COMPARATIVE ANALYSIS OF THE STANDARD LINEAR SOLID MODEL

Nevena A. Arandjelovic[1], Mihailo P. Lazarevic[1]

[1] Faculty of Mechanical Engineering,
The University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35
e-mail: narandjelovic@mas.bg.ac.rs, mlazarevic@mas.bg.ac.rs

Abstract:

In this paper we have analyzed the standard linear solid model (Zener model) of viscoelastic materials. A one-dimensional analysis was executed analytically (in MATLAB) and numerically (in ABAQUS), and corresponding graphs have been produced. The fractional differential form of the Zener model was then implemented, as it has proven to be a better match for the experimental test results for viscoelastic materials. Finally, all of the results were compared and discussed.

Key words: standard linear solid, Zener model, viscoelasticity, comparative analysis, ABAQUS, FEM, fractional calculus

1. Introduction

Viscoelastic constitutive models [1] have been in the center of attention of the scientific community, as the possibilities of computational analysis are on the rise and many materials are described more accurately by these models, in comparison to the pure elastic or viscous models.

There are many materials which display a stress relaxation or creep when a constant load or deformation is applied. These qualities, along with the occurrence of hysteresis, have been described by various viscoelastic models (Maxwell, Kelvin-Voigt, Zener, etc.) [4,5,6]. The adequacy of a certain model relies on the properties of the observed material.

In this paper, the Zener model will be analyzed. This model is also known as one of the standard linear solid models. All of the models are based on springs and dashpots, which are connected in different ways to form an approximate structure with its corresponding relationship between stress and strain (constitutive model in the form of a differential equation) [6]. The Zener model is also applied in ABAQUS, commercial software which can be used for finite element method (FEM) [2] structural analysis [7].

However, recent studies have shown that fractional viscoelastic constitutive models are in better compliance with experimental test results [4,6]. Hence, the implementation of fractional calculus [3] to these structural problems is vital and a subject of current scientific enquiry.

2. Zener model

The Zener model is shown in Fig. 1. It consists of one spring which is tied in parallel with a second spring and a dashpot, which are tied in series. This standard linear solid model is widely used, as it shows a good compliance with experimental data of many different materials.

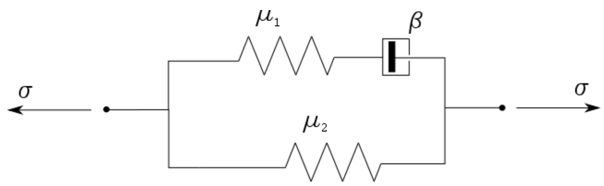

Fig. 1. Zener model.

The constitutive equation of this model is [6]:

$$\dot{\sigma} + \frac{1}{\tau_1}\sigma = \frac{\mu_1 \mu_2}{\beta}\varepsilon + (\mu_1 + \mu_2)\dot{\varepsilon}, \tag{1}$$

where $\tau_1 = \frac{\beta}{\mu_1}$. Also, if we introduce $\tau_2 = \frac{\mu_2}{\tau_1(\mu_1 + \mu_2)}$, then the previous equation takes the form:

$$\tau_1 \dot{\sigma} + \sigma = \mu_2\left(\varepsilon + \frac{1}{\tau_2}\dot{\varepsilon}\right), \tag{2}$$

These coefficients have to obey the following restrictions, so that the second law of thermodynamics would be satisfied [8,9,10]:

$$\mu_2 > 0, \ \tau_1 > 0 \ \text{and} \ \frac{1}{\tau_2} > \tau_1, \tag{3}$$

After the Laplace transform and taking into consideration that there is a constant deformation, which corresponds to a strain:

$$\varepsilon(t) = H(t), \tag{4}$$

where $H(t)$ is the Heaviside step function, the relaxation modulus is obtained:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1 \tau_2} - 1\right)e^{-\frac{t}{\tau_1}}, \tag{5}$$

The coefficients can also be written in the next form:

$$E^*(t) = k_1 + k_2 e^{-\frac{t}{\tau_1}}, \tag{6}$$

where $k_1 = \mu_2$ and $k_2 = \mu_2\left(\frac{1}{\tau_1 \tau_2} - 1\right)$.

3. Viscoelastic constitutive model in ABAQUS

In ABAQUS, the hereditary integral [4] is used to define the relationship between stress and strain, in the general sense of the possibility of a gradually increasing load:

$$\sigma(t) = \int_0^t E^*(t-\tau)\dot{\varepsilon}(\tau)d\tau, \tag{7}$$

where E^* is the relaxation function corresponding to axial loading, ε denotes the strain, t is reference time, and τ is current time.

The dimensionless relaxation modulus is defined by a Prony series expansion [7]:

$$E^*(t) = E^*(0) - E^*(0)\sum_{i=1}^{N} \overline{E}_i^{*p}\left(1 - e^{-\frac{t}{\tau_i}}\right), \tag{8}$$

which is, for the Zener model, according to the equation (5), reduced to [5]:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1\tau_2} - 1\right)e^{-\frac{t}{\tau_i}} = k_1 + k_2 e^{-\frac{t}{\tau_i}}, \tag{9}$$

According to sample material constants given in [5], the non-symblic form of the previous equation is:

$$E^*(t) = 100 + 900e^{-\frac{t}{100}}, \tag{10}$$

From the equation (10), we can calculate the material constants which are required in ABAQUS:

$$E^*(0) = E = 1000, \tag{11}$$

$$g_i = e_i = \frac{900}{E} = 0.9, \tag{12}$$

$$\tau_i = 100, \tag{13}$$

These coefficients comply with the restricitons (3). ABAQUS also requries the Poisson ratio value, which can be calulated using the known relationship among Young modulus, Poisson ratio and Bulk coefficient [5]:

$$\nu = 0.4833, \tag{14}$$

Thus, we have obtained all the neccesary material constants for numerical analysis in ABAQUS.

4. Fractional Zener model

The fractional calculus is introduced simply by substituting the integer order derivatives with derivatives of order γ, which can be any real number between 0 and 1, $\gamma \in (0,1)$. Thus, the constitutive differential equation of the Zener model takes the form:

$$\frac{d^\gamma \sigma}{dt^\gamma} + \frac{1}{\tau_1}\sigma = \frac{\mu_1\mu_2}{\beta}\varepsilon + (\mu_1 + \mu_2)\frac{d^\gamma \varepsilon}{dt^\gamma}, \tag{15}$$

After the Laplace transform, and taking into account the strain (4), relaxation modulus is obtained [6]:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1\tau_2} - 1\right)E_\gamma\left(-\left(\frac{t}{\tau_1}\right)^\gamma\right), \tag{16}$$

where E_γ is the Mittag-Leffler one-parameter function, which for the large values of the variable (t), reduces to [8]:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1\tau_2} - 1\right)\frac{1}{\Gamma(1-\gamma)}\left(-\left(\frac{t}{\tau_1}\right)^{\gamma}\right), \tag{17}$$

where Γ is the Euler's Gamma function. However, as stated in [12], recent studies have shown that the Mittag-Leffler function for the large values of the variable can also be reduced like so:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1\tau_2} - 1\right)\frac{1}{1+\left(\frac{t}{\tau_1}\right)^{\gamma}\Gamma(1-\gamma)}, \tag{18}$$

The previous equation (18) will be used for results comparison, denoted as "fractional A".

Following the generalization principle undertaken in [4] for the Maxwell model, a similar equation to the equation (17) is obtained for the Zener model:

$$E^*(t) = \mu_2 + \mu_2\left(\frac{1}{\tau_1\tau_2} - 1\right)\frac{1}{\Gamma(1-\gamma)}\left(\frac{t}{\tau_1}\right)^{-\gamma}, \tag{19}$$

This equation will be denoted as "fractional B".

The comparison between the results of the equations (18) and (19) for different values of γ is shown in Fig.2.

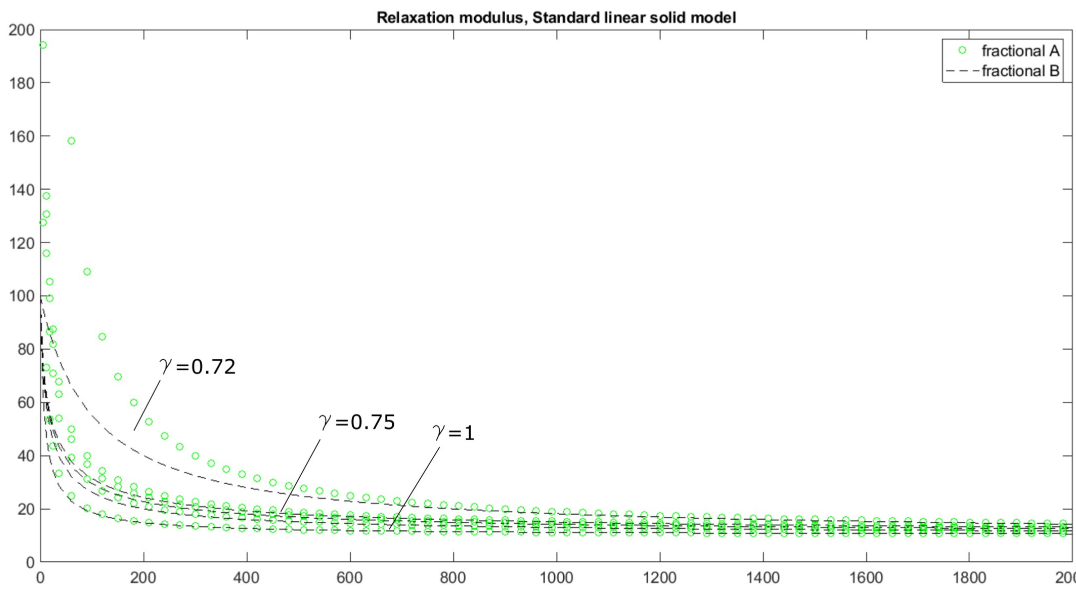

Fig. 2. Relaxation modulus comparison for analytical, ABAQUS and fractional solutions.

It is obvous that, as the value of γ approaches 1 $(\gamma \to 1)$ the values of these two equations coincide.

5. Results

The results are shown in Fig.3. As expected, the results obtained from ABAQUS coincide completely with the results of analitical calculation undertaken in MATLAB. The results obtained from the fractional forms of the constitutive model equation assimptoticly approach the previous results, for the large values of time. However, that does not mean that these results are less accurate, as it has been proven that the power law is in a better accordance with the experimental test results, than the exponential law [4,6].

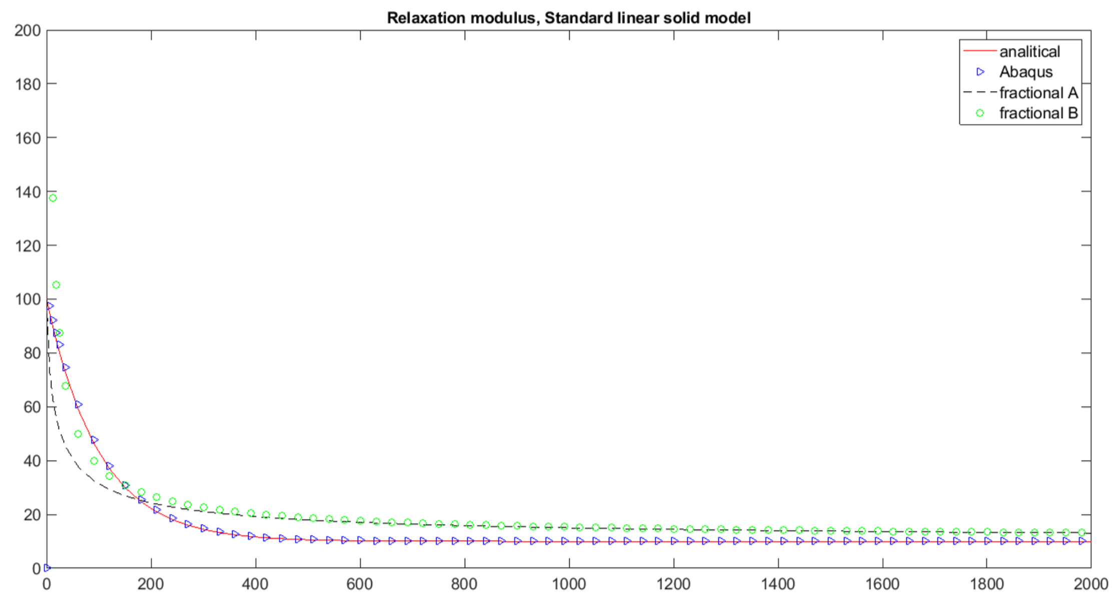

Fig. 3. Relaxation modulus comparison for analytical, ABAQUS and fractional solutions.

6. Conclusions

In this paper one of the conventional viscoelastic constitutive models was presented (the Zener model). Its implementation in ABAQUS, via Prony series, was explained and the solution for one-dimensional problem was presented. It was shown that this solution coincides completely with the analytical solution. Furthermore, fractional calculus was introduced to said model, and solved with the application with two approximations of the Mittag-Leffler function. All of the results were compared. This analysis proved to be very suggestive and, in further work, the experimental test data will be produced and the final comparison executed. The two-dimensional and three-dimensional models will be analyzed, as well.

7. Acknowledgements

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