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FINITE-TIME STABILITY OF NEUTRAL FRACTIONAL-ORDER TIME-VARYING DELAY SYSTEMS WITH NONLINEAR PARAMETER UNCERTAINTIES AND PERTURBATIONS

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Abstract:

In this contribution, the problem of finite-time stability for a class of neutral fractional-order time-varying delay systems with nonlinear parameter uncertainties and perturbations is investigated. By use of the extended form of generalized Gronwall inequality, a new sufficient condition for robust finite-time stability of such systems is obtained. Finally, a numerical example is provided to illustrate the effectiveness and applicability of the proposed theoretical results.

Key words: finite-time stability, fractional order, nonlinear, neutral, time-varying delay, perturbed systems

1. Introduction

Time-delay often appears in many real-world engineering systems either in the state, the control input, or the measurements, [1]. Stability and control design of time-delay systems are widely studied due to the effect of delay phenomena on system dynamics [2]. In this contribution, we consider system stability in the non-Lyapunov sense- *finite-time stability* (FTS) because the boundedness properties of system responses are very important from the engineering point of view. In the past decades, there has been a growing research interest in the field of finite-time stability and stabilization of time-delay systems which often leads to poor performance or even instability. Finite-time stability was first introduced in the Russian scientific community [3]. Also,due to fact that in practice, the stability of time delay systems may be destroyed by its uncertainties and nonlinear perturbations [4,5], it is necessary to study the FTS analysis of time delay systems with uncertain parameters and nonlinear perturbations.

On the other side, neutral time-delay system is common in many practical engineering application where neutral delay is the leading example of many types of time delay, which not only exists in the system state but also has to do with the derivative of the system state. Also, over the last decade, fractional-order dynamical systems with derivative presented by fractional (non-integer) differential equations have drawn much attention from researchers and engineers since fractional calculus provide an excellent tool for the description of memory and hereditary properties of various materials and processes, [6].

Recently, some authors studied a class of the fractional-delay systems (FDS) retarded type and neutral type, [7,8,9]. An example of FDS of neutral type is a viscoelastic material which is used as damping in vibration systems. The Scot-Blair model assumes that the damping is proportional to a fractional order derivative of the displacement variable with the order of derivative ranging from 0 to 1 can be presented as neutral fractional-delay systems,[10]:

$$ {}^{c}D^{\alpha}\boldsymbol{x}(t)+A^{c}D^{\alpha}\boldsymbol{x}(t-\tau)=Bx(t)+u(t),\alpha=1/2\,, \tag{1.1}$$

Besides, a lot of existing models can be remodify as a neutral fractional differential equations or neutral fractional Volterra integro-differential equations i.e the governing equation of many electromechanical and control systems with fractional terms results in more general form as multi-order fractional delay differential equations (FDDEs) given as

$$ {}^{c}D^{\alpha}\boldsymbol{x}(t)=f\left(t,x(t),{}^{c}D^{\beta}\boldsymbol{x}(t-\tau)\right),0<\beta\le\alpha<1. \tag{1.2}$$

In recent years, there have been some advances in control theory of fractional-order dynamical systems [11], particularly for different kinds of stability. Here, we are interested in FTS where FTS analysis of fractional delay systems is initially investigated and presented in [7,12] using generalized Gronwall inequality.

In literature, there are few results concerning FTS of neutral fractional order time delay systems, see [13,14,15]. Recently, we obtained and presented new criterion which is related to FTS of uncertain neutral nonhomogeneous fractional-order systems FDS with time-varying delays [16].

In this paper, based on the above motivations and discussions, at first time we shall address the finite-time stability problem of neutral fractional order time-varying delay systems with nonlinear parameter uncertainties and external disturbances.

2. Preliminaries

In this section, we consider the main definitions and properties of fractional derivative operators. The following definitions and lemmas are known and help proves our main stability criterion.

Definition 1. The gamma function $\Gamma(\cdot)$ known as the Euler's gamma function is defined as

$$\Gamma(\alpha)=\int_{0}^{\infty}\mathrm{e}^{-t}t^{\alpha-1}\mathrm{d}t,\quad \Gamma(\alpha+1)=\alpha\Gamma(\alpha),\quad \alpha\in C. \tag{2.1}$$

where C be the set of complex numbers.

Definition 2: Let $f(t)$ be a continuous function on $[a,b]$. The Riemann–Liouville fractional integral of order α is [17,18]:

$$ {}_{a}^{\mathrm{RL}}\mathrm{D}_{t}^{-\alpha}f(t)\equiv {}_{a}^{\mathrm{RL}}\mathrm{I}_{t}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-s)^{\alpha-1}f(s)\mathrm{d}s,\quad t\in[a,b],\quad \alpha\in C,\quad \mathrm{Re}(\alpha)>0. \tag{2.2}$$

where $\mathrm{Re}(\alpha)$ denotes real part of α .

The Caputo fractional derivative is defined for a function $f(\cdot):[a,b]\to C$ which belongs to the space of absolutely continuous functions: $f(t)\in AC^n[a,b]=\left\{f(t):\mathrm{d}^{n-1}f(t)/\mathrm{d}t^{n-1}\in AC[a,b]\right\},\quad n\in N.$

Definition 3: The Caputo fractional derivative of order α, $\alpha\in C$, $\mathrm{Re}(\alpha)\ge 0$, for any function $f(t)\in AC^n[a,b]$ is defined as [17,18]:

$$ {}_a^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s, & \alpha\notin N_0,\quad n=\left[\mathrm{Re}(\alpha)\right]+1,\quad n\in N.\\ f^{(n)}(t)=\dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha=n\in N_0.\end{cases} \tag{2.3}$$

Lemma 1. [19] Assume $x(t)\in C^1([0,+\infty),R), \dot{x}(t)\ge 0$ and $\alpha>0$. Then, $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds$ is monotonically increasing with respect to *t*.

Lemma 2. [20] Assume that $0<\beta<\alpha<1$,then.

$$ I_t^{\alpha}\left({}^{c}D_0^{\beta}x(t)\right)=I_t^{\alpha-\beta}x(t)-\frac{x(0)\cdot t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)},\quad t\ge 0 \tag{2.4}$$

Lemma 3. [21] (*Generalized Gronwall Inequality***)** Suppose $x(t), a(t)$ are nonnegative and local integrable on $0\le t<T, T\le+\infty$ and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0\le t<T$, $g(t)\le M=const$, $\alpha>0$ with

$$ x(t)\le a(t)+g(t)\int_0^t (t-s)^{\alpha-1}x(s)ds \tag{2.5}$$

on this interval. Then

$$ x(t)\le a(t)+\int_0^t\left[\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds,\quad 0\le t<T \tag{2.6}$$

Corollary 1 [22]: Under the hypothesis of Lemma1, let $a(t)$ be a nondecreasing function on $[0,T)$. Then it holds:

$$ x(t)\le a(t)E_{\alpha}\left(g(t)\Gamma(\alpha)t^{\alpha}\right) \tag{2.7}$$

where E_{α} is the Mittag-Leffler function defined by $E_{\alpha}(z)=\sum_{k=0}^{\infty}z^k/\Gamma(k\alpha+1)$.

Lemma 4. [19] Suppose fractional orders $\alpha>0,\beta>0$, $a(t)$ is a nonnegative function locally integrable on $[0,T)$, $g_1(t)$ and $g_2(t)$ are nonnegative, nondecreasing, continuous functions defined on $[0,T)$; $g_1(t)\le N_1, g_2(t)\le N_2, (N_1,N_2=const)$. Suppose $x(t)$ is nonnegative and locally integrable on $[0,T)$ with

$$ x(t)\le a(t)+g_1(t)\int_0^t (t-s)^{\alpha-1}x(s)ds+g_2(t)\int_0^t (t-s)^{\beta-1}x(s),\quad t\in[0,T) \tag{2.8}$$

Then,

$$x(t) \le a(t) + \int_0^t \sum_{n=1}^{\infty} \left[g(t)\right]^n \cdot \sum_{k=0}^{n} \frac{C_n^k \left[\Gamma(\alpha)\right]^{n-k} \left[\Gamma(\beta)\right]^k}{\Gamma\left((n-k)\alpha + k\beta\right)} (t-s)^{(n-k)\alpha + k\beta - 1} a(s) ds, \quad t \in [0,T) \quad (2.9)$$

where $g(t) = g_1(t) + g_2(t)$ and $C_n^k = \dfrac{n(n-1)(n-2)...(n-k+1)}{k!}$.

Corollary 2. Under the hypothesis of Lemma 4, let $a(t)$ be a nondecreasing function on $[0,T)$.

Then $x(t) \le a(t) E_\kappa \left[g(t)\left(\Gamma(\alpha) t^\alpha + \Gamma(\beta) t^\beta\right)\right]$, $\kappa = min(\alpha, \beta)$ (2.10)

.

3. Main Results

3.1 Finite-time stability of nonhomogeneous neutral fractional-order time-varying delay systems with nonlinear parameter uncertainties and perturbations

In this section we study the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the class of neutral two-term fractional order system with time-varying delays in state with nonlinear parameter uncertainties and perturbations, presented by state equation:

$$\begin{aligned} &{}^c\mathrm{D}^\alpha \boldsymbol{x}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t - \tau_x(t)) + \\ &+ A_{N1} {}^c\mathrm{D}^\beta \boldsymbol{x}(t - \tau_{xN}(t)) + B_0 \boldsymbol{u}(t) + f(x(t), t) + g(x(t - \tau_x(t)), t) + Cw(t) \end{aligned} \quad (3.1)$$

with the associated continuous function of initial state:

$$\boldsymbol{x}(t) = \boldsymbol{\psi}_x(t), t \in [-\tau_{xm}, 0], \quad (3.2)$$

Let us denote by $C\left(\left[-\tau_{x,M}, 0\right], R^n\right)$ the Banach space of all continuous real vector functions mapping the interval $\left[-\tau_{x,M}, 0\right]$, into R^n with the topology of the uniform convergence where norm of an $\boldsymbol{\psi}$ element is defined as: $\|\psi\|_C = \sup_{-\tau \le \theta \le 0} \|\psi(\theta)\|$. ${}^C\mathrm{D}_t^\alpha, {}^C\mathrm{D}_t^\beta$ denote Caputo fractional derivatives of order α, β, $0 < \beta < \alpha < 1$, a well as $x(t) \in R^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input; A_0, A_1, A_{N1}, B_0 and C are constant matrices with appropriate dimensions; $\tau_x(t), \tau_{xN}(t)$ are time-varying discrete delay and neutral delay in state respectively which satisfy:

$$0 \le \tau_x(t) \le \tau_{xM}, 0 \le \tau_{xN}(t) \le \tau_{xN}, \quad \forall t \in J = [t_0, t_0 + T], \quad t_0 \in R, \quad T > 0 \quad (3.3)$$

and τ_{xm} is defined to be $\max(\tau_{xM}, \tau_{xN})$. Behavior of system (3.1) with given initial function (3.2) is observed over time interval $J = [t_0, t_0 + T] \subset R$, where T may be either a real positive number or symbol ∞. Here, it is introduced next assumption that nonlinear parameter perturbations $f(x(t), t)$ and $g(x(t - \tau_x(t)), t)$ can be described as linear vector functions as follows, [4,5]:

$$f(x(t), t) = \Delta A_0(t) x(t), \quad g(x(t - \tau_x(t)), t) = \Delta A_1(t) x(t - \tau_x(t)) \quad (3.4)$$

where $\Delta A_0(t)$ and $\Delta A_1(t)$ are time-varying parametric structured uncertainties. Also, $w(t)\in R^n$ is the disturbance vector, which has upper bound as follows: $\|w(t)\|<\gamma_w, \forall t\in[0,T]$

In this paper, the norm $\|(\cdot)\|$ will denote any vector norm, i.e. $\|(\cdot)\|_1$, $\|(\cdot)\|_2$, or $\|(\cdot)\|_\infty$, or corresponding matrix norm induced by the equivalent vector norm, i.e. $1-$, $2-$, or $\infty-$norm, respectively. It is assumed the usual smoothness condition, which means that there are no problems with existence, uniqueness and continuity of solutions of systems with respect to initial conditions,[19].

Before proceeding further, the definition of finite-time stability will be given for nonhomogeneous system (3.1) with associated initial function (3.2).

Definition 4 [23]: The time-delay system given by nonhomogeneous state equation (3.1) satisfying initial conditions (3.2) is finite-time stable w.r.t. $\{\delta,\varepsilon,\gamma_u,t_0,J,\|(\cdot)\|\}$, $\delta<\varepsilon$, if and only if:

$$\|\boldsymbol{\psi}_x\|_C<\delta,\quad \|\boldsymbol{u}(t)\|<\gamma_u,\quad \forall t\in J \quad\Rightarrow\quad \|\boldsymbol{x}(t)\|<\varepsilon,\quad \forall t\in J. \tag{3.5}$$

Theorem 1: The nonhomogeneous nonlinear neutral two-term fractional order time varying delay system (3.1) satisfying initial conditions (3.2) is finite-time stable w.r.t. $\{\delta,\varepsilon,\gamma_u,J_0,\|(\cdot)\|\}$, $\delta<\varepsilon$, if the following condition holds:

$$\begin{aligned}&\left[1+\frac{\sigma_{\max}(A_{N1})|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\eta_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\right]E_\kappa\left[\left(\frac{\sigma_{\max}(A_{N1})}{\Gamma(\alpha-\beta)}+\frac{\eta_\Sigma}{\Gamma(\alpha)}\right)\left(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^\alpha\right)\right]\\&+\frac{\gamma_{0u}|t|^\alpha}{\Gamma(\alpha+1)}+\frac{\gamma_{0w}|t|^\alpha}{\Gamma(\alpha+1)}\le\frac{\varepsilon}{\delta},\quad \forall t\in J_0,\end{aligned} \tag{3.6}$$

where:

$$\begin{aligned}&\eta_\Sigma=\sum_{i=0}^{n=1}\eta_{A_i},\quad \eta_{A_i}=\sigma_{\max}(A_i)+\sigma_{\max}(\Delta A_i),\quad i=0,1,\quad b_0=\|B_0\|c=\|C\|\\&\gamma_{0u}=b_0\gamma_u/\delta,\gamma_{0w}=c\gamma_w/\delta,\ \sup_{t\in[0,T]}\|\Delta A_0(t)\|=\Delta a_0,\quad \sup_{t\in[0,T]}\|\Delta A_1(t)\|=\Delta a_1,\end{aligned} \tag{3.7}$$

with $\sigma_{\max}(\cdot)$ being the largest singular value of a matrix $(\cdot)$.

Proof: Following the property of the fractional order $0<\beta<\alpha<1$, a solution can be obtained in the form of the equivalent Volterra integral equation, where is $t_0=0$:

$$\begin{aligned}\boldsymbol{x}(t)&=\psi_x(0)-A_{N1}\cdot\psi_x(-\tau_{xm})\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}A_{N1}x(s-\tau_{xN}(s))ds+\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\begin{bmatrix}A_0\boldsymbol{x}(s)+A_1\boldsymbol{x}(t-\tau_x(s))\\+B_0u(s)+Cw(s)+f(x(s),s)+g(x(s-\tau_x(s)),s)\end{bmatrix}ds\end{aligned} \tag{3.8}$$

Now, using the norm $\|(\cdot)\|$ on equation (3.8), we can obtain an estimate of the solution $\boldsymbol{x}(t)$:

$$\begin{aligned}\|\boldsymbol{x}(t)\| &\le \|\psi_x(0)\| + \|A_{N1}\|\|\psi_x(-\tau_{xm})\|\frac{|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\|A_{N1}\|\|x(s-\tau_{xN}(s))\|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\left[\|A_0\boldsymbol{x}(s)+A_1\boldsymbol{x}(t-\tau_x(s))+B_0u(s)+Cw(s)+f(x(s),s)+g(x(s-\tau_x(s)),s)\|\right]ds\end{aligned} \tag{3.9}$$

Also, taking into account previous assumption, one can obtain: (3.10)

$$\begin{aligned}&\left\|\begin{matrix}A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}(t-\tau_x(t))+ \\ +B_0\boldsymbol{u}(t)+Cw(t)+f(x(t),t)+g(x(t-\tau_x),t)\end{matrix}\right\| = \left\|\begin{matrix}(A_0+\Delta A_0)\boldsymbol{x}(t)+(A_1+\Delta A_1)\boldsymbol{x}(t-\tau_x(t)) \\ +B_0\boldsymbol{u}(t)+Cw(t)\end{matrix}\right\| \\ &\le (\sigma_{\max}(A_0)+\Delta a_0)\|\boldsymbol{x}(t)\| + (\sigma_{\max}(A_1)+\Delta a_1)\|\boldsymbol{x}(t-\tau_x(t))\| + \|B_0\|\|\boldsymbol{u}(t)\| + \|C\|\|\boldsymbol{w}(t)\|\end{aligned}$$

where: $\|\boldsymbol{x}(t-\tau_x(t))\| \le \sup\left\{\|\boldsymbol{x}(t^{\square})\| : t^{\square}\in[t-\tau_{xm},t]\right\}$. Applying this inequality, expression (3.10) can be rewritten as follows:

$$\begin{aligned}&\|(A_0+\Delta A_0)\boldsymbol{x}(t)+(A_1+\Delta A_1)\boldsymbol{x}(t-\tau_x(t))+B_0\boldsymbol{u}(t)+Cw(t)\| \le \\ &\le \eta_\Sigma\left(\sup_{t^{\square}\in[t-\tau_{xm},t]}\|\boldsymbol{x}(t^{\square})\| + \|\boldsymbol{\psi}_x\|_C\right) + b_0\|\boldsymbol{u}(t)\| + c\|w(t)\|, \quad t>0^+,\end{aligned} \tag{3.11}$$

Then,taking into account (3.9) and (3.11) we can get (3.12)

$$\begin{aligned}\|\boldsymbol{x}(t)\| &\le \|\psi_x\|_C\left[1+\frac{\sigma_{\max}(A_{N1})|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\sigma_{\max}(A_{N1})\|x(s-\tau_{xN}(s))\|ds + \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\left[\eta_\Sigma\left(\sup_{s^{\square}\in[s-\tau_{xm},s]}\|\boldsymbol{x}(s^{\square})\| + \|\boldsymbol{\psi}_x\|_C\right) + b_0\|\boldsymbol{u}(s)\| + c\|w(s)\|,\right]ds\end{aligned}$$

In view of the conditions for $\|\boldsymbol{u}(s)\| < \gamma_u, \|w(s)\| < \gamma_w$, one may rewrite the above inequality as

$$\begin{aligned}\|\boldsymbol{x}(t)\| &\le \|\psi_x\|_C\left[1+\frac{\sigma_{\max(A_{N1})}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\eta_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\right] + \frac{\sigma_{\max(A_{N1})}}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\sup_{s^{\square}\in[s-\tau_{xm},s]}\|\boldsymbol{x}(s^{\square})\|ds + \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\left[\eta_\Sigma\left(\sup_{s^{\square}\in[s-\tau_{xm},s]}\|\boldsymbol{x}(s^{\square})\|\right)\right]ds + \frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)} + \frac{c\gamma_w|t|^\alpha}{\Gamma(\alpha+1)}\end{aligned} \tag{3.13}$$

Note that $e(t)$ (3.14) is nondecreasing function on $J_0=[0,T]$ and $\frac{\sigma_{\max}(A_{N1})}{\Gamma(\alpha-\beta)}, \frac{\eta_\Sigma}{\Gamma(\alpha)}$ are monotonically increasing, nonnegative continuous functions on $J_0=[0,T]$.

$$e(t) = \|\psi_x\|_C\left[1+\frac{\sigma_{\max}(A_{N1})|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\eta_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\right]. \tag{3.14}$$

Then, using Lemma 3, [21], we have:

$$\begin{aligned}\sup_{t^*\in[t-\tau_{x,M},t]}\left\|\boldsymbol{x}\left(t^*\right)\right\| \le e(t)+\frac{\sigma_{\max}\left(A_{N1}\right)}{\Gamma(\alpha-\beta)}\int_0^t\left|(t-s)\right|^{\alpha-\beta-1}\sup_{s^*\in[s-\tau_{xm},s]}\left\|\boldsymbol{x}\left(s^*\right)\right\|ds+\\ +\frac{\eta_\Sigma}{\Gamma(\alpha)}\int_0^t\left|(t-s)\right|^{\alpha-1}\left[\left(\sup_{s^*\in[s-\tau_{xm},s]}\left\|\boldsymbol{x}\left(s^*\right)\right\|\right)\right]ds+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{c\gamma_w|t|^\alpha}{\Gamma(\alpha+1)}\end{aligned} \tag{3.15}$$

From Lemma 4 [19], we obtain:

$$\|\boldsymbol{x}(t)\|\le\sup_{t^*\in[t-\tau_{x,M},t]}\left\|\boldsymbol{x}\left(t^*\right)\right\|\le e(t)E_\kappa\left[g(t)\left(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^\alpha\right)\right] \tag{3.16}$$

where $g(t)=g_1(t)+g_2(t),\ g_1=\dfrac{\sigma_{\max}\left(A_{N1}\right)}{\Gamma(\alpha-\beta)},g_2=\dfrac{\eta_\Sigma}{\Gamma(\alpha)}$ and $\kappa=min(\alpha,\alpha-\beta)$ .So, it yields:

$$\|\boldsymbol{x}(t)\|\le\delta\left[1+\frac{\sigma_{\max}\left(A_{N1}\right)|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\eta_\Sigma|t|^\alpha}{\Gamma(\alpha+1)}\right]E_\kappa\left[g(t)\left(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^\alpha\right)\right]+\frac{b_0\gamma_u|t|^\alpha}{\Gamma(\alpha+1)}+\frac{c\gamma_w|t|^\alpha}{\Gamma(\alpha+1)} \tag{3.17}$$

Finally, using the basic condition of Theorem 1,we can obtain the required finite time stability condition:

$$\|\boldsymbol{x}(t)\|<\varepsilon,\quad\forall t\in J_0. \tag{3.18}$$

From Theorem 1, it follows the next result.

Theorem 2: The *homogeneous* system is given by (3.1), when $\boldsymbol{u}(t)\equiv 0,\ \forall t\in J_0,$ and without perturbations and disturbance $f\left(x(t),t\right)\equiv 0,\ g\left(x\left(t-\tau_x(t)\right),t\right)\equiv 0,\ w(t)\equiv 0$, satisfying function of initial state (3.2) is finite-time stable w.r.t. $\left\{\delta,\varepsilon,J_0,\|(\cdot)\|\right\},\ \delta<\varepsilon,$ if the following condition is satisfied:

$$\begin{aligned}&\left[1+\frac{\sigma_{\max}\left(A_{N1}\right)|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\left(\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right)\right)|t|^\alpha}{\Gamma(\alpha+1)}\right]\cdot\\ &E_\kappa\left[\left(\frac{\sigma_{\max}\left(A_{N1}\right)}{\Gamma(\alpha-\beta)}+\frac{\left(\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right)\right)}{\Gamma(\alpha)}\right)\left(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha)t^\alpha\right)\right]\le\frac{\varepsilon}{\delta},\quad\forall t\in J_0,\end{aligned} \tag{3.19}$$

Proof: The proof immediately follows from the proof of the previous Theorem 1.

4. Numerical example

To demonstrate the effectiveness of the previously obtained FTS result, it is considered nonhomogeneous nonlinear perturbed neutral fractional time-varying delay system with disturbance as follows:

$$\begin{aligned}&{}^c\mathrm{D}^{0.5}\boldsymbol{x}(t)=A_0\boldsymbol{x}(t)+A_1\boldsymbol{x}\left(t-\tau_x(t)\right)+\\ &+A_{N1}{}^c\mathrm{D}^{0.1}\boldsymbol{x}\left(t-\tau_{xN}(t)\right)+B_0\boldsymbol{u}(t)+f\left(x(t),t\right)+g\left(x\left(t-\tau_x(t)\right),t\right)+Cw(t)\end{aligned} \tag{4.1}$$

where:

$$A_0 = \begin{bmatrix} -0,2 & 0 \\ -0,1 & 0,3 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & -0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad \Delta A_1(t) = \begin{bmatrix} 0.02(1-\sin t) & 0 \\ 0 & 0.03\cos t \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0,2 & 0,1 \\ 0 & -0,1 \end{bmatrix}, \quad A_{N1} = \begin{bmatrix} 0,3 & 0 \\ -0,05 & 0,2 \end{bmatrix}, \quad \Delta A_0(t) = \begin{bmatrix} 0.02\cos t & 0 \\ 0 & 0.04\cos t \end{bmatrix}, \tag{4.2}$$

and $t_0 = 0, \tau_x = \tau_{xN} = 0.1, \tau_{xm} = 0.1$, with associated functions: $\boldsymbol{x}(t) = \boldsymbol{\psi}_x(t) = \begin{bmatrix} 0,05 & 0,05 \end{bmatrix}^{\mathrm{T}}$, $t \in [t_0 - \tau_{xm}, t_0] = [-0,1\ 0]$. The task is to analyze the FTS with respect to $\{\delta = 0.2,\ \varepsilon = 50,\ \gamma_u = 2,\ J_0 = [0,3]\ \mathrm{s}\}$. From the initial functions and given state equation, we have:

$$\|\boldsymbol{\psi}_x\|_{\mathrm{C}} = \max_{t\in[-0.1,\ 0]} \|\boldsymbol{\psi}_x(t)\| = \|\boldsymbol{\psi}_x\| = \left(0,05^2 + 0,05^2\right)^{1/2} = 0,071 < \delta = 0,1, \tag{4.3}$$

as well as:

$$\begin{gathered}
\sigma_{\max}(A_0) = 0.3257, \sigma_{\max}(A_1) = 0.2288,\ \sigma_{\max}(A_{N1}) = 0.3071, \\
\Delta a_0 = \sup_{t\in[0,T]} \|\Delta A_0(t)\| = \begin{Vmatrix} 0.02 & 0 \\ 0 & 0.04 \end{Vmatrix} = 0.04, \Delta a_1 = \sup_{t\in[0,T]} \|\Delta A_1(t)\| = \begin{Vmatrix} 0.02 & 0 \\ 0 & 0.03 \end{Vmatrix} = 0.03, \\
\eta_{A_0} = \sigma_{\max}(A_0) + \Delta a_0 = 0,3657, \quad \eta_{A_1} = \sigma_{\max}(A_1) + \Delta a_1 = 0.2588, \\
\eta_{\Sigma} = \eta_{A_0} + \eta_{A_1} = 0.6245, \quad \gamma_w = 1 \\
b_0 = \|B_0\| = \sigma_{\max}(B_0) = 3, c = \|C\| = 0.2395, \gamma_{u0} = b_0\gamma_u / \delta = 60, \quad \gamma_{0w} = c\gamma_w / \delta = 2.395,
\end{gathered} \tag{4.4}$$

Applying the condition of the Theorem 1, it follows:

$$\begin{gathered}
\left[1 + \frac{0.3071|T_e|^{0.4}}{\Gamma(1.4)} + \frac{0.6245|T_e|^{0.5}}{\Gamma(1.5)}\right] E_{0.4}\left[\left(\frac{0.3071}{\Gamma(0.4)} + \frac{0.6245}{\Gamma(0.5)}\right)\left(\Gamma(0.4)T_e^{\ 0.4} + \Gamma(0.5)T_e^{\ 0.5}\right)\right] \\
+ \frac{60|T_e|^{0.5}}{\Gamma(1.5)} + \frac{2.395 \cdot |T_e|^{0.5}}{\Gamma(1.5)} \le \frac{50}{0.2},
\end{gathered} \tag{4.5}$$

so, we obtain the estimated time of finite-time stability $T_{\mathrm{e}} \approx 0,635$ s.

5. Conclusion

In this paper, finite-time stability analysis for a class of nonhomogeneous nonlinear perturbed neutral fractional system with multiple time-varying delays and disturbance has been investigated. By use of the generalized Gronwall inequality, new criterion for the FTS have been developed. A sufficient condition for robust FTS for this class of neutral fractional time-varying delay systems has been proposed. Finally, a numerical example has been provided to illustrate the effectiveness and the benefit of the proposed novel stability criterion of FTS.

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