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CONTRIBUTION TO THE DETERMINATION OF THE GLOBAL MINIMUM TIME FOR THE BRACHISTOCHRONIC MOTION OF A HOLONOMIC MECHANICAL SYSTEM

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Abstract:	The problem of the brachistochronic motion of a holonomic scleronomic mechanical system is analyzed. The system moves in an arbitrary field of known potential forces. The problem is formulated as an optimal control task, where generalized speeds are taken as control variables. The problem considered is reduced to solving the corresponding two-point boundary-value problem (TPBVP). In order to determine the global minimal solution of the TPBVP, an appropriate numerical procedure based on the shooting method is presented. The global minimal solution represents the solution with the minimum time of motion. The procedure is illustrated by an example of determining the brachistochronic motion of a disk that performs plane motion in a vertical plane in a homogeneous field of gravity.
Response to Reviewers:	Reviewer #2 1. The 4th remark in the last comments still has not been solved well, the language is even not official English in the part of “Response to Reviewers”. The numerical example has just three degrees of freedom. It's simple and suspect. Therefore, an example which has four or more degrees of freedom is necessary, or readers may doubt the availability of the method. Authors

The authors do apologize for having sent the response to the 4th remark of Reviewer #2, which was not translated into English. Further below, we provide the English translation of our response to this remark, which is included in the manuscript in Section 3, Lines 21-36:
“Based on the global evaluation of all initial values1(0) , 3(0) , , n(0) it can be asserted that all solutions of corresponding TPBVP are located within the specified intervals, where t f 0 . For the case of multiple solutions of Pontryagin’s maximum principle, the global minimum is a solution corresponding to the minimum value of the terminal time t f . The solutions of TPBVP for mechanical systems with 3DOF can be graphically represented in the space 3 by means of built-in ContourPlot3D() Mathematica function (see e.g. [22]), which will be shown in a numerical example to follow. Although we are now able to determine the estimation of all initial values for 1(0) , 3(0) , , n(0) , the global minimum time can be determined for mechanical system up to 3DOF by visual observation. For the mechanical systems with the higher than 3 DOF, the methods presented in papers [23,24] can be applied to determine all possible solutions of the system of nonlinear algebraic equations (26) with the condition (25).”

Authors’ comments: The reason for considering the example with 3DOF in Section 4 is our wish to present how the global minimum time for the brachistochronic motion of the system can be determined by direct visual observation. Certainly, for the mechanical systems with the higher than 3DOF we are deprived of the possibility to use visual observation to determine the requested solution. However, the problem of determining the global minimum time is then based on the determination of all possible solutions of the nonlinear algebraic equations (26) that satisfy the condition (25) by applying methods described in works [23] and [24]. This way, the applicability of our method is enabled for the systems with the higher than 3 DOF.

Reviewer #2 2. In the part 4 (Conclusions), the sentence “the approach described in this paper is the most convenient” may be not reasonable. Besides, in my point of view, it seems not good to compare with the reference [12] in the conclusions.

Authors
The last sentence in Section 5 (Conclusions) has been deleted because its essence is contained in the first sentence below equation (7).

Reviewer #2
3. In this paper, a few representations of the vectors are still not common. For example, the vector q should be in italic and bold.

Authors
The requested corrections have been made.

CONTRIBUTION TO THE DETERMINATION OF THE GLOBAL MINIMUM TIME FOR THE BRACHISTOCHRONIC MOTION OF A HOLONOMIC MECHANICAL SYSTEM

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Abstract. The problem of the brachistochronic motion of a holonomic scleronomic mechanical system is analyzed. The system moves in an arbitrary field of known potential forces. The problem is formulated as an optimal control task, where generalized speeds are taken as control variables. The problem considered is reduced to solving the corresponding two-point boundary-value problem (TPBVP). In order to determine the global minimal solution of the TPBVP, an appropriate numerical procedure based on the shooting method is presented. The global minimal solution represents the solution with the minimum time of motion. The procedure is illustrated by an example of determining the brachistochronic motion of a disk that performs plane motion in a vertical plane in a homogeneous field of gravity.

Key words: brachistochronic motion, optimal control, holonomic, mechanical system, shooting method

1. Introduction

In [1], the classical brachistochrone problem (find a smooth curve down which a particle slides from rest at a point A to a point B in a vertical plane influenced by its own gravity in the least time) was formulated for the first time as the problem of the optimal control theory. By using the calculus of variation, in [2] generalization of the classical brachistochrone problem for a conservative holonomic dynamical system with n degrees of freedom was performed. In our paper, the brachistochronic motion of such dynamical system is analized in the framework of the optimal control theory.

Solving the formulated problem of optimal control can deploy the methods that are generally arranged into two major groups such as direct and indirect ones. For the survey of literature about these methods refer to, e.g., [3]. The direct methods are based on the discretization of the formulated optimal problem in order to obtain a nonlinear programming problem, while in indirect ones a corresponding two point boundary value problem (TPBVP) is numerically solved. The TPBVP is derived by means of the calculus of variations and Pontryagin's maximum principle.

The shooting technique [4] is applied in this paper to solve the TPBVP. The disadvantage of the shooting method is that convergence to a numerical solution of the TPBVP is very sensitive to initial guess for the initial (or final) values of the costate (adjoint) variables. In other words, to realize convergence, it is necessary to provide that initial guess for the costates is sufficiently close to the optimal solution. Since the costate variables, as a rule, have no physical interpretations, the estimation of initial (or final) values of the costates represents a challenging task. In that regard, various methods were proposed for suitable estimation of initial values of costates. Paper [5] describes a method where initial

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costates are expressed in terms of state variables for which the values can be guessed. Paper [6] employs a method based on dynamic programming, whereas in [7] the costates estimation is achieved by means of a Legendre pseudospectral method. A concise review of mentioned method is given in [8]. On the other hand, papers [9,10] depict two different methods for determining inital costates based on the homotopy approach. By using the theory of canonical transformation and generating functions in Hamiltonian dynamics, paper [11] presents a method for determining initial and/or terminal costates via known initial and/or terminal values of states. In [12] the necessity for evaluating initial costates is avoided by adequate choice of control variables. Finally, in recently published paper [13], after a TPBVP is formed corresponding to an optimal control problem considered, it is appropriately modified by means of the continuation method and discretized by the Gauss pseudospectral method, whereby a system of nonlinear algebraic equations is obtained. Numerical solution of this system yielded approximate values of costates.

Our paper was motivated by the fact that, in a general case, the brahistohrone problem of the considered mechanical system may have more than one solution. In such systems it is needed to establish all possible solutions that satisfy neccessary optimality conditions and to choose the one to which the minimum time of the brachistochronic motion of a system corresponds. Further considerations will refer to this time as the global minimum time. In this paper, for considered brachistochrone problem a relative simple method for the estimation of both minimal and maximal values of inital costates is presented. On the basis of this estimation, all possible solutions of the brachistochrone problem were established and consequently the global minimum time was defined. The paper is organized as follows: In Section 2 the brachistochrone problem of a conservative holonomic mechanical system with n degrees of freedom is formulated in the framework of optimal control theory. The corresponding TPBVP is defined. In Section 3 the approach for the estimation of the upper and lower bounds of initial costates values is given. This estimation was the basis for presenting one procedure to establish all possible solutions of the considered brachistochrone problem, and therefore to define the global minimum time. A numerical example is solved in Section 4. Conclusions are drawn in Section 5.

2. The brachistochronic motion problem as an optimal control problem

Let us consider the motion of a holonomic scleronomic conservative mechanical system with n degrees of freedom. The configuration of the system is defined by n generalized coordinates $\boldsymbol{q} = [q_1, \ldots, q_n]^T$. The kinetic and potential energies of the system are given as [14,15]:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \mathbf{M}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \Pi = \Pi(\boldsymbol{q}), \tag{1}$$

where $\mathbf{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the symmetric positive definite mass matrix and $\dot{\boldsymbol{q}} = [\dot{q}_1, \ldots, \dot{q}_n]^T$ is the vector of generalized velocities. Let the following initial and terminal constraints be given:

$$t_0 = 0, \;\; \boldsymbol{q}(t_0) = \boldsymbol{q}_0, \tag{2}$$

$$t = t_f, \ \boldsymbol{\psi} = \left[\psi_1(\boldsymbol{q}_f), \ldots, \psi_m(\boldsymbol{q}_f)\right]^T = \boldsymbol{0}, \ m < n, \tag{3}$$

where $\boldsymbol{q}_0 = \left[q_{1(0)}, \ldots, q_{n(0)}\right]^T$, $q_i(t_0) = q_{i(0)} (i = \overline{1,n})$, $\boldsymbol{q}_f = \left[q_{1(f)}, \ldots, q_{n(f)}\right]^T$, $q_i(t_f) = q_{i(f)} (i = \overline{1,n})$, the final time t_f is free, and

$$\text{rank}\left[\partial \psi_k / \partial q_{i(f)}\right] = m \, (i = \overline{1,n}; k = \overline{1,m}) \ , \tag{4}$$

where $\left[\partial \psi_k / \partial q_{i(f)}\right] \in \mathbb{R}^{m \times n}$ is the constraint Jacobian matrix [14,15].

Also, the total mechanical energy of the system is conserved during the motion [14,15], that is:

$$\Phi(\boldsymbol{q}, \dot{\boldsymbol{q}}) \equiv T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \Pi(\boldsymbol{q}) - E_0 = 0, \tag{5}$$

where E_0 is the total mechanical energy of the system at the initial instant $t_0 = 0$.

The brachistochrone problem of the mechanical system may be formulated as the optimal control task. Namely, taking the generalized velocities as control variables

$$\dot{q}_i = u_i, \quad i = \overline{1,n}, \tag{6}$$

the brachistochrone problem of the mechanical system consists in determining the optimal controls $\boldsymbol{u} = [u_1, \ldots, u_n]^T$ as well as the system motion $q_i(t) (i = \overline{1,n})$ corresponding to them, so that the mechanical system moves in the shortest time t_f from the initial state (2) to the terminal state (3) with unchanged value of the total mechanical energy of the system (5). This can be expressed as the minimization of the functional

$$I = \int_0^{t_f} dt, \tag{7}$$

subject to (2), (3), and (5). Note that in [12] the controls are taken as $u_i = \ddot{q}_i \ (i = \overline{1, n-1})$, which required the application of the singular control theory [16, 17]. In order to solve the optimal control problem formulated by means of Pontryagin's maximum principle, the Hamiltonian and an augmented terminal function are defined as follows respectively (see [18,19]):

$$H = -1 + \boldsymbol{\lambda}^T \boldsymbol{u} + \mu \Phi(\boldsymbol{q}, \boldsymbol{u}), \tag{8}$$

$$G = \boldsymbol{\nu}^T \boldsymbol{\psi} \ , \tag{9}$$

where $\boldsymbol{\nu} = \left[\nu_1, \ldots, \nu_m\right]^T$ is an m-dimensional constant Lagrange multiplier vector, $\boldsymbol{\lambda} = \left[\lambda_1(t), \ldots, \lambda_n(t)\right]^T$ is an n-dimensional costate (adjoint) vector, and $\mu(t)$ are the Lagrange multiplier. The well-known first-order necessary conditions of optimality [18,19] are the state equations (6), the terminal conditions (3) as well as:

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T, \tag{10}$$

$$\left(\frac{\partial H}{\partial \boldsymbol{u}}\right)^T = 0, \tag{11}$$

$$H(t_f) = 0, \tag{12}$$

$$\boldsymbol{\lambda}_f = \left(\frac{\partial G}{\partial \boldsymbol{q}_f}\right)^T, \tag{13}$$

where $\boldsymbol{\lambda}_f = \left[\lambda_{1(f)}, \ldots, \lambda_{n(f)}\right]^T$, $\lambda_i(t_f) = \lambda_{i(f)}$ $(i = \overline{1,n})$.

From (10) and (11) it follows that:

$$\boldsymbol{\lambda} = -\mu \mathbf{M}(\boldsymbol{q})\boldsymbol{u}. \tag{14}$$

Since the Hamiltonian is not dependent explicitly on time and taking into account (12), the system has the first integral of motion [18,19], $H(t) \equiv 0$, or in the developed form:

$$-1 + \boldsymbol{\lambda}^T \boldsymbol{u} + \mu \Phi(\boldsymbol{q}, \boldsymbol{u}) = 0. \tag{15}$$

Combining Eqs. (5), (14), and (15), the multiplier μ is determined by:

$$\mu = -\frac{1}{2\left[E_0 - \Pi(\boldsymbol{q})\right]}. \tag{16}$$

Now, Eqs. (10), (14), and (16) yield:

$$\dot{\boldsymbol{\lambda}} = \frac{1}{2\left[E_0 - \Pi(\boldsymbol{q})\right]}\left(\frac{\partial \boldsymbol{\Phi}(\boldsymbol{q}, \boldsymbol{u})}{\partial \boldsymbol{q}}\right)^T, \tag{17}$$

$$\boldsymbol{u} = 2\left[E_0 - \Pi(\boldsymbol{q})\right]\mathbf{M}^{-1}(\boldsymbol{q})\boldsymbol{\lambda}. \tag{18}$$

Finally, in accordance with the above relations, the problem posed is reduced to solving a system of $2n$ first-order differential equations of the form:

$$\dot{\boldsymbol{q}} = 2\left[E_0 - \Pi(\boldsymbol{q})\right]\mathbf{M}^{-1}(\boldsymbol{q})\boldsymbol{\lambda}, \tag{19}$$

$$\dot{\boldsymbol{\lambda}} = \frac{1}{2\left[E_0 - \Pi(\boldsymbol{q})\right]}\left(\frac{\partial \boldsymbol{\Phi}(\boldsymbol{q}, \boldsymbol{u})}{\partial \boldsymbol{q}}\right)^T. \tag{20}$$

The general solution of this equation system consists of $2n$ unknown integration constants, and to determine them the relations (2) and (3) provide $n+m$ conditions. The remaining $n-m$ necessary conditions can be obtained from (13). Namely, according to (3), in the equation system (13) there is an equation subsystem containing m equations for which the relation (4) holds. Solving this system for $\nu_k (k = \overline{1,m})$ and substituting the thus obtained multipliers into the remaining $n-m$ equations of the system, Eq. (13) yields the required additional $n-m$ conditions to determine the integration constants as follows:

$$\boldsymbol{\psi}_\ell^*\left(\boldsymbol{q}_f, \boldsymbol{\lambda}_f\right) = 0, \quad \ell = \overline{1, n-m}. \tag{21}$$

3. Shooting method and determination of the upper and lower bounds for the values of initial costates

The numerical procedure for determining a particular solution of the equation system (19)-(20), corresponding to the boundary conditions (2), (3) and (21), is based on the shooting method [4]. The application of this method requires the evaluation of the interval of the values of the multipliers $\lambda_i\,(i=\overline{1,n})$ at the initial instant, that is, $\lambda_{i(0)}=\lambda(t_0)(i=\overline{1,n})$. This task can be realized in the following way. Namely, applying the first integral (15) at the initial instant of the motion and using Eqs. (16) and (18), a positive definite quadratic form can be written as follows:

$$\boldsymbol{\lambda}_0^T\mathbf{M}^{-1}(\boldsymbol{q}_0)\boldsymbol{\lambda}_0=\frac{1}{2\left(E_0-\Pi_0\right)}, \tag{22}$$

where $\boldsymbol{\lambda}_0=\left[\lambda_{1(0)},\ldots,\lambda_{n(0)}\right]^T$, $\lambda_{i(0)}=\lambda_i(t_0)\,(i=\overline{1,n})$, $\boldsymbol{q}_0=\left[q_{1(0)},\ldots,q_{n(0)}\right]^T$, and $\Pi_0=\Pi(\boldsymbol{q}_0)$. In order to reduce the quadratic form (22) to the canonical form [21], a linear orthogonal transformation of the Lagrange multipliers is introduced as follows:

$$\boldsymbol{\lambda}_0=\mathbf{S}\boldsymbol{\lambda}_0^*, \tag{23}$$

where $\boldsymbol{\lambda}_0^*=\left[\lambda_{1(0)}^*,\ldots,\lambda_{n(0)}^*\right]^T$, $\lambda_i^*(t_0)=\lambda_{i(0)}^*\ (i=\overline{1,n})$, and $\mathbf{S}\in R^{n\times n}$ is an orthogonal transformation matrix (also called eigenvector matrix [21]) such that $\mathbf{S}^{-1}\mathbf{M}^{-1}(\boldsymbol{q}_0)\mathbf{S}=diag\left(e_1,\ldots,e_n\right)$, where $e_i\,(i=\overline{1,n})$ are the eigenvalues of the matrix $\mathbf{M}^{-1}(\boldsymbol{q}_0)$. After the application of the above coordinate transformation, the following canonical form of (22) is obtained:

$$e_1\left(\lambda_{1(0)}^*\right)^2+e_2\left(\lambda_{2(0)}^*\right)^2+\ldots+e_n\left(\lambda_{n(0)}^*\right)^2=\frac{1}{2\left(E_0-\Pi_0\right)}. \tag{24}$$

Finally, the canonical form (24) implies the following estimations of the values of the Lagrange multipliers $\lambda_{i(0)}^*(i=\overline{1,n})$:

$$\begin{gathered}
-\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_1}}\le\lambda_{1(0)}^*\le\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_1}},\\
-\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_2}}\le\lambda_{2(0)}^*\le\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_2}},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_n}}\le\lambda_{n(0)}^*\le\sqrt{\frac{1}{2\left(E_0-\Pi_0\right)e_n}},
\end{gathered} \tag{25}$$

where the global estimation of the initial values $\lambda_{i(0)}(i=\overline{1,n})$ can be provided based on (23) and (25). Note that for $t_0=0$ from the first integral (15) one initial costate, let's say $\lambda_{2(0)}$, can be expressed through the rest of $n-1$ initial costates. In the case when $\mathbf{M}(\boldsymbol{q}_0)$ represents a diagonal matrix, the quadratic form (22) has the canonical form, and it is then possible from this canonical form to directly express the initial costate $\lambda_{2(0)}$ through the rest of initial costates. In this way, the number of initial costates participating in the shooting process is reduced by one.

For the initial costates $\lambda_{i(0)}(i=\overline{1,n})$ and the initial states $q_{i(0)}(i=\overline{1,n})$, the final states and costates can be obtained through the forward integration of differential equations (19) and (20). In this manner, taking into account that the final states and costates should satisfy the conditions (3) and (21), the following relations can be established in numerical form:

$$\boldsymbol{\Gamma}(\boldsymbol{z})=\left[\psi_1(\boldsymbol{q}_f),\ldots,\psi_m(\boldsymbol{q}_f),\,\psi_1^*\left(\boldsymbol{q}_f,\boldsymbol{\lambda}_f\right),\ldots,\psi_{n-m}^*\left(\boldsymbol{q}_f,\boldsymbol{\lambda}_f\right)\right]^T=\mathbf{0}_{n\times1}\,, \tag{26}$$

where $\boldsymbol{\Gamma}(\boldsymbol{z})\equiv\left[\Gamma_1(\boldsymbol{z}),\ldots,\Gamma_n(\boldsymbol{z})\right]^T\in\mathbb{R}^{n\times1}$ is called the shooting function [10] and

$\boldsymbol{z}=\left[t_f,\,\lambda_{1(0)},\,\lambda_{3(0)},\ldots,\,\lambda_{n(0)}\right]$.

Based on the global evaluation of all initial values $\lambda_{1(0)},\,\lambda_{3(0)},\ldots,\,\lambda_{n(0)}$ it can be asserted that all solutions of corresponding TPBVP are located within the specified intervals, where $t_f>0$. For the case of multiple solutions of Pontryagin's maximum principle, the global minimum is a solution corresponding to the minimum value of the terminal time t_f. The solutions of TPBVP for mechanical systems with 3DOF can be graphically represented in the space $\mathbb{R}^3$ by means of built-in **ContourPlot3D**() Mathematica function (see e.g. [22]), which will be shown in a numerical example to follow. Although we are now able to determine the estimation of all initial values for $\lambda_{1(0)},\,\lambda_{3(0)},\ldots,\,\lambda_{n(0)}$, the global minimum time can be determined for mechanical system up to 3DOF by visual observation. For the mechanical systems with the higher than 3 DOF, the methods presented in papers [23,24] can be applied to determine all possible solutions of the system of nonlinear algebraic equations (26) with the condition (25).

4. Numerical example

Let us consider a circular disk with a radius R and mass m. The disk is moving in a uniform gravitational field in a vertical plane coinciding with the plane Oxy (see Fig. 1) of the inertial reference frame $Oxyz$,where y represents the vertical axis directed downwards. A linear spring with a natural (unstretched) length of $\ell_0=R$ and modulus c is attached to the mass center C of the disk, as depicted in Fig. 1. The local coordinate frame $C\xi\eta\zeta$ has its origin at point C and it is attached to the disk in the manner shown in Fig. 1. At time t_0, the axes of the frame $C\xi\eta\zeta$ are denoted with $C_0\xi_0\eta_0\zeta_0$. The disk position relative to the frame $Oxyz$ defines a set of Lagrange coordinates $\boldsymbol{q}=[q_1,\,q_2,\,q_3]^T$ where q_1 and q_2 are the Cartesian coordinates of the mass center C, while q_3 represents the angle of rotation of the disk (see Fig. 1).

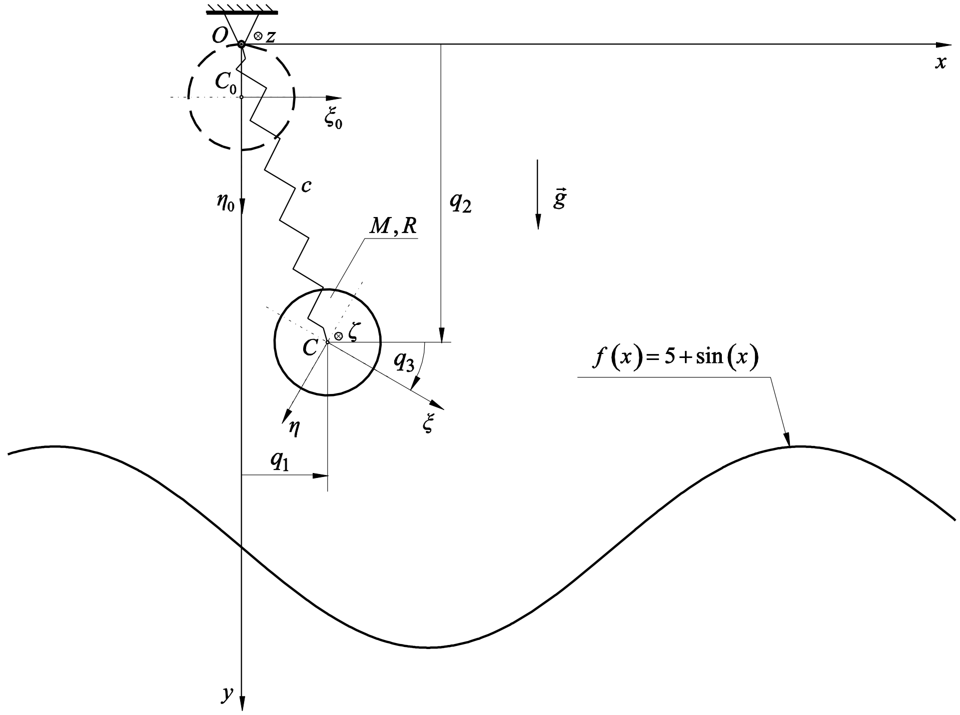

Fig.1. Motion of the disk in the vertical plane *Oxy*.

The kinetic and potential energies of the disk, respectively, are:

$$T=\frac{1}{2}M\left(\dot{q}_1^2+\dot{q}_2^2+\frac{1}{2}R^2\dot{q}_3^2\right), \tag{27}$$

$$\Pi=-Mgq_2+\frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2, \tag{28}$$

where $\vec{g}=-g\,\vec{j}$ and g is the gravity acceleration. In the case considered, the conditions (3) and (4) read:

$$t_0=0,\ \ q_{1(0)}=0,\ \ q_{2(0)}=R,\ \ q_{3(0)}=0, \tag{29}$$

$$t=t_f,\quad \psi_1\equiv q_{2(f)}-5-\sin q_{1(f)}=0, \tag{30}$$

$$\psi_2\equiv q_{3(f)}-2\pi=0, \tag{31}$$

where (31) means that, at the terminal instant t_f, the disk mass center must be positioned on the sinusoide $f(x)=5+\sin(x)$. The relation (5), based on (27) and (28), takes the following form:

$$\Phi(\boldsymbol{q},\ \dot{\boldsymbol{q}})\equiv\frac{1}{2}M\left(\dot{q}_1^2+\dot{q}_2^2+\frac{1}{2}R^2\dot{q}_3^2\right)-Mgq_2+\frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2-E_0=0. \tag{32}$$

For the problem considered, the system of equations (19) and (20) reads:

$$\dot{q}_1=\frac{2\left[E_0+Mgq_2-\frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2\right]}{M}\lambda_1, \tag{33}$$

$$\dot{q}_2 = \frac{2\left[E_0 + Mgq_2 - \frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2\right]}{M}\lambda_2, \tag{34}$$

$$\dot{q}_3 = \frac{4\left[E_0 + Mgq_2 - \frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2\right]}{MR^2}\lambda_3, \tag{35}$$

$$\dot{\lambda}_1 = \frac{c}{2\left[E_0 + Mgq_2 - \frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2\right]}\frac{\sqrt{q_1^2+q_2^2}-R}{\sqrt{q_1^2+q_2^2}}q_1, \tag{36}$$

$$\dot{\lambda}_2 = \frac{1}{2\left[E_0 + Mgq_2 - \frac{1}{2}c\left(\sqrt{q_1^2+q_2^2}-R\right)^2\right]}\left(\frac{\sqrt{q_1^2+q_2^2}-R}{\sqrt{q_1^2+q_2^2}}cq_2 - Mg\right), \tag{37}$$

$$\dot{\lambda}_3 = 0, \tag{38}$$

where $\boldsymbol{u} = \left[\dot{q}_1, \dot{q}_2, \dot{q}_3\right]^T$.

Further, based on (27), (28), and (29), the quadratic form (22) reads:

$$\frac{1}{M}\lambda_{1(0)}^2 + \frac{1}{M}\lambda_{2(0)}^2 + \frac{2}{MR^2}\lambda_3^2 = \frac{1}{2\left(E_0 + Mgq_{2(0)}\right)}, \tag{39}$$

where, in accordance with (38), one has that $\lambda_3(t) = \text{const.}$

Now, solving Eq. (39) for the multiplier $\lambda_{2(0)}$ yields:

$$\lambda_{2(0)} = \pm\sqrt{\frac{M}{2\left(E_0 + Mgq_{2(0)}\right)} - \left(\lambda_{1(0)}^2 + \frac{2}{R^2}\lambda_3^2\right)}, \tag{40}$$

and, for the system considered, Eq. (21) is reduced to the form:

$$\psi_1^* \equiv \lambda_{1(f)} + \lambda_{2(f)}\cos q_{1(f)} = 0. \tag{41}$$

Finally, the TPBVP corresponding to the problem considered in this example is determined by Eqs. (33)-(38) and the boundary conditions (29), (30), (31), and (41). If TPBVP is solved by the shooting method [4], choosing the values (29), (40), $\lambda_1(t_0) = \lambda_{1(0)}$, and $\lambda_3(t_0) = \lambda_3$ and solving Cauchy's problem of Eqs. (33)-(38), the following dependences can be established in the numerical form:

$$\Gamma_1\left(\lambda_{1(0)}, \lambda_3, t_f\right) = \psi_1 = 0, \quad \Gamma_2\left(\lambda_{1(0)}, \lambda_3, t_f\right) = \psi_2 = 0, \quad \Gamma_3\left(\lambda_{1(0)}, \lambda_3, t_f\right) = \psi_1^* = 0. \tag{42}$$

The determination of all solutions of the TPBVP with this method requires the estimation of values of the multipliers $\lambda_{1(0)}$ and λ_3 where $t_f > 0$. Hence, according to (39), the following estimations may be given:

$$-\sqrt{\frac{M}{2\left(E_0+Mgq_{2(0)}\right)}}\le\lambda_{1(0)}\le\sqrt{\frac{M}{2\left(E_0+Mgq_{2(0)}\right)}},$$
$$-\sqrt{\frac{M}{2\left(E_0+Mgq_{2(0)}\right)}}\le\lambda_{2(0)}\le\sqrt{\frac{M}{2\left(E_0+Mgq_{2(0)}\right)}}, \tag{43}$$
$$-\sqrt{\frac{MR^2}{4\left(E_0+Mgq_{2(0)}\right)}}\le\lambda_{3(0)}\le\sqrt{\frac{MR^2}{4\left(E_0+Mgq_{2(0)}\right)}}.$$

On the other hand, solutions of the TPBVP may be determined in another way. Namely, now it is possible to determine the intersections of the surfaces (42) as:

$$p_f=\Gamma_1\left(\lambda_{1(0)},\lambda_3,t_f\right)\cap\Gamma_3\left(\lambda_{1(0)},\lambda_3,t_f\right), \tag{44}$$

$$r_f=\Gamma_2\left(\lambda_{1(0)},\lambda_3,t_f\right)\cap\Gamma_3\left(\lambda_{1(0)},\lambda_3,t_f\right), \tag{45}$$

where p_f and r_f are the space curves represented by the following dependencies established in the numerical form:

$$p_f=f_p\left(\lambda_{1(0)},t_f\right),\quad r_f=f_r\left(\lambda_{1(0)},t_f\right). \tag{46}$$

Now, the solution of the TPBVP can be geometrically represented by the crossing points of the curves (46) as:

$$f_p\left(\lambda_{1(0)},\ t_f\right)\cap f_r\left(\lambda_{1(0)},t_f\right)=\{M_1,...,M_r\}. \tag{47}$$

The number of elements of the set (47) is equal to the number of possible solutions of the TPBVP, while the coordinates of crossing points in the space $\left(\lambda_{1(0)},\lambda_3,t_f\right)$ represent the TPBVP solutions. It should be pointed out that from the viewpoint of the ease of observation of the crossing points M_i and visual estimation of their coordinates, the method of crossing curves (46) is more convenient to use than the surface crossing method (42).

Now, it is possible by applying the space curves crossing method (46) to perform the estimation of the values of coordinates $\left(\lambda_{1(0)},\lambda_3,t_f\right)$ of all crossing points (47). The estimated values of coordinates $\left(\lambda_{1(0)},\lambda_3,t_f\right)$ of the crossing points can be used as initial iteration for finding accurate values of the quantities $\lambda_{1(0)}$, λ_3, and t_f by applying the shooting method.

In this paper, the implementation of the method of crossing of the curves (46) is achieved by using the built-in ContourPlot3D() Mathematica function (see e.g. [22]). The TPBVP is solved for the following values of the system parameters:

$$E_0=3500\frac{\text{kgm}^2}{\text{s}^2},\ M=3\text{kg},\ R=0.2\text{m},\ c=0.2\frac{\text{kN}}{\text{m}},\ q_{2(0)}=0.2\text{m},\ \ g=9.80665\frac{\text{m}}{\text{s}^2}. \tag{48}$$

In accordance with (43), the following global estimation of the initial values of the corresponding multipliers can be given:

$$-0.0207 \le \lambda_{1(0)} \le 0.0207, \quad -0.0207 \le \lambda_{2(0)} \le 0.0207, \quad -0.0029 \le \lambda_3 \le 0.0029. \tag{49}$$

In Fig. 2, the space curves (46) and the crossing points (47) are shown. By observing Fig. 2, it can be concluded that the TPBVP has not a unique solution.
Visual estimation of the values of coordinates of the crossing points M_1, M_2, and M_3, respectively, from Fig. 2 are $(0, 0, 0.1)$, $(0.015, 0, 0.15)$, and $(0.005, 0, 0.18)$ which represent the initial iteration for finding accurate values by applying the shooting method.

All solutions are shown in Table 1. It should be stressed that all possible solutions of the TPBVP correspond to the values of the terminal time t_f for which it holds $0 < t_f < t_f^*$, where $t_f^* = 0.1954\text{s}$. The reason for this lies in the fact that for specified values of the multipliers $\lambda_{1(0)}$ and λ_3 and $t_f > t_f^*$, the function ψ_3^* takes the infinite values.

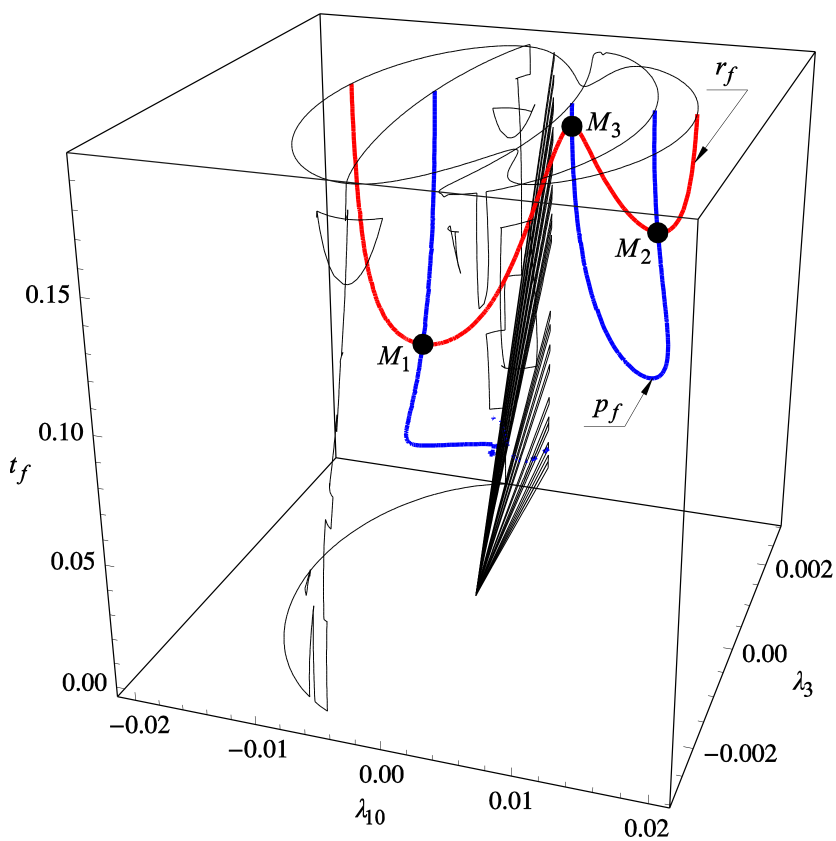

Fig. 2. Crossing of the curves $p_f = f_p\left(\lambda_{1(0)}, t_f\right)$ and $r_f = f_r\left(\lambda_{1(0)}, t_f\right)$.

Table 1. The TPBVP solutions

Solutions	$\lambda_{1(0)}\,[\mathrm{s/m}]$	$\lambda_3\,[\mathrm{s/m}]$	$t_f\,[\mathrm{s}]$
First solution (M_1)	–0.006182	0.000677	0.09326
Second solution (M_2)	0.013277	0.000545	0.150368
Third solution (M_3)	0.006419	0.000534	0.187075

Based on the values shown in Table 1, it can be concluded that the global minimum time for the brachistochronic motion of the disk in the vertical plane corresponds to the first solution (point M_1 shown in Fig. 2) and it is $t_f = 0.09326\ \mathrm{s}$. In Figs. 3 and 4, the graphs of the state and control variables corresponding to the first solution (M_1) given in Table 1 are shown.

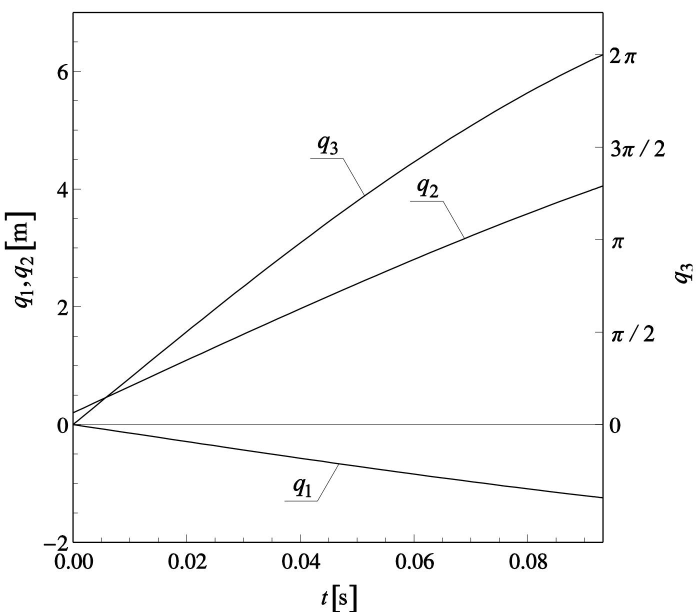

Fig. 3. Graphs of the generalized coordinates corresponding to the first solution (M_1).

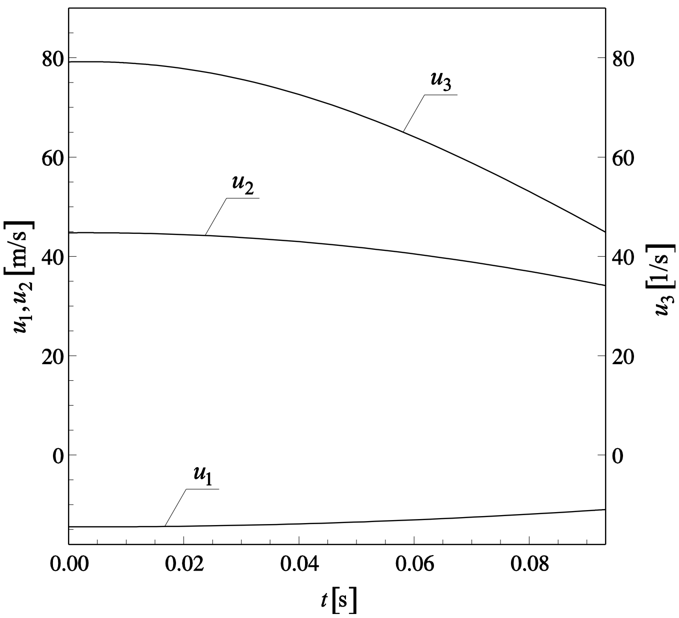

Fig. 4. Graphs of the optimal controls corresponding to the first solution (M_1).

In general, the brachistochronic motion of a mechanical system can be realized by means of the control forces, whose power is equal to zero. These control forces can be either active forces (see e.g. [25,26,27]) or reactions of additionaly imposed independent ideal stationary constraints to the system, which must be in accordance with the system's brachistochronic motion (see e.g. [12,28]). The combinations of these types of control forces are also allowed. As in [12], the brachistochronic motion of the disk is realized by means of the fixed and moving centrodes of the disk [29,30,31]. The motion of the disk is equivalent to the rolling without slipping of the moving centrode on the fixed centrode at angular velocity equal to the angular velocity of disk [29,30,31]. The immovable centrode implies the geometric locus of momentary centers of disk rotation relative to immovable plane , whereas the movable centrode means the geometric locus of momentary poles of disk velocity relative to movable plane . Although at any moment the momentary pole of velocities coincides with the momentary center of rotation, it should be taken into account that the momentary pole is the velocity of the point on movable plane , while the momentary center of rotation is the point of immovable plane.

The parametric equations of the fixed centrode read [29,30,31]:

$$x = q_1 - \frac{\dot{q}_2}{\dot{q}_3}, \quad y = q_2 + \frac{\dot{q}_1}{\dot{q}_3}, \tag{50}$$

and those of the moving centrode:

$$\xi = \frac{1}{\dot{q}_3}\left(\dot{q}_1 \sin q_3 - \dot{q}_2 \cos q_3\right), \quad \eta = \frac{1}{\dot{q}_3}\left(\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3\right). \tag{51}$$

The fixed and moving centrodes for the positions of the disk corresponding to the time instances $t_0 = 0$ and t_f as well as the trajectory of point C are shown in Figs. 5, 6, and 7.

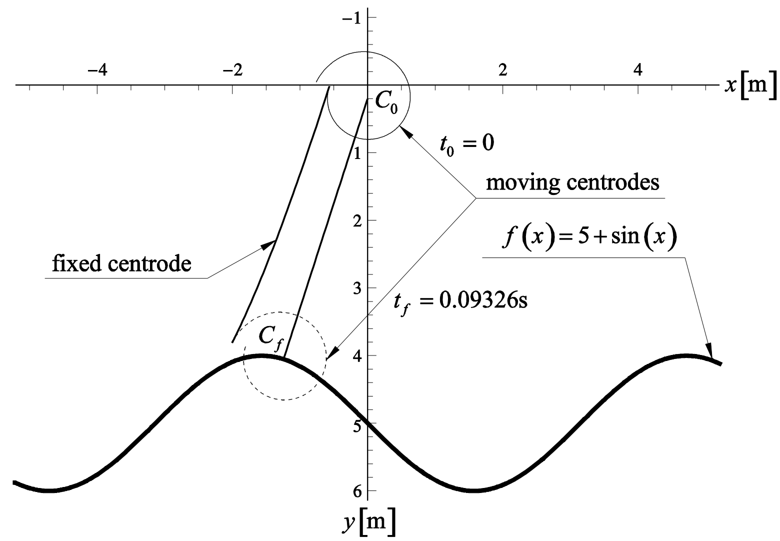

Fig. 5. Centrodes and the trajectory of point C corresponding to the first solution (M_1).

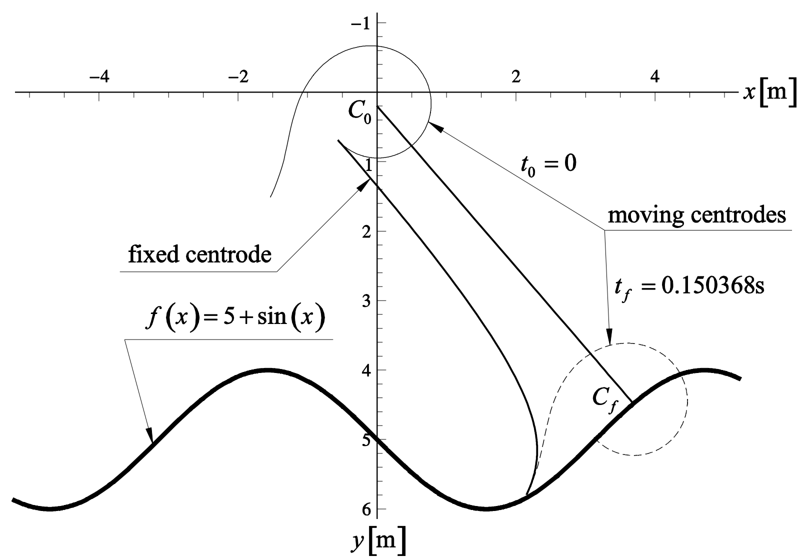

Fig. 6. Centrodes and the trajectory of point C corresponding to the second solution (M_2).

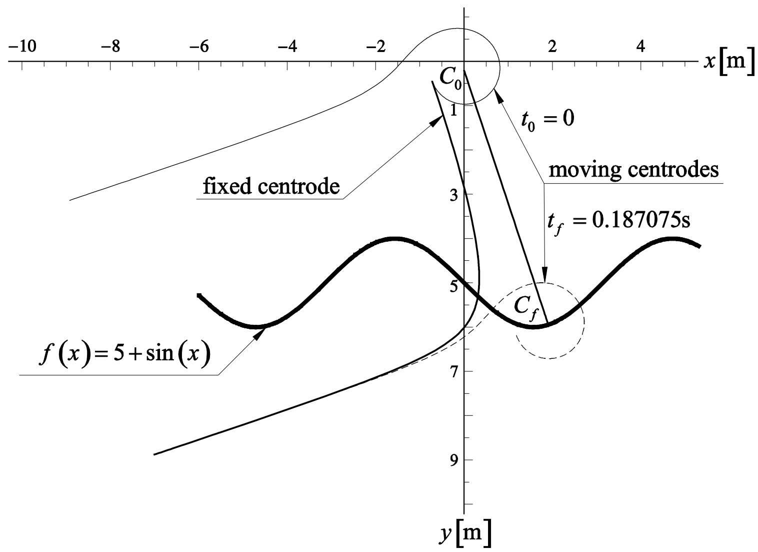

Fig. 7. Centrodes and the trajectory of point *C* corresponding to the third solution (M_3).

5. Conclusions

In this paper, a method for determination of the global minimal solution for the brachistochronic motion of holonomic scleronomic conservative mechanical systems with n degrees of freedom is presented. The method also includes an interesting way for estimating the values of the multipliers $\lambda_i (i = \overline{1,n})$. This is of great importance in the applications of the shooting method, because the multipliers $\lambda_i (i = \overline{1,n})$ usually have no physical interpretations, which implies difficulties in the estimation of their initial or terminal values. Our work is distinct from those enlisted in references, which consider the problem of determining costates values, because our work determines the upper and lower bounds of possible values of costates. These estimations enable to relatively simply determine all possible solutions of the considered problem and thereby define the global minimum time. In the case of systems with three degrees of freedom, the method allows for the geometric visual representation of possible solutions in the form of crossing of the curves in 3D space. Note that the approach from [12] developed for the variable-mass mechanical systems does not require the estimation of costates values. However, this approach is not suitable for determining the global minimum time for the brachistochronic motion of the considered mechanical system, because it does not allow for determining the upper and lower bounds of the values of state variables, which is necessary to do when determining the global minimum time.

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Figure 1

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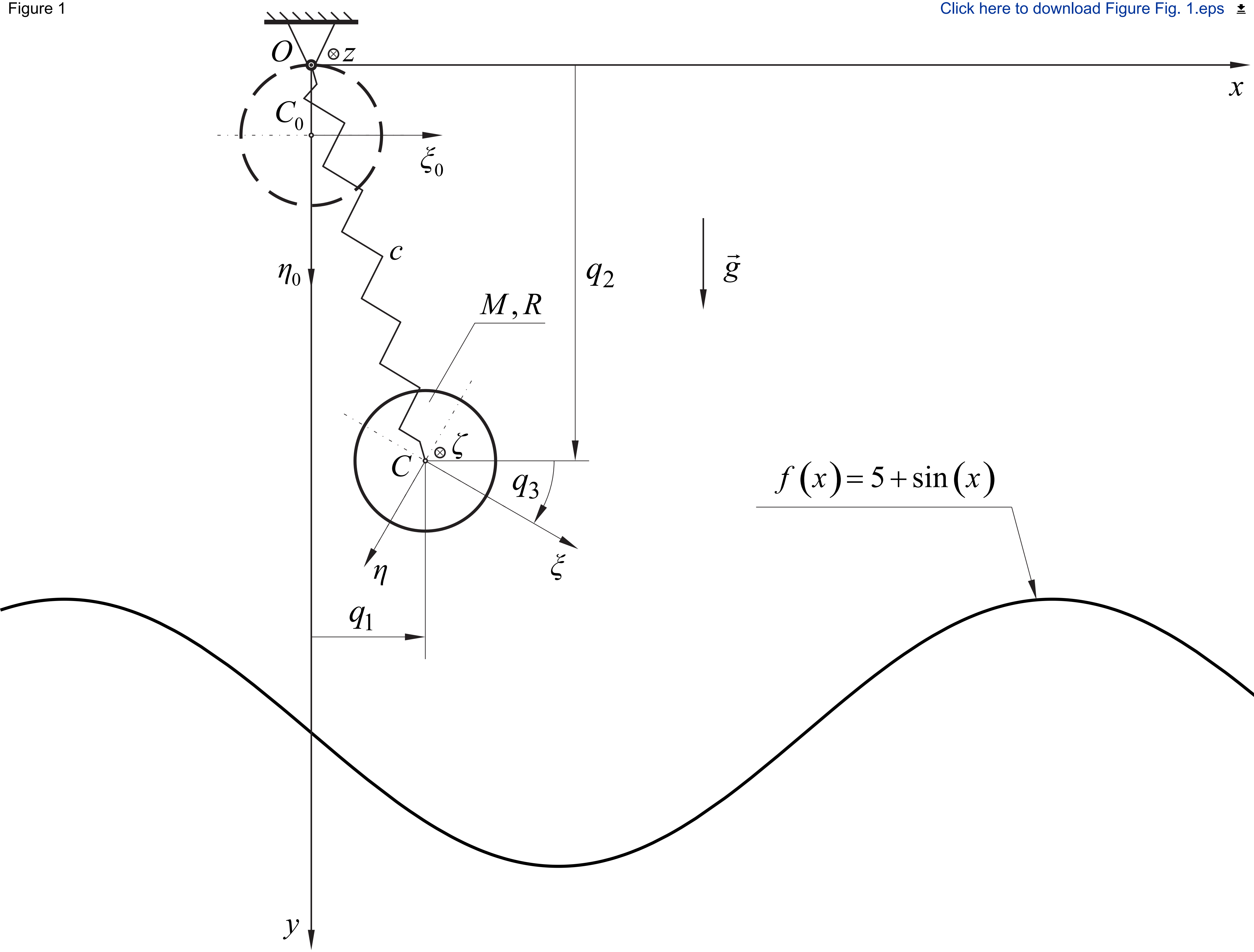

Figure 2

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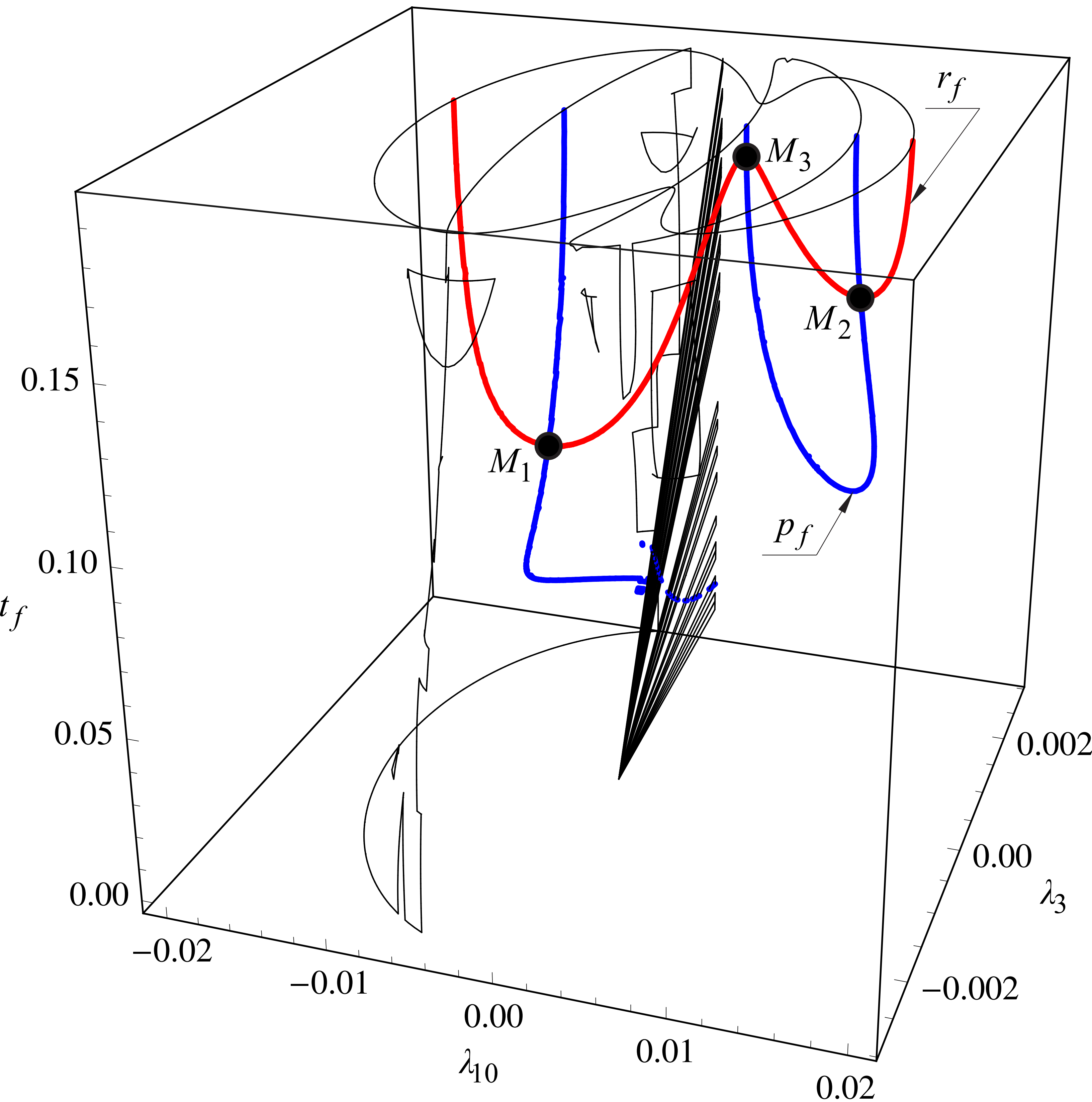


Figure 3

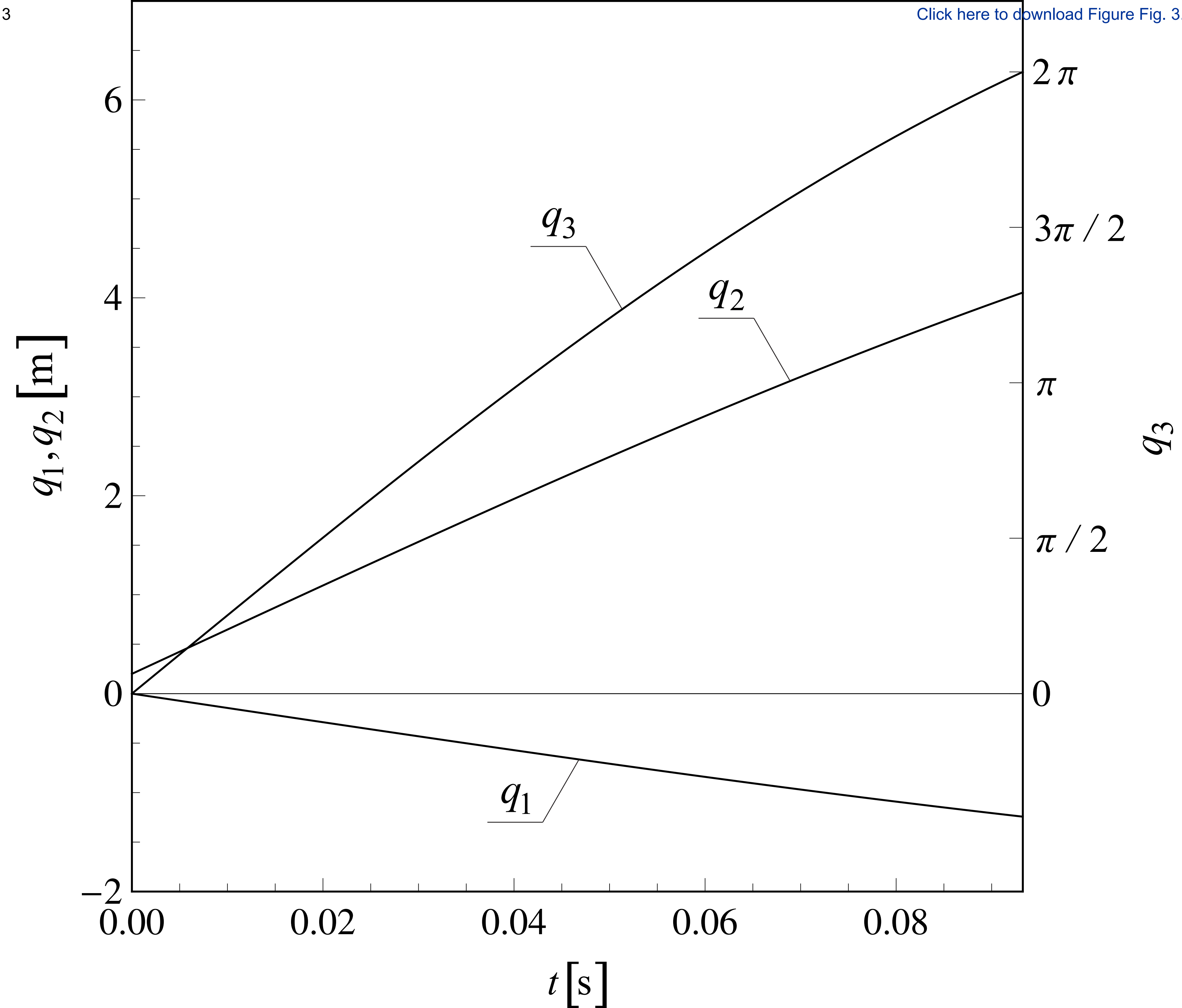
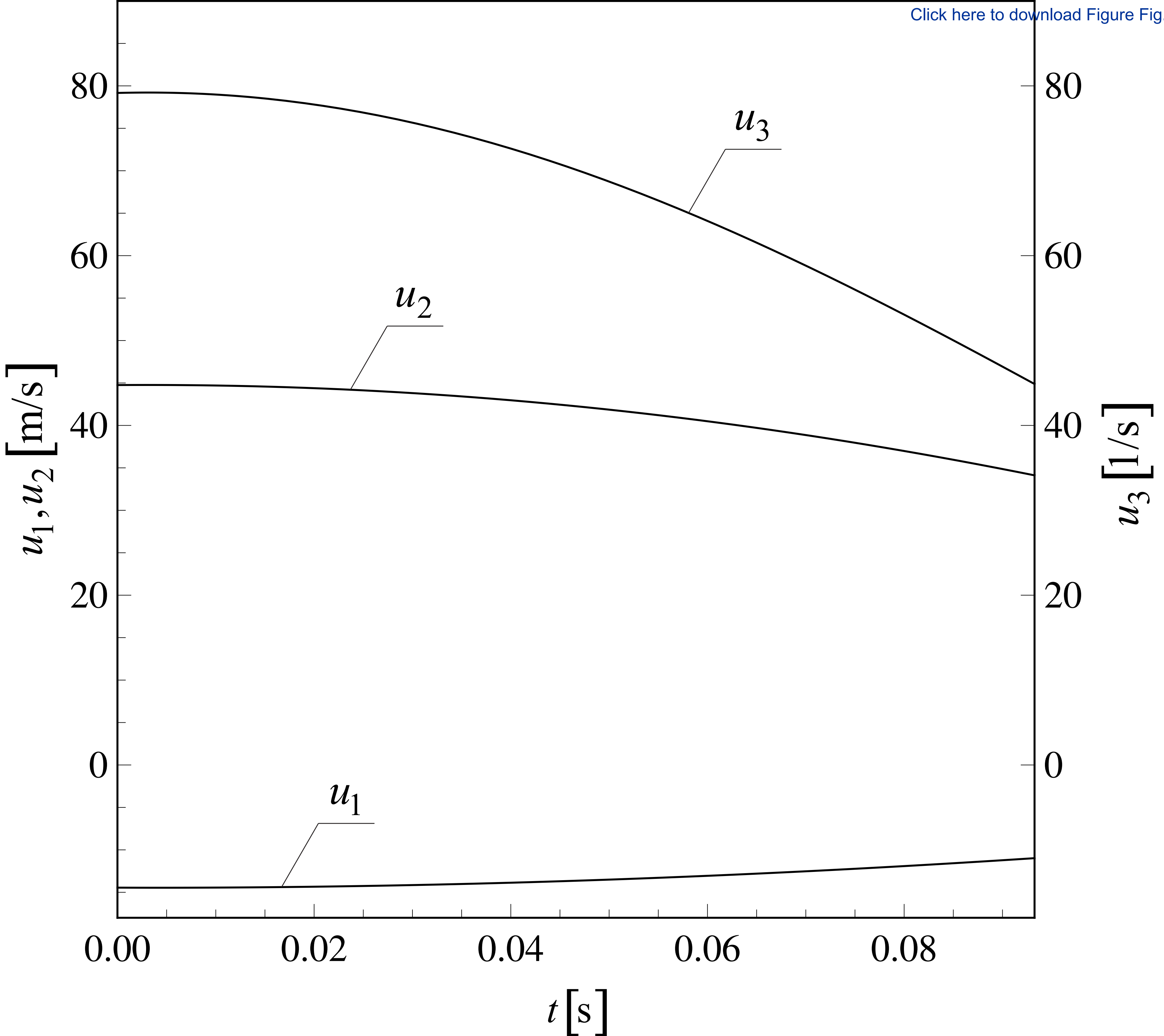


Figure 4



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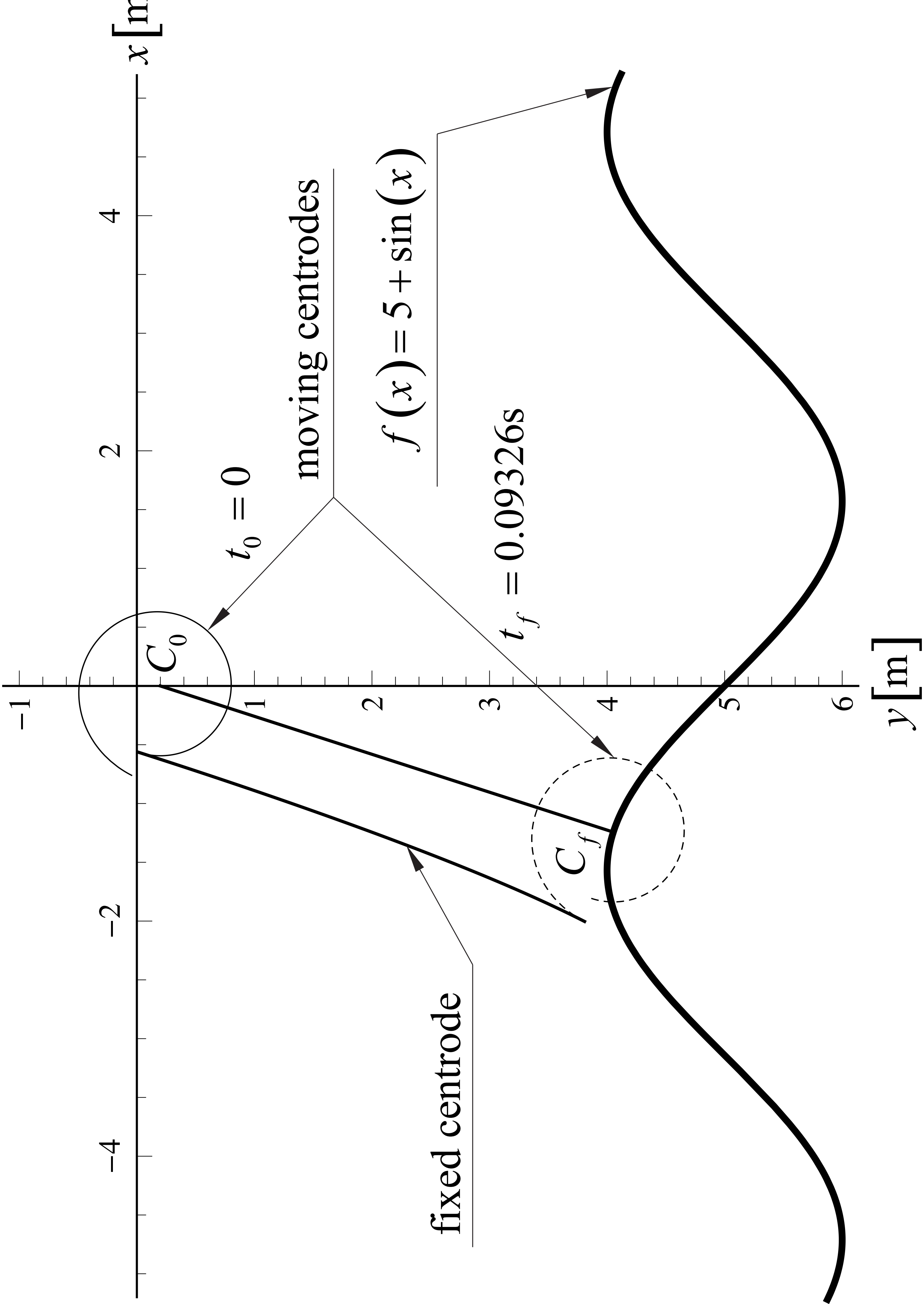

Figure 6

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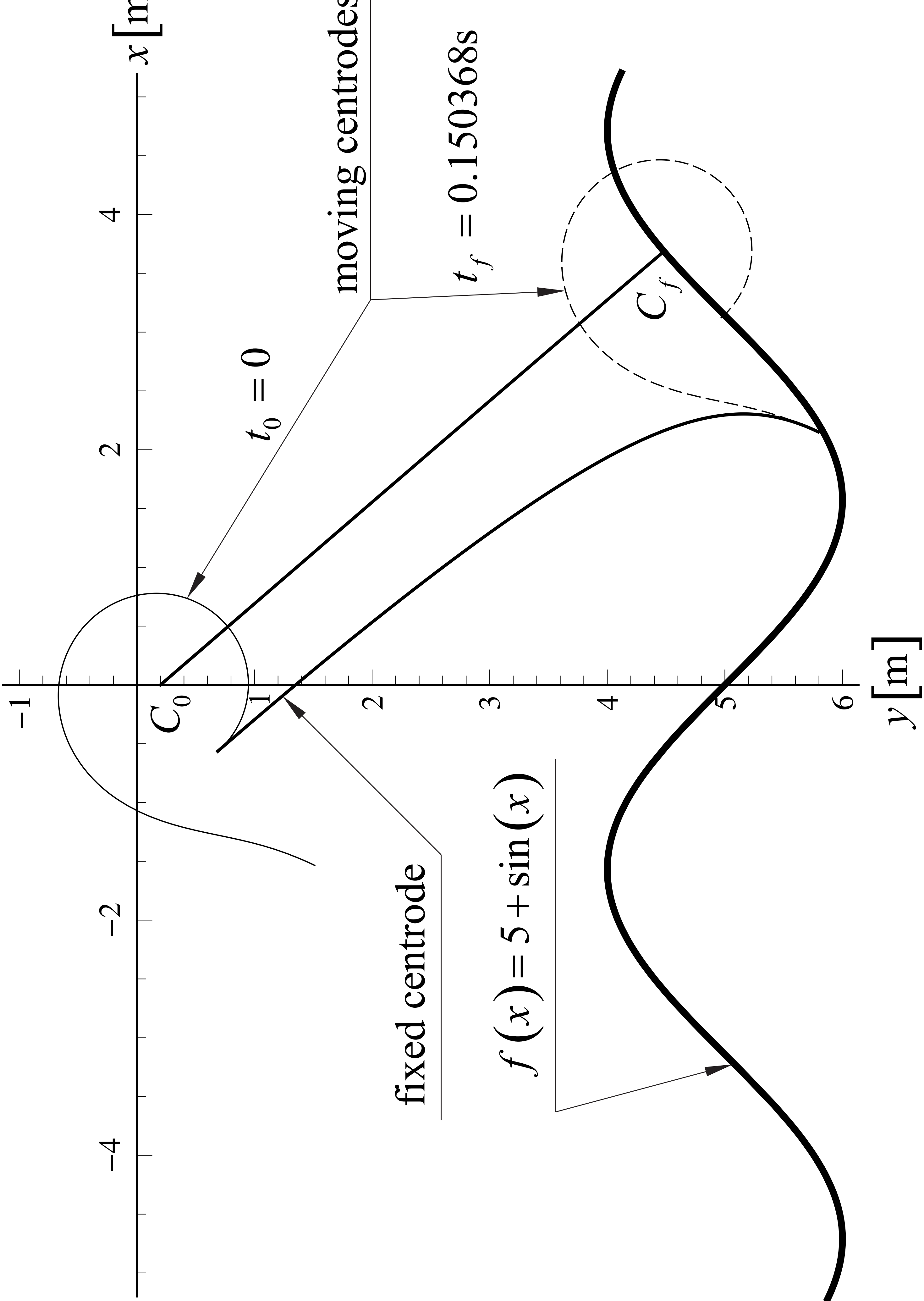

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