D-decomposition method for stabilization of inverted pendulum using fractional order PD controller

Petar D. Mandić, Mihailo P. Lazarević, and Tomislav B. Šekara

***Abstract*— This paper deals with stability problem of inverted pendulum controlled by a fractional order PD controller. D-decomposition method for determining stability region in controller parameters space is hereby presented. The D-decomposition problem for linear systems is extended for linear fractional systems and for the case of linear parameters dependence. Knowledge of stability regions enables tuning of the fractional order PD controller.**

***Index Terms*— fractional order PID, D-decomposition, asymptotic stability, inverted pendulum.**

I. Introduction

The inverted pendulum is one of the most interesting problems in control theory and has been studied through many researches in control community. It is nonlinear, unstable and underactuated system, and thus an excellent benchmark for testing different control algorithms. On the other hand, in recent years considerable attention has been paid to fractional calculus and its application[1,2]. In control theory fractional order controllers are used to improve the performance of closed loop systems. Among them, fractional order PID controllers are the ones most frequently used and were first introduced in [3,4]. It has been shown that fractional order PID controller enhances the system control performance when used with integer order and fractional order plants[5].

One of the basic requirements in control systems is their asymptotic stability. There are several methods for determining stability region of a closed loop system. D-decomposition is one of them. Using this method, parameter plane is decomposed by the so called boundaries of D-decomposition into finite number regions D(k). The region D(0), if existing, is the stability region, i.e. it guarantees the asymptotic stability of the closed loop system. In this paper, D-decomposition method is applied to the inverted pendulum case, and determining its stability regions in parameters space of a fractional order PD controller is presented. D-decomposition for linear fractional systems is investigated, and for the case of linear parameters dependence. This technique enables efficient computational method for determining the asymptotic stability region. When stability regions are known, tuning of the fractional order controller can be carried out.

First, mathematical model of rotational inverted pendulum is presented. Then, a fractional order PD controller is introduced in order to stabilize the pendulum. Method for tuning the parameters of fractional order controller is given, using the abovementioned D-decomposition method. At the end, example is given and tests with different controller parameters are compared and analyzed in Matlab Simulink environment.

II. Mathematical model and controller design

A. Mathematical model of Furuta pendulum

In Fig.1 a schematic view of rotational inverted pendulum, also known as Furuta pendulum, is shown. It is a mechanical system with two degrees of freedom, where angular position of the arm and the pendulum are denoted as θ and φ, respectively. The arm is driven with a torque, while no torque is applied directly to the pendulum. Hence, it is an underactuated mechanical system because it has only one control input and two degrees of freedom.

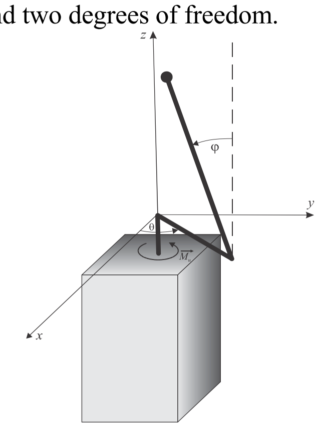

Fig. 1. A schematic view of the Furuta pendulum

Parameters of the system are: m_1 - mass of the arm, m_2 - mass of the pendulum, R_1 - distance of the arm's pivot point to the pendulum's pivot point, R_2 - distance of the pendulum's pivot point to its end (extreme), $2r_1$, $2r_2$ - total length of the arm, and pendulum respectively, $J_{\varsigma 1}$ - moment of inertia of the arm with respect to its center of mass, $J_{\xi 2}, J_{\eta 2}, J_{\zeta 2}$ -axial

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moments of inertia of the pendulum with respect to its center of mass.

Herein, the Rodriguez method is proposed for modeling the dynamics of the system where configuration of the mechanical model can be defined by generalized coordinates q_1 and q_2 represent by θ and φ, respectively. The equations of motion of the inverted pendulum can be expressed in a covariant form of Langrange's equation of second kind as follows [6,7]:

$$\sum_{\alpha=1}^{n} a_{\gamma\alpha}\ddot{q}_\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}_\alpha\dot{q}_\beta = Q_\gamma \quad \gamma = 1,2 \tag{1}$$

wherein the coefficients $a_{\alpha\beta}$ are the covariant coordinates of the basic metric tensor $\left[a_{\gamma\alpha}\right] \in R^{2\times 2}$ and $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2$ presents Christoffel symbols of the first kind. The generalized forces Q_γ can be presented in the following expression (2), wherein Q_γ^g, Q_γ^a denote the generalized gravitational and control forces, respectively.

$$Q_\gamma = Q_\gamma^g + Q_\gamma^a, \;\; \gamma = 1,2 \tag{2}$$

M will denote external torque which is applied to the arm. The equations of motion of our system can be rewritten in full form:

$$a_{11}\ddot{\theta} + a_{12}\ddot{\varphi} + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{22,1}\dot{\varphi}^2 = M \tag{3}$$

$$a_{12}\ddot{\theta} + a_{22}\ddot{\varphi} - \Gamma_{12,1}\dot{\theta}^2 = Q_2^g \tag{4}$$

wherein:

$$\begin{aligned}
a_{11} &= J_{\varsigma 1} + J_{\eta 2}\sin^2(\varphi) + J_{\varsigma 2}\cos^2(\varphi) + m_2 R_1^2 + m_1(R_1 - r_1)^2 \\
&\quad + m_2(R_2 - r_2)^2\sin^2(\varphi), \\
a_{12} &= -m_2 R_1(R_2 - r_2)\cos(\varphi) = -K_3\cos(\varphi), \\
a_{22} &= J_{\xi 2} + m_2(R_2 - r_2)^2 = K_4, \\
\Gamma_{12,1} &= 0.5\left(m_2(R_2 - r_2)^2 + J_{\eta 2} - J_{\varsigma 2}\right)\sin(2\varphi) = K_2\sin(2\varphi), \\
\Gamma_{22,1} &= m_2 R_1(R_2 - r_2)\sin(\varphi) = K_3\sin(\varphi), \\
Q_2^g &= m_2 g(R_2 - r_2)\sin(\varphi) = K_1\sin(\varphi)
\end{aligned} \tag{5}$$

For simplicity, we introduce physical parameters K_1, K_2, K_3, K_4 which are defined as shown above.

B. Controller design

In this section a control strategy is developed to stabilize the pendulum in upright position. This problem is usually divided into two different control problems. The first one is to swing up the pendulum from the down to the upright position, and it is usually solved with energy control strategies. Once the pendulum is close to the desired upright position, the swing up controller switches to balancing controller and stabilizes the pendulum. The swing up strategy will not be considered here since the goal of this paper is not building an accurate swing up controller, but stabilizing the pendulum and finding stability region by using fractional order PD controller.

A nonlinear technique known as inverse dynamic control is used for pendulum stabilization. It is basically a partial feedback linearization procedure [8,9], which simplifies the control design. The first step of this procedure is to calculate $\ddot{\theta}$ from (4) and plug it into (3). After rearranging, (3) now reads:

$$\frac{a_{11}}{a_{12}}\left(Q_2^g + \Gamma_{12,1}\dot{\theta}^2\right) + \left(a_{12} - \frac{a_{11}a_{22}}{a_{12}}\right)\ddot{\varphi} + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{22,1}\dot{\varphi}^2 = M\,. \tag{6}$$

We can see that $\ddot{\theta}$ has been cancelled out in (6). Now, control input M can be chosen as follows:

$$M = \frac{a_{11}}{a_{12}}\left(Q_2^g + \Gamma_{12,1}\dot{\theta}^2\right) + \left(a_{12} - \frac{a_{11}a_{22}}{a_{12}}\right)M_R + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{22,1}\dot{\varphi}^2\,. \tag{7}$$

wherein M_R stands for new control input. Now, (3) and (4) become:

$$\ddot{\theta} = -\frac{K_1}{K_3}\tan(\varphi) - 2\frac{K_2}{K_3}\sin(\varphi)\dot{\theta}^2 + \frac{K_4}{K_3}\frac{M_R}{\cos(\varphi)} \tag{8}$$

$$\ddot{\varphi} = M_R\,, \tag{9}$$

wherein physical parameters K_1, K_2, K_3, K_4 are defined in (5). The control signal is defined in each position of the pendulum except for the horizontal, i.e. $|\varphi| < \pi/2$. Now, we can linearize system described with (8)-(9) around equilibrium point $\left(\theta,\dot{\theta},\varphi,\dot{\varphi}\right) = (0,0,0,0)$. A controller derived from a linearized system will work for a nonlinear system, provided that region of attraction is not too large. Under this condition, linearization allows us to neglect nonlinear, quadratic term $\dot{\theta}^2$ in (8). So, linearization around desired equilibrium point leads to:

$$\ddot{\theta} = -\frac{K_1}{K_3}\varphi + \frac{K_4}{K_3}M_R\,, \tag{10}$$

$$\ddot{\varphi} = M_R\,. \tag{11}$$

If we only want to achieve asymptotic stability for $(\varphi,\dot{\varphi})$, a simple PD controller with the form $M_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi}$ can be used. It stabilizes the inverted pendulum for each $K_{P\varphi}, K_{D\varphi} > 0$. However, it does not stabilize the arm. The reason for this is that underactuated systems such as inverted pendulum are difficult to implement full feedback linearization procedure. Therefore, the new goal is to improve M_R so that asymptotic stability for $\left(\varphi,\dot{\varphi},\theta,\dot{\theta}\right)$ can be accomplished. To achieve this, an extended fractional order PID controller is proposed, as a generalization of the PID controller [10]. The control feedback law will be extended as follows:

$$M_R = -\left(K_{P\theta}\theta + K_{D\theta}\theta^{\alpha}\right) - \left(K_{P\varphi}\varphi + K_{D\varphi}\varphi^{\beta}\right) + \frac{K_1}{K_4}\varphi , \quad (12)$$

wherein $K_{P\theta}, K_{D\theta}, K_{P\varphi}, K_{D\varphi}$ denote proportional and differential gains of the controller, and α, β real differentiator parameters. After substituting (12) into (10) and (11), we obtain:

$$\ddot{\theta} + \frac{K_4}{K_3}K_{D\theta}\theta^{\alpha} + \frac{K_4}{K_3}K_{P\theta}\theta = -\frac{K_4}{K_3}K_{D\varphi}\varphi^{\beta} - \frac{K_4}{K_3}K_{P\varphi}\varphi , \quad (13)$$

$$\ddot{\varphi} + K_{D\varphi}\varphi^{\beta} + \left(K_{P\varphi} - \frac{K_1}{K_4}\right)\varphi = -K_{D\theta}\theta^{\alpha} - K_{P\theta}\theta . \quad (14)$$

We can notice that the last term on the right side of (12) is introduced to cancel out term which contains φ in (10). Taking $\alpha = 1$ and $\beta = 1$ we obtain classical PD controller. Six parameters $\left(K_{P\theta}, K_{D\theta}, K_{P\varphi}, K_{D\varphi}, \alpha, \beta\right)$ in (12) can be changed in order to achieve asymptotic or relative stability of closed loop system. The goal of this paper is to determine the influence of $K_{D\theta}$ and $K_{D\varphi}$ parameters on asymptotic stability of system described with (13)-(14).

III. D-DECOMPOSITION METHOD

Using the classical D-decomposition method [11,12] the stability region in the parameter plane $\left(K_{D\theta}, K_{D\varphi}\right)$ may be determined. The characteristic polynomial of the closed loop system described with (13)-(14) is given by:

$$\begin{aligned} f(s) = s^4 + s^2\left(K_4K_{D\theta}s^{\alpha}/K_3 + K_{D\varphi}s^{\beta}\right) - K_1K_{D\theta}s^{\alpha}/K_3 + \\ + s^2\left(K_4K_{P\theta}/K_3 + K_{P\varphi} - K_1/K_4\right) - K_1K_{P\theta}/K_3 \end{aligned} . \quad (15)$$

The plane $\left(K_{D\theta}, K_{D\varphi}\right)$ is decomposed by the boundaries of the D-decomposition into finite number regions D(*k*). Any point in D(*k*) corresponds to such values of $K_{D\theta}$ and $K_{D\varphi}$ that polynomial (15) has exactly *k* zeroes with positive real parts [13,14]. The region D(0) represents the stability region. The stability boundaries are curves on which each point corresponds to polynomial (15) having zeroes on the imaginary axes[15]. It may be the real zero boundary, the complex zero boundary, or the singular line [16].

Real zero boundary is defined by the equation $f(0) = 0$. It is easy to see that that polynomial (15) has no zero $s = 0$ if $K_{P\theta} \neq 0$, which will be the case in this paper. The complex zero boundary corresponds to the pure imaginary zeroes of (15). We obtain this boundary by solving the equation:

$$\begin{aligned} & w^4 - w^2\left(K_4K_{D\theta}\left(jw\right)^{\alpha}/K_3 + K_{D\varphi}\left(jw\right)^{\beta}\right) - K_1K_{D\theta}\left(jw\right)^{\alpha}/K_3 \\ & -w^2\left(K_4K_{P\theta}/K_3 + K_{P\varphi} - K_1/K_4\right) - K_1K_{P\theta}/K_3 = 0 \end{aligned} \quad (16)$$

which we get by substituting $s = jw$ in polynomial (15) and equating it to 0. The complex equation (16) can be rewritten as:

$$f(jw) = u\left(w,\alpha,\beta\right) + jv\left(w,\alpha,\beta\right) = 0, \quad (17)$$

where $u\left(w,\alpha,\beta\right)$ and $v\left(w,\alpha,\beta\right)$ denote the real and imaginary part of (16). Terms $\left(jw\right)^{\alpha}$ and $\left(jw\right)^{\beta}$ which are required for (16) can be expressed as [17,18]:

$$\left(jw\right)^{\alpha} = w^{\alpha}\left(\cos\left(\alpha\pi/2\right) + j\sin\left(\alpha\pi/2\right)\right), \quad w \geq 0 \quad (18)$$

Then, equating the real and imaginary part of (17) to zero, one obtains the following 2-D system:

$$\begin{bmatrix} U_1\left(w,\alpha,\beta\right) & U_2\left(w,\alpha,\beta\right) \\ V_1\left(w,\alpha,\beta\right) & V_2\left(w,\alpha,\beta\right) \end{bmatrix} \begin{pmatrix} K_{D\theta} \\ K_{D\varphi} \end{pmatrix} = \begin{pmatrix} Q_1(w) \\ Q_2(w) \end{pmatrix} \quad (19)$$

wherein

$$\begin{aligned} & a = K_4/K_3, \quad b = K_1/K_3, \\ & U_1\left(w,\alpha,\beta\right) = \left(aw^2 + b\right)w^{\alpha}\cos(0.5\alpha\pi), \\ & U_2\left(w,\alpha,\beta\right) = w^{2+\beta}\cos(0.5\beta\pi), \\ & V_1\left(w,\alpha,\beta\right) = \left(aw^2 + b\right)w^{\alpha}\sin(0.5\alpha\pi), \\ & V_2\left(w,\alpha,\beta\right) = w^{2+\beta}\sin(0.5\beta\pi), \\ & Q_1(w) = w^4 - w^2(aK_{P\theta} + K_{P\varphi} - b/a) - \mathrm{b}\,K_{P\theta}, \\ & Q_2(w) = 0. \end{aligned} \quad (20)$$

Solving it for parameters $\left(K_{D\theta}, K_{D\varphi}\right)$, we obtain:

$$K_{D\theta} = \frac{\Delta_{\theta}}{\Delta}, \quad K_{D\varphi} = \frac{\Delta_{\varphi}}{\Delta}, \quad (21)$$

wherein

$$\Delta = \begin{vmatrix} U_1\left(w,\alpha,\beta\right) & U_2\left(w,\alpha,\beta\right) \\ V_1\left(w,\alpha,\beta\right) & V_2\left(w,\alpha,\beta\right) \end{vmatrix}, \quad (22)$$

$$\Delta_{\theta} = \begin{vmatrix} Q_1(w) & U_2\left(w,\alpha,\beta\right) \\ 0 & V_2\left(w,\alpha,\beta\right) \end{vmatrix}, \quad \Delta_{\varphi} = \begin{vmatrix} U_1\left(w,\alpha,\beta\right) & Q_1(w) \\ V_1\left(w,\alpha,\beta\right) & 0 \end{vmatrix}. \quad (23)$$

It can be easily shown that:

$$\Delta = \left(aw^2 + b\right)w^{\alpha+\beta+2}\sin(0.5\left(\beta - \alpha\right)\pi) \quad (24)$$

For $\Delta \neq 0$, (21) describe a curve in the $(K_{D\theta}, K_{D\varphi})$ plane representing the complex zero boundary, for the fixed values $K_{P\theta}, K_{P\varphi}, \alpha$ and β, as w runs from 0 to ∞. In crossing this curve, two roots move from one half plane to another.

Now, a more detailed analysis must be done when $\Delta = 0$. It follows from (24) that this is true for $w = 0$ or $\beta - \alpha = k, \forall k = 0, \pm 2, \pm 4, \ldots$. For the first case when $w = 0$, (20) can be written as:

$$\begin{aligned} &U_1(0,\alpha,\beta) = 0,\ U_2(0,\alpha,\beta) = 0,\ Q_1(0) = -\mathrm{b}\,K_{P\theta}, \\ &V_1(0,\alpha,\beta) = 0,\ V_2(0,\alpha,\beta) = 0,\ Q_2(w) = 0. \end{aligned} \tag{25}$$

It follows from (19) and (25) that $0 = -\mathrm{b}\,K_{P\theta}$. This cannot be true for $K_{P\theta} \neq 0$, so the system (19) have no real solutions when $w = 0$. In the second case, $\Delta = 0$ for $\beta - \alpha = k$, $\forall k = 0, \pm 2, \pm 4, \ldots$. The fractional orders α and β are in the range from 0 to 1, and therefore it follows $\beta - \alpha = 0$. Now, for $\beta = \alpha$, (20) reads:

$$\begin{aligned} &U_1(w,\alpha,\alpha) = (aw^2 + b)\, w^\alpha \cos(0.5\alpha\pi), \\ &U_2(w,\alpha,\alpha) = w^{2+\alpha} \cos(0.5\alpha\pi), \\ &V_1(w,\alpha,\alpha) = (aw^2 + b)\, w^\alpha \sin(0.5\alpha\pi), \\ &V_2(w,\alpha,\alpha) = w^{2+\alpha} \sin(0.5\alpha\pi), \\ &Q_1(w) = w^4 - w^2(aK_{P\theta} + K_{P\varphi} - b/a) - \mathrm{b}\,K_{P\theta}, \\ &Q_2(w) = 0. \end{aligned} \tag{26}$$

Equation (19) can be rewritten as:

$$\begin{aligned} &w^\alpha \cos(0.5\alpha\pi)\left[(aw^2 + b)K_{D\theta} + w^2 K_{D\varphi}\right] = \\ &w^4 - w^2(aK_{P\theta} + K_{P\varphi} - b/a) - \mathrm{b}\,K_{P\theta}, \end{aligned} \tag{27}$$

$$(aw^2 + b)K_{D\theta} + w^2 K_{D\varphi} = 0. \tag{28}$$

which leads to:

$$d(w) = w^4 - w^2(aK_{P\theta} + K_{P\varphi} - b/a) - \mathrm{b}\,K_{P\theta} = 0. \tag{29}$$

Frequency w_s for which $d(w_s) = 0$ determines singular line. In this case $\Delta = \Delta_\theta = \Delta_\varphi = 0$, and D-decomposition contains not a point, but a whole line. This singular line can be obtained from either (27) or (28), and it reads:

$$K_{D\varphi} = -\left(a + \frac{b}{w_s^2}\right) K_{D\theta} \tag{30}$$

Equations (21) and (30) determine the stability boundaries in parameter space $(K_{D\theta}, K_{D\varphi})$ for the fixed values $K_{P\theta}, K_{P\varphi}, \alpha$ and β.

IV. SIMULATION RESULTS

In this section, simulation results of system described with (13)-(14) are presented. Using the D-decomposition method, parameter space $(K_{D\theta}, K_{D\varphi})$ can be divided into stable and unstable regions. The stable region can be found by checking one arbitrary test point within each region, and testing the stability of polynomial (15) using the inverse Laplace transformation. In this paper, only the stability region D(0) is presented.

Physical parameters K_1, K_2, K_3, K_4 are defined in (5) and are taken from the real laboratory model of Furuta pendulum. They have the following values: $K_1 = 6.514e-2$, $K_2 = 9.186e-4$, $K_3 = 1.428e-3$, and $K_4 = 1.837e-3$. Also, controller gains $K_{P\theta}$ and $K_{P\varphi}$ are predetermined and chosen as: $K_{P\theta} = -0.0219$, $K_{P\varphi} = 41.48$. Now, the influence of $(K_{D\theta}, K_{D\varphi})$ parameters on stability property of system can be investigated using the D-decomposition approach. For the case $\alpha = [0,1]$, $\beta = 1$, stability regions are plotted as shown in Fig. 2.

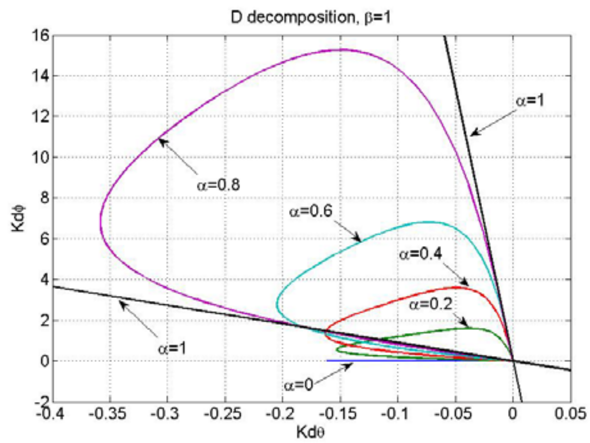

Fig. 2. Stability regions for $\alpha = [0,1]$, $\beta = 1$.

By varying α and repeating the D-partition procedure, different stability regions are obtained. The global stability region can then be visualized in a 3D plot as shown in Fig.3. It can be seen from these figures that larger values of α provide bigger stability region. For $\alpha = 1$ stability region is determined with following singular lines:

$$K_{D\varphi} = -267.2 K_{D\theta},\ K_{D\varphi} = -9.11 K_{D\theta} \tag{31}$$

Based on Lyapunov's indirect method theorem [5], we can draw the conclusion about stability of the nonlinear system by

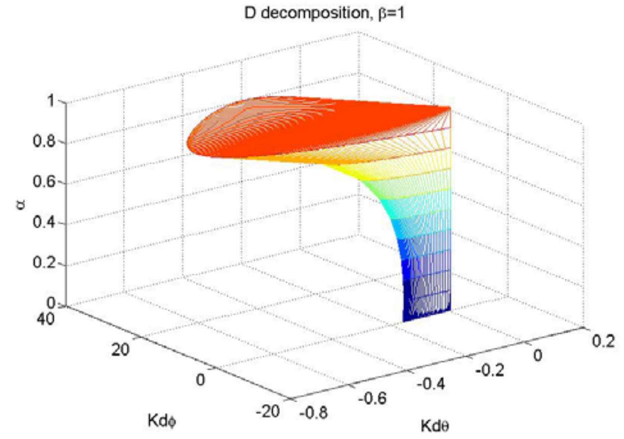

Fig. 3. 3D Stability regions for $\alpha = [0,1)$, $\beta = 1$.

investigating the stability of its linearized model. In other words, stability regions obtained in above examples for the system (10)-(11), will be the same as for the nonlinear system (8)-(9), but only in the small area of the equilibrium point.

By varying β parameter while α remains constant, following stability regions are obtained, as shown in Fig. 4.

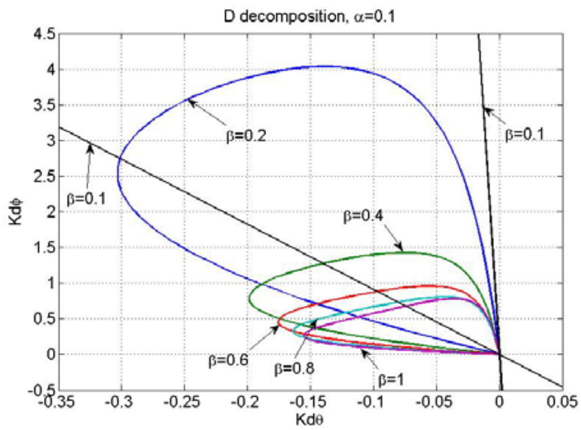

Fig. 4. Stability regions for $\beta = [0.1,1]$, $\alpha = 0.1$.

3D representation of above figure can be seen in Fig. 5.

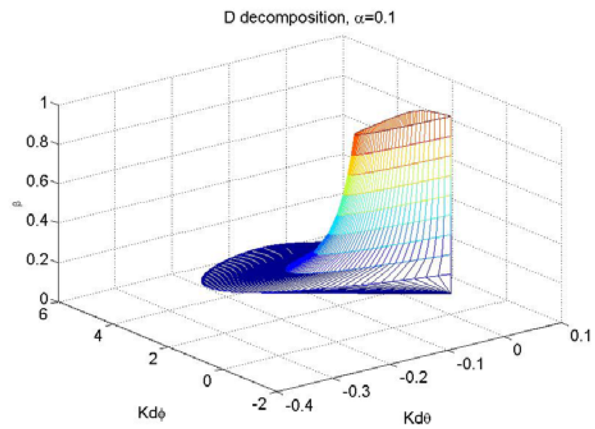

Fig. 5. 3D Stability regions for $\beta = (0.1, 1]$, $\alpha = 0.1$.

In these figures, the values of β are taken in the range (0.1,1] for better visibility. As mentioned earlier, the stability region D(0) in all examples is chosen by testing an arbitrary point and checking the stability of polynomial (15). Herein, an example of the aforementioned procedure will be shown. In Fig. 6 three different points for $\alpha = 0.7$, $\beta = 1$ are chosen and tested using the impulse response. All simulations presented are performed using Matlab software.

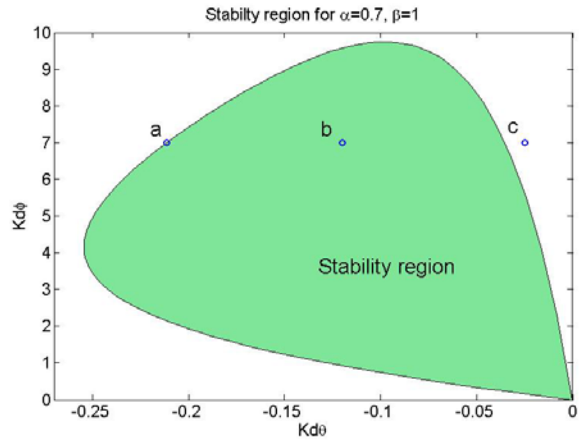

Fig. 6. Stability region testing.

Points are marked as *a*, *b* and *c*. As we can see, points *c* lies outside the stability region. System whose parameters $(K_{D\theta}, K_{D\varphi})$ are determined by point *b* should be stable, while the one with point *a* should be on the stability margin.

Impulse response of $1/f(s)$ for each depicted point is obtained using the inverse Laplace transformation. The results are shown in Fig. 7.

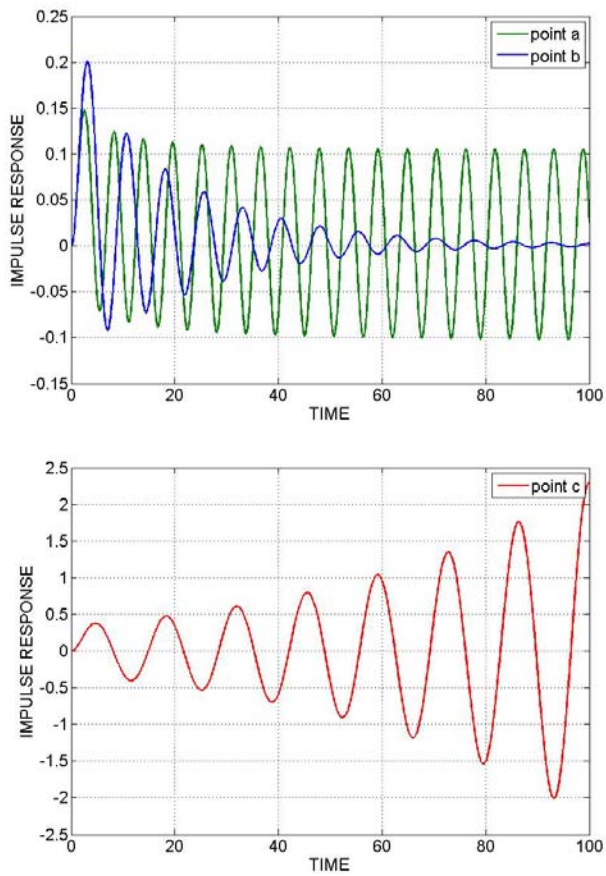

Fig. 7. Impulse response for points *a*, *b* and *c*.

Impulse responses are as we expected, which confirms that D-decomposition procedure is well derived and stability domain is properly chosen.

V. Conclusion

In this paper, the stability problem of Furuta pendulum controlled by fractional order PD controller is given. Mathematical model of rotational inverted pendulum is derived and fractional order PD controller is introduced in order to stabilize it. The problem of asymptotic stability of closed loop system is solved using the D-decomposition approach. On the basis of this method, analytical forms expressing the boundaries of stability regions in the parameters space were determined. The D-decomposition technique is extended for linear fractional order systems and for the case of linear parameter dependence. An example is given and tests are made in order to confirm that stability domains are well calculated. Knowledge of these stability regions enables tuning of the fractional order PD controller.

More detailed analysis of D-decomposition technique for fractional order systems will be a subject of future research. Also, a transfer from simulation to real laboratory model of inverted pendulum will be considered.

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