case when the friction coefficient μ takes the values in the interval $\mu^* \le \mu \le \mu_{1\,\max}$ has not been considered.

In Fig. 17, where the switching function H_1 for different values of μ is graphically represented, it is evident that $H_1(\tau) > 0, \forall \tau \in (t_1,\ t_\mathrm{f}]$. It can be shown that the Kelley optimality condition (56) is satisfied here as well.

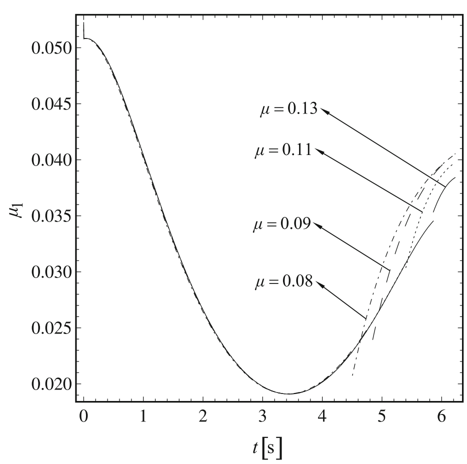


Fig. 16 Graphs of minimum required sliding friction coefficient μ_1 for different values of the coefficient μ

Journal: **11071** MS: **3035** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2016/8/24** Pages: **15** Layout: **Medium**

Author Proof

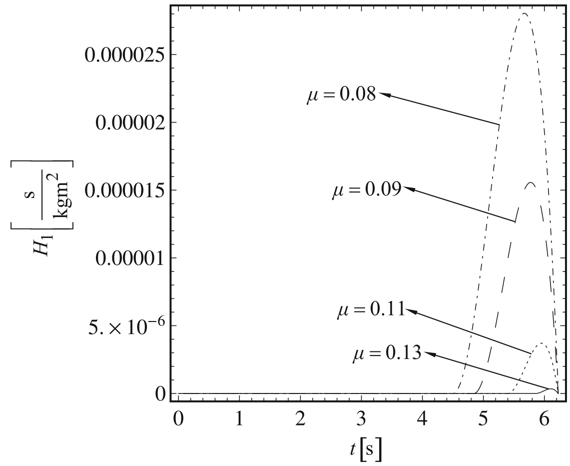


Fig. 17 Graphs of the switching function H_1 for different values of the coefficient μ

5 Conclusions

This paper considers the problem of brachistochronic motion of a wheeled vehicle. The presented procedure can be deployed to establish, for the specified vehicle parameters, the necessary values of dry friction coefficients between the vehicle wheels and horizontal plane of motion to realize the brachistochronic motion of the vehicle without side slipping of the wheels. It has been shown that the brachistochronic motion of the vehicle is not possible to realize without using singular control. Namely, in satisfying the conditions (74), the optimal extremal begins with a singular subarc and ends with a nonsingular subarc (see the optimal control policy 75). Furthermore, for sufficiently high value of the friction coefficient, optimal control is singular in its entirety. In accordance with the analysis presented in Sect. 3, it can be deduced that the front vehicle wheels require a higher value of the friction coefficient than needed by the rear wheels to prevent side slipping. This fact is justifiable in decision making to take for control variable the projection of the reaction force applied to the front vehicle wheels from the horizontal plane onto the axis of the front vehicle axle. The law of change of the angle of rotation of the front axle, $\theta(t)$, is determined from Eq. (4) as $\theta(t) = \arctan(l\omega(t)/V(t))$, and its direction is anticlockwise during the brachistochronic motion of the vehicle (see Figs. 5, 12). Due to the relation $J_2 << J_1$, the brachistochronic motion of the vehicle is realized by the control torque $\overrightarrow{L}_1$ which has a very low magnitude (see Eq. 15 and Figs. 7, 14). Also, since from Eq. (19) one has that $F_1 = -J_2\dot{\theta}(\dot{\omega}+\ddot{\theta})/V$, an identical conclusion can be drawn for the magnitude of the control force $\overrightarrow{F}_1$ (see Figs. 7, 14). Also, from Figs. 7 and 14 it can be observed that during the brachistochronic motion the force $\overrightarrow{F}_1$ and the torque $\overrightarrow{L}_1$ have variable directions. Finally, it should be pointed out that for the case of the vehicle considered in our work, moving along a horizontal surface, the potential energy of the system is constant. Due to this fact, it follows from the law of conservation of the vehicle total energy that the vehicle kinetic energy must be constant, which is realized by optimal energy transfer between the translational kinetic energy and rotational kinetic energy of the vehicle. On the other hand, note that in the case of classical brachistochrone problem (a particle moving in a vertical plane) one has an optimal trade between the potential and the kinetic energy of the particle during the brachistochronic motion.

Acknowledgments Support for this research was provided by the Ministry of Education, Science and Technological Development of the Republic of Serbia under Grants Nos. ON17400 and TR35006. This support is gratefully acknowledged.

References

1. Neimark Ju., I., Fufaev, N.A.: Dynamics of Nonholonomic Systems. Nauka, Moscow (1967)
2. Soltakhanov, ShKh, Yushkov, M.P., Zegzhda, S.A.: Mechanics of Non-Holonomic Systems: A New Class of Control Systems. Springer-Verlag, Berlin Heidelberg (2009)
3. Baruh, H.: Analytical Dynamics. McGraw-Hill, Columbus, OH (1999)
4. Martynenko, YuG: The theory of the generalized Magnus effect for non-holonomic mechanical systems. PMM-J. Appl. Math. Mech. **68**, 847–855 (2004)
5. Šalinić, S., Obradović, A., Mitrović, Z., Rusov, S.: Brachistochrone with limited reaction of constraint in an arbitrary force field. Nonlinear Dyn. **69**, 211–222 (2012)
6. Šalinić, S., Obradović, A., Mitrović, Z., Rusov, S.: Erratum: Brachistochrone with limited reaction of constraint in an arbitrary force field (Nonlinear Dynamics 69 (211–222). doi:10.1007/s11071-011-0258-1). Nonlinear Dyn. 70, 891-892
7. Šalinić, S., Obradović, A., Mitrović, Z., Rusov, S.: On the brachistochronic motion of the Chaplygin sleigh. Acta Mech. **224**, 2127–2141 (2013)
8. Matyukhin, V.I.: The control of a wheeled mechanical system. PMM-J. Appl. Math. Mech. **71**, 208–220 (2007)
9. Matyukhin, V.I.: Control of a wheeled system taking into account its inertial properties. Mech. Solids+ **48**, 243–253 (2013)

Author Proof

10. Hull, D.G.: Sufficient conditions for a minimum of the free-final-time optimal control problem. J. Optimiz. Theory App. **68**, 275–287 (1991)
11. Chowdhry, R.S., Cliff, E.M.: Optimal rigid body motions, Part 2: Minimum time solutions. J. Optimiz. Theory App. **70**, 255–276 (1991)
12. Seywald, H., Kumar, R.R.: Singular control in minimum time spacecraft reorientation. J. Guid. Control Dynam. **16**, 686–694 (1993)
13. Shen, H., Tsiotras, P.: Time-optimal control of axi-symmetric rigid spacecraft using two controls. J. Guid. Control Dynam. **22**, 682–694 (1999)
14. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: The Mathematical Theory of Optimal Processes. Wiley, New Jersey (1962)
15. Bryson, A.E., Ho, Y.C.: Applied Optimal Control. Hemisphere, New York (1975)
16. Gelfand, I.M., Fomin, S.V.: Calculus of Variations. Prentice Hall, Englewood Cliffs, New Jersey (1963)
17. Gabasov, R., Kirillova, F.M.: High order necessary conditions for optimality. SIAM J. Control **10**, 127–168 (1972)
18. Kelley, H., Kopp, R.E., Moyer, G.H.: Singular extremals. In: G., Leitmann (ed.) Mathematics in Science and Engineering, vol.31, Topics in Optimization, pp. 63-101, Academic Press, New York (1967)
19. Stoer, J., Bulirsch, J.: Introduction to Numerical Analysis. Springer, Berlin (1993)
20. McDanell, J.P., Powers, W.F.: Necessary conditions for joining optimal singular and nonsingular subarcs. SIAM J. Control **9**, 161–173 (1971)
21. Odia, A., Bell, D.J.: Junction point on partially singular trajectories. Int. J. Control **85**, 1996–2003 (2012)

uncorrected proof

Journal: **11071** MS: **3035** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2016/8/24** Pages: **15** Layout: **Medium**

Journal: 11071
Article: 3035

Springer
the language of science

Author Query Form

Please ensure you fill out your response to the queries raised below and return this form along with your corrections

Author Proof

Dear Author

During the process of typesetting your article, the following queries have arisen. Please check your typeset proof carefully against the queries listed below and mark the necessary changes either directly on the proof/online grid or in the 'Author's response' area provided below

Query	Details required	Author's response
1.	Please confirm if the corresponding author is correctly identified. Amend if necessary.	

uncorrected proof