BRACHISTOCHRONIC MOTION OF A VARIABLE MASS NONHOLONOMIC MECHANICAL SYSTEM

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Abstract

The paper considers the brachistochronic motion of a variable mass nonholonomic mechanical system [4] in a horizontal plane, between two specified positions. Variable mass particles are interconnected by a lightweight mechanism of the 'pitchfork' type. The law of the time-rate of mass variation of the particles, as well as relative velocities of the expelled particles, as a function of time, are known. Differential equations of motion, where the reactions of nonholonomic constraints and control forces figure, are created based on the general theorems of dynamics of a variable mass mechanical system [6]. The formulated brachistochrone problem, with adequately chosen quantities of state, is solved, in this case, as the simplest task of optimal control by applying Pontryagin's maximum principle [1]. A corresponding two-point boundary value problem (TPBVP) of the system of ordinary nonlinear differential equations is obtained, which, in a general case, has to be numerically solved [2]. Numerical procedure for solving the TPBVP is performed by the shooting method. On the basis of thus obtained brachistochronic motion, the active control forces, along with the reactions of nonholonomic constraints, are determined. The analysis of the brachistochronic motion for different values of the initial position of a variable mass particle *B* is presented.

Keywords: Brachistochrone, variable mass, nonholonomic system, Pontryagin's maximum principle, optimal control

1. Introduction

A nonholonomic mechanical system [4] is composed of two variable mass particles, *A* and *B*, whose motion is constrained by the imposition of perpendicularity of the velocities by means of the Chaplygin blades of negligible masses, as shown in Fig. 1a. In order to develop the differential equations of motion of a variable mass nonholonomic mechanical system (henceforth referred to as 'the system'), as well as for the needs of further considerations, first, two Cartesian reference coordinate systems must be introduced: the stationary coordinate system $Oxyz$, whose coordinate plane Oxy coincides with the horizontal plane of motion, and the non-stationary coordinate system $A\xi\eta\varsigma$ that is rigidly attached to point *A* of the system, so that the coordinate plane $A\xi\eta$ coincides with the plane Oxy (refer to Fig. 1a). The axis of the non-stationary coordinate system $A\xi$ is determined by the direction AB, that is $B \in A\xi$, whereas unit vectors of the non-stationary coordinate system axes are $\vec{\lambda}, \vec{\mu}$ and $\vec{\nu}$, respectively. Variable mass particles A and B are interconnected by a lightweight mechanism of the 'pitchfork' type, which allows the distance $\overline{AB} = \xi \neq \text{const.}$ to change. The configuration of the considered system relative to the system $Oxyz$ is defined by a set of Lagrangian coordinates $\left(q^1, q^2, q^3, q^4\right)$, where $q^1 = x$ and $q^2 = y$ are Cartesian coordinates of the point A, $q^3 = \varphi$ is the angle between the axis Ox and the axis $A\xi$, whereas $q^4 = \xi$ is the relative coordinate of the variable mass point B relative to the non-stationary coordinate system.

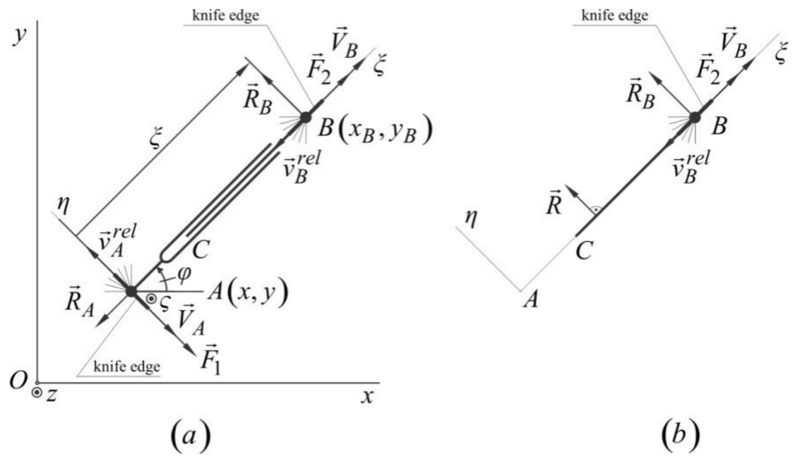

Fig. 1. Variable-mass nonholonomic mechanical system

In accordance with the restriction of motion of the points A and B of the system, homogeneous nonholonomic constraints can be written in the following form [4], [5]

$$\begin{aligned} &\dot{x}\cos\varphi + \dot{y}\sin\varphi = 0, \\ -&\dot{x}\sin\varphi + \dot{y}\cos\varphi + \xi\dot{\varphi} = 0. \end{aligned} \tag{1}$$

The velocity of the variable mass point A relative to the system $Oxyz$, which has the axis $A\eta$ direction, has the following form

$$V_A = \dot{x}\sin\varphi - \dot{y}\cos\varphi, \tag{2}$$

where $V_A = \vec{V}_A \cdot \vec{\mu}$. The coordinates of the variable mass point B relative to the coordinate system $Oxyz$ are

$$x_B = x + \xi\cos\varphi, \quad y_B = y + \xi\sin\varphi, \quad z_B = 0. \tag{3}$$

Now, based on the second nonholonomic constraint (1), and taking into account the relation (2), the angular velocity of the system is determined in the form

$$\dot{\varphi} = \frac{V_A}{\xi}. \tag{4}$$

The velocity of the variable mass point B relative to the system $Oxyz$, which has the axis $A\xi$ direction, is determined based on relations (1) and (3),

$$V_B = \dot{\xi}, \tag{5}$$

where $V_B = \vec{V}_B \cdot \vec{\lambda}$.

Differential equations of motion of the variable mass system will be developed based on general theorems of the dynamics of variable mass system [6], that is, based on the momentum change theorem as well as on the moment of momentum theorem for the moving point *A*,

$$\begin{aligned} &\frac{d\vec{K}}{dt} = \vec{F}_R^s + \left(\vec{V}_A + \vec{v}_A^{rel}\right)\dot{m}_A + \left(\vec{V}_B + \vec{v}_B^{rel}\right)\dot{m}_B, \\ &\frac{d\vec{L}_A}{dt} + \vec{V}_A \times \vec{K} = \vec{M}_A^s + \vec{\rho}_A \times \left(\vec{V}_A + \vec{v}_A^{rel}\right)\dot{m}_A + \vec{\rho}_B \times \left(\vec{V}_B + \vec{v}_B^{rel}\right)\dot{m}_B. \end{aligned} \tag{6}$$

where $\vec{v}_A^{rel}$ and $\vec{v}_B^{rel}$ are relative velocities of the particles expelled from points *A* and *B* of the system, whose directions coincide with the axes $A\eta$ and $A\xi$ respectively (directions represented in Fig. 1a), whereas $\vec{\rho}_A$ and $\vec{\rho}_B$ are relative vectors of the variable mass points *A* and *B* relative to the origin of the non-stationary coordinate system $A\xi\eta\zeta$.
The law of the time-rate of masses variation of the particles *A* and *B* as a function of time are

$$m_A(t) = m_B(t) = m(t) = m_0 e^{-k_m t}, \tag{7}$$

where k_m is the determined positive constant, whereas m_0 is a mass of the particles *A* and *B* at the initial time moment $t_0 = 0$. Relative velocities of the particles expelled from points *A* and *B* of the system are

$$v_A^{rel} = v_B^{rel} = v, \tag{8}$$

where v is the determined positive constant.
For vector relations (6) there are the following corresponding scalar differential equations relative to the axes of the defined non-stationary coordinate system $A\xi\eta\zeta$, which, after a brief rearrangement, can be written in the form as follows

$$\begin{aligned} &m\left(V_A\dot{\varphi} + \dot{V}_B\right) = F_2 - R_A + k_m v m, \\ &m\left(V_B\dot{\varphi} - \dot{V}_A\right) = -F_1 + R_B - k_m v m, \\ &m V_A V_B = R_B \xi, \end{aligned} \tag{9}$$

where F_1 and F_2 are control forces. Now, based on the momentum change theorem, differential equations of motion can be generated for the *BC* segment of the system (see Fig. 1b), the rod *BC* being of negligible mass, relative to the axes of the system $A\xi\eta\zeta$

$$\begin{aligned} &m\dot{V}_B = F_2 + k_m v m, \\ &m V_B \dot{\varphi} = R_B + R, \end{aligned} \tag{10}$$

where R is the projection of the resultant of a system of internal forces.
Solving the system of equations (9) and (10) determines the reactions of nonholonomic constraints R_A and R_B, the control forces F_1 and F_2, as well as the resultant of a system of internal forces R, to realize motion as a function of defined quantities of state and a corresponding derivative

$$\begin{aligned} & R_A = -m\frac{V_A^2}{\xi}, \\ & R_B = m\frac{V_A V_B}{\xi}, \\ & F_1 = m\left(\dot{V}_A - k_m v\right), \\ & F_2 = m\left(\dot{V}_B - k_m v\right), \\ & R = 0. \end{aligned} \tag{11}$$

As it is known, the realization of the brachistochronic motion of the meachanical systems can, in general, be accomplished by the control forces, whose total power during brachistochronic motion equals zero, and which can be represented in the form of active control forces, the reactions forces of constraints, or by their mutual combinations. In our case, the brachistochronic motion is realized by the active control forces $\vec{F}_1 = F_1(t)\vec{\mu}$ and $\vec{F}_2 = F_2(t)\vec{\lambda}$ whose power during brachistochronic motion equals zero

$$P^G = \vec{F}_1 \cdot \vec{V}_A + \vec{F}_2 \cdot \vec{V}_B = 0, \tag{12}$$

that is

$$F_1 V_A + F_2 V_B = 0. \tag{13}$$

2. Brachistochronic motion as the problem of optimal control

In this section, the problem of brachistochronic motion of the system is formulated as the problem of optimal control [1]. In order to define the equations of state which describe the motion of the considered system in the state space, first, from conditions (13), taking into account the law of change in the control forces F_1 and F_2 given in (11), the following relation can be established

$$\dot{\Phi} = 2k_m v\left(V_A + V_B\right), \tag{14}$$

where

$$\Phi = V_A^2 + V_B^2. \tag{15}$$

Now, based on (1), (2), (4), (5) and (14), the equations of state can be created in the form

$$\begin{aligned} & \dot{x} = V_A \sin\varphi, \\ & \dot{y} = -V_A \cos\varphi, \\ & \dot{\varphi} = \frac{V_A}{\xi}, \\ & \dot{\xi} = V_B, \\ & \dot{\Phi} = 2k_m v\left(V_A + V_B\right). \end{aligned} \tag{16}$$

The coordinates of the initial state x, y, φ and ξ, as well as the function of the quadratic form of velocities Φ, are determined at the initial position of the system on manifolds:

$$t_0=0,\quad x(t_0)=0,\quad y(t_0)=0,\quad \varphi(t_0)=0,$$
$$\xi(t_0)=\xi_0,\quad \Phi(t_0)=\Phi_0, \tag{17}$$

as well as the coordinates of the end state x, y, φ and ξ at the terminal position on manifolds:

$$t=t_f,\quad x(t_f)=x_f,\quad y(t_f)=y_f,$$
$$\varphi(t_f)=\varphi_f,\quad \xi(t_f)=\xi_f, \tag{18}$$

where t_f is the in advance unknown value of the final time moment corresponding to the end state of the system on manifolds (18). The brachistochrone problem of the system motion described by differential equations of state (16), consists of determining the coordinates of optimal control V_A and V_B, as well as their corresponding state coordinates x, y, φ, ξ and Φ, so that the system starting from the initial state on manifolds (17) moves to the end state on manifolds (18) in a minimum time. This can be expressed in the form of condition so that the functional

$$I=\int_{t_0}^{t_f} dt, \tag{19}$$

on the interval $[t_0, t_f]$ has a minimum value.

In order to solve the problem of optimal control, formulated by Pontryagin's maximum principle [1], the Pontryagin function is created in the form as follows

$$H=\lambda_0+\lambda_x V_A\sin\varphi-\lambda_y V_A\cos\varphi+\lambda_\varphi\frac{V_A}{\xi}+\lambda_\xi V_B+2k_m\nu\lambda_\Phi\left(V_A+V_B\right)+\mu\left(V_A^2+V_B^2-\Phi\right), \tag{20}$$

where $\lambda_0=\text{const.}\le 0$, λ_x, λ_y, λ_φ, λ_ξ and λ_Φ are the conjugate vector coordinates, where it can be taken that $\lambda_0=-1$, whereas μ is a multiplier corresponding to relation (15). Taking into account the boundary conditions (17) and (18), as well as the fact that time does not figure explicitly in equations of state (16), the defined problem of optimal control can be solved by a straightforward application of Theorem 22 [1].

Based on the Pontryagin function (20), the conjugate system of differential equations has the form

$$\dot\lambda_x=0,\quad \dot\lambda_y=0,$$
$$\dot\lambda_\varphi=-V_A\left(\lambda_x\cos\varphi+\lambda_y\sin\varphi\right), \tag{21}$$
$$\dot\lambda_\xi=\frac{V_A}{\xi^2}\lambda_\varphi,\quad \dot\lambda_\Phi=\mu,$$

wherefrom it follows that $\lambda_x=\text{const.}$ and $\lambda_y=\text{const.}$

Having in mind that the initial state (17) is completely defined, the transversality conditions corresponding to the initial position of the system are identically satisfied.

The transversality conditions at the terminal position of the system on manifolds (18) have the following form

$$\lambda_x \delta x\left(t_f\right)+\lambda_y \delta y\left(t_f\right)+\lambda_\varphi\left(t_f\right)\delta\varphi\left(t_f\right)+\lambda_\xi\left(t_f\right)\delta\xi\left(t_f\right)+\lambda_\Phi\left(t_f\right)\delta\Phi\left(t_f\right)=0, \tag{22}$$

whereas, in accordance with (18), the variations of coordinates corresponding to the terminal position of the system are

$$\delta x\left(t_f\right)=0,\ \delta y\left(t_f\right)=0,\ \delta\varphi\left(t_f\right)=0, \delta\xi\left(t_f\right)=0. \tag{23}$$

Taking into account the independence of variation $\delta\Phi\left(t_f\right)$, based on (22) and (23), it is arrived at the following boundary condition

$$\lambda_\Phi\left(t_f\right)=0. \tag{24}$$

If controls belong to an open set, as in this case, the conditions based on which the optimal control is defined can be expressed in the form [1]

$$\left(\frac{\partial H}{\partial u_i}\right)_{\boldsymbol{u}^{opt}}=0, \quad \left(\frac{\partial^2 H}{\partial u_i \partial u_j}\right)_{\boldsymbol{u}^{opt}} u_i u_j \le 0, \quad (i,\, j=1,2). \tag{25}$$

When time t_f is not determined in advance, as in this case, in solving the system of equations (16) and (21) in the final form, the condition should be added, following from a straightforward application of Theorem 1 [1], that the value of the Pontryagin function on the optimal trajectory equals zero for $\forall t \in \left[t_0, t_f\right]$

$$H(t)=0, \tag{26}$$

that is, in accordance with the Pontryagin function (20)

$$-1+\lambda_x V_A \sin\varphi-\lambda_y V_A \cos\varphi+\lambda_\varphi \frac{V_A}{\xi}+\lambda_\xi V_B+2k_m \nu \lambda_\Phi\left(V_A+V_B\right)+\mu\left(V_A^2+V_B^2-\Phi\right)=0. \tag{27}$$

Now, based on (20), (25) and (27), the value of the multiplier μ is determined, as well as of the control functions V_A and V_B in the following form

$$\begin{aligned} &\mu=-\frac{1}{2\Phi}, \quad V_A=\left(\lambda_x \sin\varphi-\lambda_y \cos\varphi+2k_m \nu\lambda_\Phi+\frac{1}{\xi}\lambda_\varphi\right)\Phi, \\ &V_B=\left(\lambda_\xi+2k_m \nu \lambda_\Phi\right)\Phi. \end{aligned} \tag{28}$$

Based on condition (26) defined at the initial time moment, as well as (17), (27) and (28), the conjugate vector coordinate λ_φ is determined at the initial time moment

$$\lambda_\varphi\left(t_0\right)_{1/2}=\xi\left(t_0\right)\left(\lambda_y-2k_m \nu\lambda_\Phi\left(t_0\right) \pm \sqrt{\frac{1}{\Phi_0}-\left(\lambda_\xi\left(t_0\right)+2k_m \nu\lambda_\Phi\left(t_0\right)\right)^2}\right). \tag{29}$$

Now, based on (16), (21) and (28), the basic and conjugate system of differential equations can be created in the form

$$\begin{aligned}
&\dot{x} = \frac{\sin\varphi\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 2k_m \nu \lambda_\Phi\right)\right]\Phi}{\xi},\\
&\dot{y} = -\frac{\cos\varphi\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 2k_m \nu \lambda_\Phi\right)\right]\Phi}{\xi},\\
&\dot{\varphi} = \frac{\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 2k_m \nu \lambda_\Phi\right)\right]\Phi}{\xi^2}, \dot{\xi} = \left(\lambda_\xi + 2k_m \nu \lambda_\Phi\right)\Phi,\\
&\dot{\Phi} = \frac{2k_m \nu\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 4k_m \nu \lambda_\Phi + \lambda_\xi\right)\right]\Phi}{\xi}, \dot{\lambda}_x = 0,\\
&\dot{\lambda}_y = 0, \dot{\lambda}_\varphi = -\frac{\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right)\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 2k_m \nu \lambda_\Phi\right)\right]\Phi}{\xi},\\
&\dot{\lambda}_\xi = \frac{\lambda_\varphi\left[\lambda_\varphi + \xi\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi + 2k_m \nu \lambda_\Phi\right)\right]\Phi}{\xi^3}, \dot{\lambda}_\Phi = -\frac{1}{2\Phi},
\end{aligned} \tag{30}$$

whereas the state coordinates, as well as the conjugate vector coordinates, based on (17) and (29), are determined at the initial time moment

$$\begin{aligned}
&t_0 = 0, \quad x(t_0) = 0, \quad y(t_0) = 0, \quad \varphi(t_0) = 0,\\
&\xi(t_0) = \xi_0, \quad \Phi(t_0) = \Phi_0, \quad \lambda_\xi(t_0) = \lambda_{\xi 0}, \quad \lambda_\Phi(t_0) = \lambda_{\Phi 0}\\
&\lambda_\varphi(t_0)_{1/2} = \xi_0\left(\lambda_y - 2k_m \nu \lambda_{\Phi 0} \pm \sqrt{\frac{1}{\Phi_0} - \left(\lambda_{\xi 0} + 2k_m \nu \lambda_{\Phi 0}\right)^2}\right).
\end{aligned} \tag{31}$$

Numerical procedure for solving the corresponding TPBVP of the system of ordinary nonlinear differential equations of the first kind is based on the shooting method [2]. The five-parameter shooting consists of determining the unknown coordinates of the conjugate vector λ_x, λ_y, $\lambda_{\xi 0}$ and $\lambda_{\Phi 0}$ as well as a minimum required time t_f .

The TPBVP is solved for the following values of the parameters

$$\begin{aligned}
&\Phi_0 = 2\frac{\mathrm{m}^2}{\mathrm{s}^2}, \ m_0 = 2\mathrm{kg}, \ k_m = 0.2\frac{1}{\mathrm{s}}, \nu = 1\frac{\mathrm{m}}{\mathrm{s}},\\
&x_f = 1.5\mathrm{m}, \ y_f = -1\mathrm{m}, \varphi_f = \frac{\pi}{2}\mathrm{rad}, \xi_f = 3\mathrm{m}.
\end{aligned} \tag{32}$$

Table 1 shows the TPBVP solutions for different values of the initial position of the variable mass point *B*.

Solutions	$\lambda_x\,[\mathrm{s/m}]$	$\lambda_y\,[\mathrm{s/m}]$	$\lambda_{\xi 0}\,[\mathrm{s/m}]$	$\lambda_{\Phi 0}\,[\mathrm{s^2/m^2}]$	$t_f\,[\mathrm{s}]$
$\xi_0 = 1\mathrm{m}$	0.710261	1.258352	–0.753975	0.387281	1.999254
$\xi_0 = 0.8\mathrm{m}$	0.317475	0.413259	–0.290266	0.351329	1.888149
$\xi_0 = 0.6\mathrm{m}$	–0.121316	–0.543731	0.326574	0.344384	1.894710

Table 1. TPBVP solutions for different values of ξ_0

Figures 2 - 6 show the laws of change in the state coordinates, the reactions of nonholonomic constraints, and the control forces at different values of the initial position of variable mass point *B* displayed in Table 1.

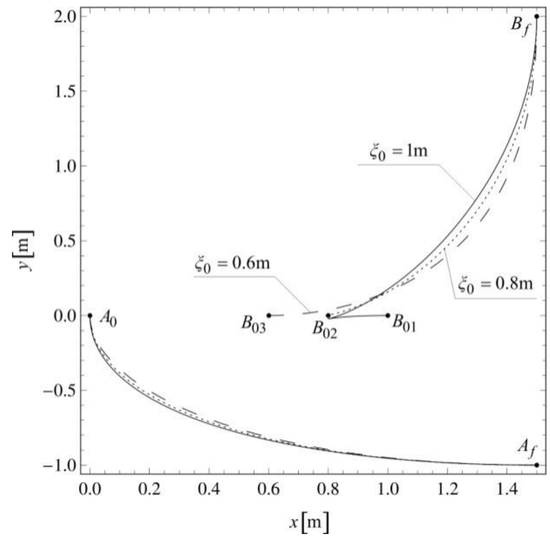

Fig. 2. Trajectories of variable mass points *A* and *B*

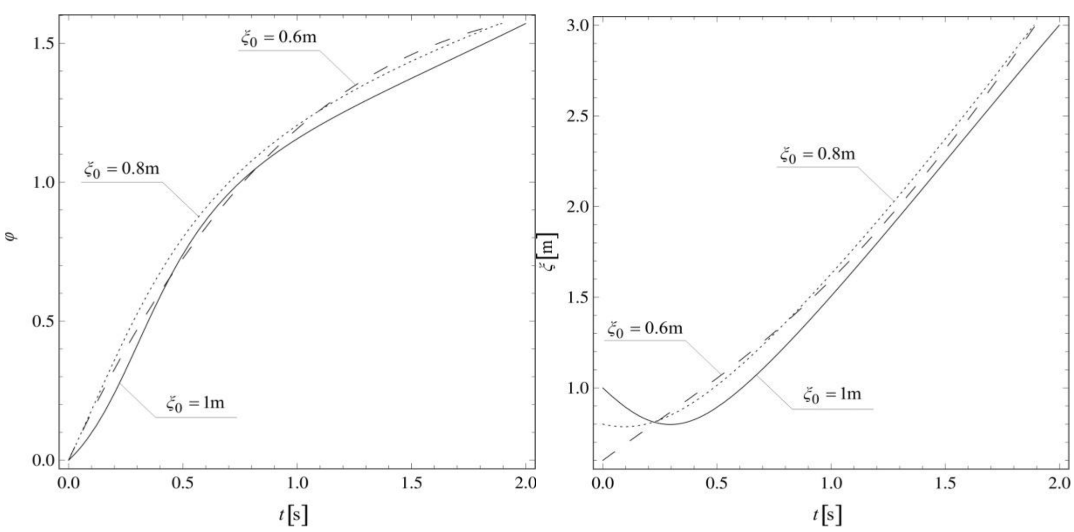

Fig. 3. Graphs of angle φ and relative coordinate ξ

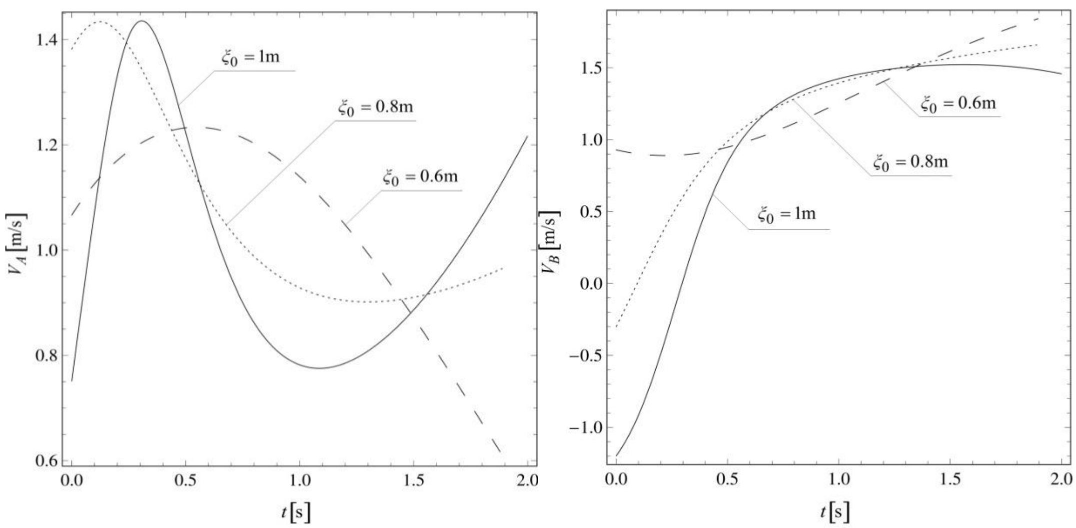

Fig. 4. Graphs of control functions V_A and V_B

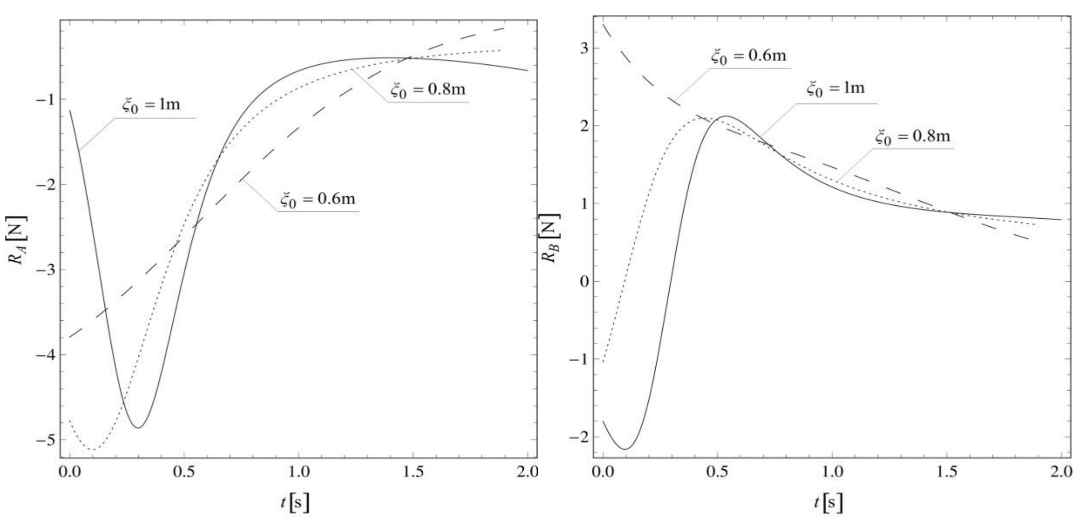

Fig. 5. Graphs of reactions of nonholonomic constraints R_A and R_B

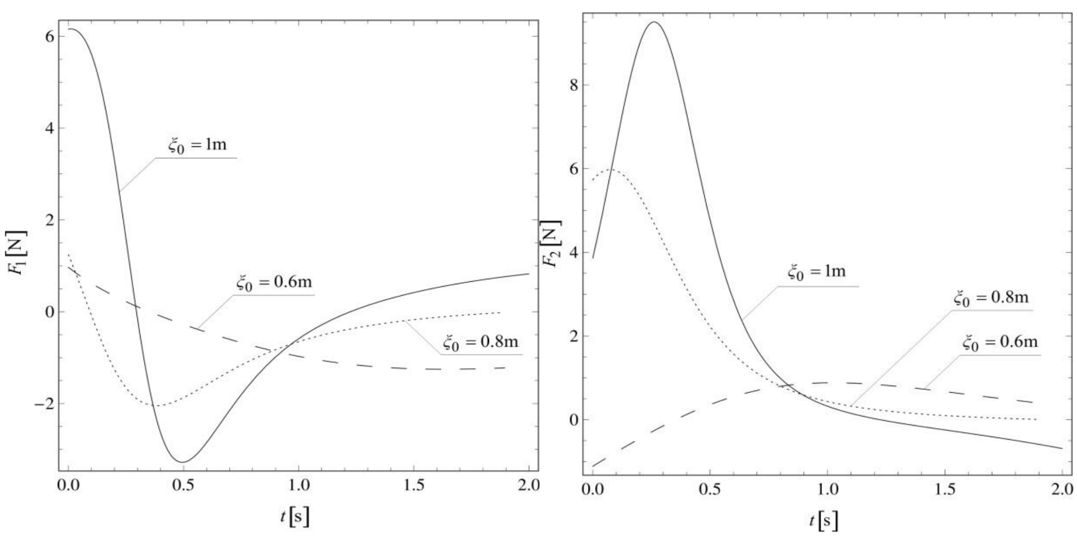

Fig. 6. Graphs of control forces F_1 and F_2

3. Conclusions

This paper considers the brachistochronic planar motion of a variable mass nonholonomic mechanical system, with specified initial and final positions. The procedure for creating differential equations of motion based on the general theorems of dynamics of a variable-mass mechanical system is presented. The formulated brachistochrone problem, along with adequately chosen quantities of state, is solved as a task of optimal control by applying Pontryagin's maximum principle. Numerical procedure for solving the TPBVP is based on the shooting method. Afterwards, the reactions of nonholonomic constraints as well as the control forces are determined to realize the brachistochronic motion. The analysis of brachistochronic motion at different values of the initial position of the variable mass point *B* is performed. Authors consider that the results obtained in this work can be extended to the general case of brachistochronic motion of a variable mass nonholonomic mechanical system, which will be the subject of further investigations.

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