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GLOBAL MINIMUM TIME FOR THE BRACHISTOCHRONIC MOTION OF A PARTICLE IN AN ARBITRARY FIELD OF POTENTIAL FORCES

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Abstract

The problem of the brachistochronic motion of a particle in space is considered. Particle *M* moves in the field of known potential forces. The brachistochrone problem is formulated as an optimal control task, where the particle velocity projections are taken as control variables. The problem considered is reduced to solving the corresponding two-point boundary–value problem (TPBVP).The appropriate numerical procedure to apply in determining the solutions to the TPBVP is based on the shooting method. The paper presents the procedure for estimating the interval of initial values of the conjugate vector coordinates. Based on given estimation, it may be claimed that all solutions to the corresponding TPBVP are certainly located within given intervals, and thereby the global minimum time too for the brachistochronic motion of a particle. In the case of multiple solutions of the principle of maximum, the global minimum is the solution corresponding to the minimum time.

Key words: Particle, Brachistochronic motion, Optimal control, Pontryagin's maximum principle,Global minimum time, Shooting method

1. Introduction

The motion of particle *M* of mass $m = 3\text{kg}$ in the reference frame *Oxyz* of the Cartesian coordinate system, where axis *Oz* is directed vertically downward, is considered. The unit vectors of axes *Ox*, *Oy* and *Oz* are $\vec{i}, \vec{j}$ and $\vec{k}$ respectively. Particle *M* moves in space along a smooth curve, which is treated as a bilateral constraint in an arbitrary field of known potential forces. The particle is attached to the spring of stiffness $c = 200\text{N/m}$, of free length $l_0 = 0{,}2\text{m}$. The other end of the spring is attached to the stationary particle *O*. The particle initiates motion at position

$$t_0 = 0, \quad x(t_0) = 0, \quad y(t_0) = 0, \quad z(t_0) = 0{,}2\text{m}. \tag{1}$$

Potential energy of the particle considered in the example is determined by the expression

$$\Pi = -mgz + \frac{1}{2}c\left(\sqrt{x^2 + y^2 + z^2} - l_0\right)^2. \tag{2}$$

Taking into account that particle *M* moves in the arbitrary field of known potential forces, the *'law'* of conservation of total mechanical energy holds [1; 2]

$$\Phi(x, y, z, \dot{x}, \dot{y}, \dot{z}) = T(\dot{x}, \dot{y}, \dot{z}) + \Pi(x, y, z) - E_0 = 0, \tag{3}$$

where $E_0 = 5500\,\text{kgm}^2/\text{s}^2$ value of the particle's mechanical energy at the initial time moment $t_0 = 0$.

The brachistochrone problem for a particle may be formulated as an optimal control task. Namely, by taking the particle *M* velocity projections onto axes *Ox*, *Oy* and *Oz* for control variables u_x, u_y and u_z respectively

$$\dot{x} = u_x, \quad \dot{y} = u_y, \quad \dot{z} = u_z, \tag{4}$$

the brachistochrone problem for a particle consists of determining the extremal controls $u_x = u_x(t), u_y = u_y(t)$ and $u_z = u_z(t)$, as well as their corresponding finite equations of motion $x = x(t), y = y(t)$ and $z = z(t)$, so that the particle which initiates motion at position (1) moves to the final position on the manifold

$$t = t_f, \quad \Psi_f = z_f - a\left(5 + \sin\left(\frac{x_f}{a}\right) + \sin\left(\frac{y_f}{a}\right)\right) = 0, \tag{5}$$

where $a = 1\text{m}$, with unchanged value of the particle's mechanical energy, in a minimum time t_f. This may be expressed in the form of condition that the functional

$$J\left(x, y, z, u_x, u_y, u_z\right) = \int_0^{t_f} dt, \tag{6}$$

over the interval $\left[0, t_f\right]$ has the minimum value.

2. Solving the optimal control problem

In order to solve the problem of optimal control, formulated by Pontryagin's maximum principle [4], the Pontryagin function is created in the form as follows

$$H\left(x, y, z, u_x, u_y, u_z, \boldsymbol{\lambda}\right) = \lambda_0 + \lambda_x u_x + \lambda_y u_y + \lambda_z u_z, \tag{7}$$

where $\boldsymbol{\lambda} = \left(\lambda_0, \lambda_x, \lambda_y, \lambda_z\right)^T$ is a conjugate vector, $\lambda_0 = \text{const.} \leq 0$ (see[4; 5; 6]) and where it can be taken that $\lambda_0 = -1$. Taking into account the boundary conditions (1) and (5), as well as the fact that time does not figure explicitly in the equations of state (4), the posed problem of optimal control can be solved by directly applying Theorem 22 [4]. The conjugate system of differential equations, taking into account Pontryagin's function (7) as well asthe relation of bound (3), is of the form [4]

$$\begin{aligned}
\dot{\lambda}_x &= -\frac{\partial H}{\partial x} + \mu\frac{\partial \Phi}{\partial x} = \mu c x\left(1 - \frac{l_0}{\sqrt{x^2 + y^2 + z^2}}\right), \\
\dot{\lambda}_y &= -\frac{\partial H}{\partial y} + \mu\frac{\partial \Phi}{\partial y} = \mu c y\left(1 - \frac{l_0}{\sqrt{x^2 + y^2 + z^2}}\right), \\
\dot{\lambda}_z &= -\frac{\partial H}{\partial z} + \mu\frac{\partial \Phi}{\partial z} = \mu\left(cx\left(1 - \frac{l_0}{\sqrt{x^2 + y^2 + z^2}}\right) - mg\right),
\end{aligned} \tag{8}$$

where μ is the Lagrange multiplier.

The transversality conditions at the initial time moment $t_0 = 0$ are identically satisfied, taking into account that the initial position (1) of the particle is completely determined. The transversality conditions at the final time moment t_f are

$$\lambda_x(t_f)\Delta x(t_f) + \lambda_y(t_f)\Delta y(t_f) + \lambda_z(t_f)\Delta z(t_f) = 0, \tag{9}$$

where, in accordance with (5)

$$\frac{\partial \Psi_f}{\partial x_f}\Delta x\left(t_f\right) + \frac{\partial \Psi_f}{\partial y_f}\Delta y\left(t_f\right) + \frac{\partial \Psi_f}{\partial z_f}\Delta z\left(t_f\right) = 0, \tag{10}$$

where $\Delta\left(\cdot\right)$ represents asynchronous variation [8; 9] of the quantity $\left(\cdot\right)$. Now, having in mind the independence of variations $\Delta x(t_f)$ and $\Delta y(t_f)$, based on (9) and (10) the following boundary conditions can be created

$$\begin{aligned} \Psi^*_{f1} &= \lambda_{xf} + \lambda_{zf}\cos\left(\frac{x_f}{a}\right) = 0, \\ \Psi^*_{f2} &= \lambda_{yf} + \lambda_{zf}\cos\left(\frac{y_f}{a}\right) = 0, \end{aligned} \tag{11}$$

If the permissible controls belong to an open set, as is the case here, the conditions for determining extremal controls may be expressed in the form [4]

$$\frac{\partial H}{\partial u_i} = \mu\frac{\partial \Phi}{\partial u_i}, \quad i = x, y, z. \tag{12}$$

When time t_f is not defined beforehand, as is the case here, it is necessary to add the condition, which follows by directly applying Theorem 1 [4], that the value of Pontryagin's function on the extremal trajectory equals zero at any time moment [4; 5; 6]

$$H\left(t\right) = 0, \tag{13}$$

that is, in accordance with Pontryagin's function (7), we can write

$$\lambda_0 + \lambda_x u_x + \lambda_y u_y + \lambda_z u_z = 0. \tag{14}$$

Based on (3), (12) and (14), the Lagrange multiplier μ and extremal controls u_x, u_y and u_z are defined

$$\mu = \frac{1}{2\left[E_0 - \Pi\left(x,y,z\right)\right]}, u_i = \frac{2\left[E_0 - \Pi\left(x,y,z\right)\right]}{m}\lambda_i, \quad i = x, y, z, \tag{15}$$

among which the optimal ones should be sought. Now, based on (4), (8) and (15), we can generate the fundamental and conjugate system of differential equations of the first kind in the normal form

$$\begin{aligned} \dot{x} &= \frac{2\left[E_0 + mgz - \frac{1}{2}c\left(\sqrt{x^2+y^2+z^2} - l_0\right)^2\right]}{m}\lambda_x, \\ \dot{y} &= \frac{2\left[E_0 + mgz - \frac{1}{2}c\left(\sqrt{x^2+y^2+z^2} - l_0\right)^2\right]}{m}\lambda_y, \\ \dot{z} &= \frac{2\left[E_0 + mgz - \frac{1}{2}c\left(\sqrt{x^2+y^2+z^2} - l_0\right)^2\right]}{m}\lambda_z, \end{aligned} \tag{16}$$

$$\dot{\lambda}_x = \frac{cx\left(1-\dfrac{l_0}{\sqrt{x^2+y^2+z^2}}\right)}{2\left[E_0+mgz-\dfrac{1}{2}c\left(\sqrt{x^2+y^2+z^2}-l_0\right)^2\right]},$$

$$\dot{\lambda}_y = \frac{cy\left(1-\dfrac{l_0}{\sqrt{x^2+y^2+z^2}}\right)}{2\left[E_0+mgz-\dfrac{1}{2}c\left(\sqrt{x^2+y^2+z^2}-l_0\right)^2\right]}, \tag{17}$$

$$\dot{\lambda}_z = \frac{cz\left(1-\dfrac{l_0}{\sqrt{x^2+y^2+z^2}}\right)-mg}{2\left[E_0+mgz-\dfrac{1}{2}c\left(\sqrt{x^2+y^2+z^2}-l_0\right)^2\right]}.$$

Based on relation (14) defined at the initial time moment $t_0 = 0$, and taking into account (15), the following positive definite quadratic form reduced to canonical form is obtained [10]

$$\lambda_{x0}^2+\lambda_{y0}^2+\lambda_{z0}^2 = \frac{m\lambda_0^2}{2\left(E_0-\Pi_0\right)} = \text{const.} \tag{18}$$

Numerical procedure to be used for determining unknown boundary values of the corresponding TPBVP, is based on the shooting method [12]. A three-parameter shooting consists of determining unknown coordinates λ_{x0} and λ_{y0} of the conjugate vector at the initial time moment, if it is taken into account that one of the conjugate vector coordinates, let's say λ_{z0}, may be expressed from relation (18) in the form as follows

$$\lambda_{z0} = \pm\sqrt{\frac{m\lambda_0^2}{2\left(E_0-\Pi_0\right)}-\left(\lambda_{x0}^2+\lambda_{y0}^2\right)}, \tag{19}$$

as well as by a minimum required time t_f. Numerical procedure involves '*shooting*' of the ultimate boundary conditions (5) and (11), in accordance with (16) and (17). When applying the method of shooting, it is necessary to determine the estimations of the intervals of values of the unknown boundary values for λ_{x0}, λ_{y0} and t_f. Global estimations of the intervals of values of coordinates λ_{x0}, λ_{y0} and λ_{z0} of the conjugate vector at the initial time moment may be determined based on the canonical form (18) as follows

$$-0.0165 \le \lambda_{i0} \le 0.0165, \quad i = x, y, z. \tag{20}$$

In the case of multiple solutions to the maximum principle, the global minimum time is that solution which corresponds to the minimum time. Given that the ultimate goal is to determine that solution to the TPBVP corresponding to the minimum time, we will determine the solution to the TPBVP in the interval of the final time moment

$$0 \le t_f \le 0.18\,\text{s}. \tag{21}$$

To this end, let us establish the following functional relations in the numerical form

$$\mathbf{\Gamma}(z)=\left[\Psi_f\left(x_f,y_f,z_f\right),\Psi^*_{f1}\left(x_f,y_f,z_f,\lambda_{xf},\lambda_{zf}\right),\Psi^*_{f2}\left(x_f,y_f,z_f,\lambda_{yf},\lambda_{zf}\right)\right]^T = \mathbf{0}_{3\times1}, \tag{22}$$

where $\mathbf{\Gamma}(z)\equiv\left[\Gamma_1(z),\Gamma_2(z),\Gamma_3(z)\right]^T\in\mathbb{R}^{3\times1}$ is the shooting function [11] and $z=\left[\lambda_{x0},\lambda_{y0},t_f\right]$. The TPBVP solutions may be geometrically represented in space $\mathbb{R}^3$ with axes λ_{x0}, λ_{y0} and t_f, taking into account (20) and (21), by means of the nested **ContourPlot3D()** *Mathematica* function (see[13]). Namely, it is now possible to determine in the space $\left(\lambda_{x0},\lambda_{y0},t_f\right)$ the crossing of surfaces (22) as

$$\begin{aligned} p_f &= \Gamma_1\left(\lambda_{x0},\lambda_{y0},t_f\right)\cap\Gamma_3\left(\lambda_{x0},\lambda_{y0},t_f\right),\\ q_f &= \Gamma_2\left(\lambda_{x0},\lambda_{y0},t_f\right)\cap\Gamma_3\left(\lambda_{x0},\lambda_{y0},t_f\right), \end{aligned} \tag{23}$$

where p_f and q_f are space curves represented by the following functional dependencies in the numerical form

$$p_f=f_p\left(\lambda_{x0},t_f\right),\quad q_f=f_q\left(\lambda_{x0},t_f\right). \tag{24}$$

Now, the TPBVP solutions may be geometrically represented by the points obtained at the crossings of space curves (24), in space $\left(\lambda_{x0},\lambda_{y0},t_f\right)$, as

$$f_p\left(\lambda_{x0},t_f\right)\cap f_q\left(\lambda_{x0},t_f\right)=\{M_1,\ldots,M_r\}. \tag{25}$$

The number of the elements of a set (25) equals the number of the TPBVP solutions, whereas the coordinates of crossing points in space $\left(\lambda_{x0},\lambda_{y0},t_f\right)$ represent the TPBVP solutions.

Figure 2.1 shows space curves p_f and q_f, while Figure 2.2 displays crossing points of space curves. It is evident that TPBVP does not have single solution.

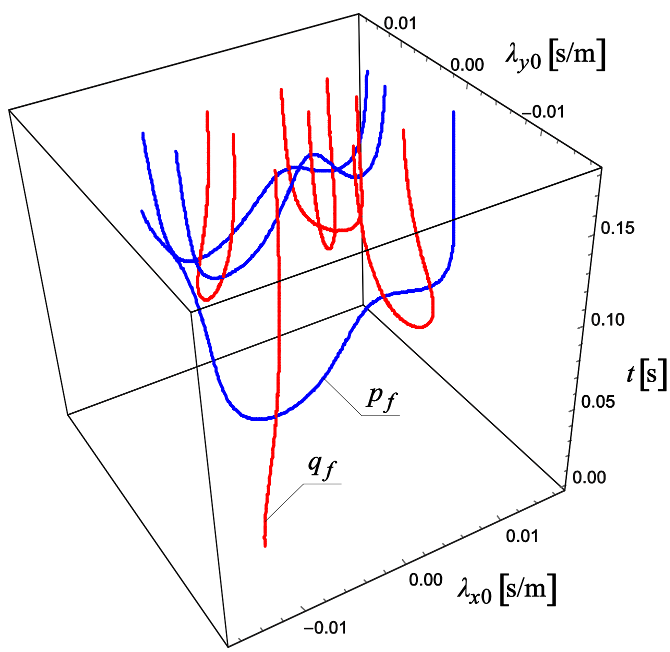

Figure 2.1 Crossing of space curves $p_f=f_p\left(\lambda_{x0},t_f\right)$ and $q_f=f_q\left(\lambda_{x0},t_f\right)$

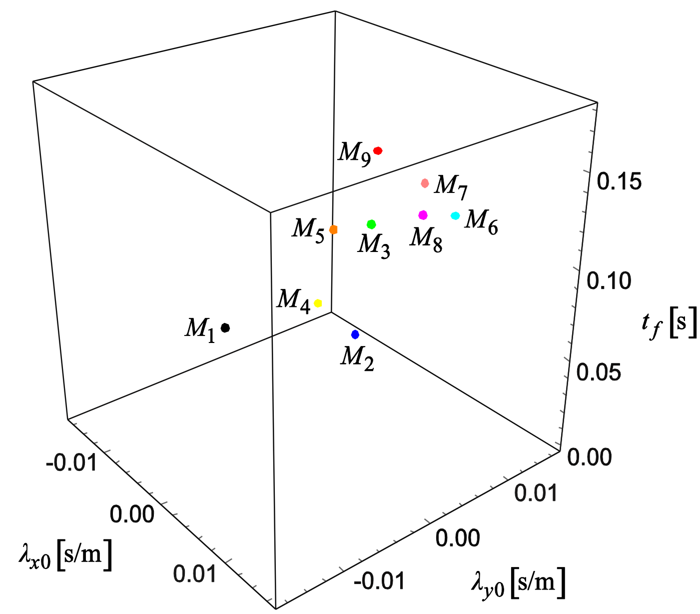

Figure 2.2 Crossing points $M_i\left(i=\overline{1,9}\right)$ of space curves $p_f = f_p\left(\lambda_{x0}, t_f\right)$ and $q_f = f_q\left(\lambda_{x0}, t_f\right)$

Table 2.1 displays the solutions to TPBVP for $\lambda_{z0} = \sqrt{\frac{m}{2\left(E_0 - \Pi_0\right)} - \left(\lambda_{x0}^2 + \lambda_{y0}^2\right)}$, whereas the solutions to TPBVP for $\lambda_{z0} = -\sqrt{\frac{m}{2\left(E_0 - \Pi_0\right)} - \left(\lambda_{x0}^2 + \lambda_{y0}^2\right)}$ do not exist.

Solutions	t_f [s]	λ_{x0} [s/m]	λ_{y0} [s/m]
1st solution (M_1)	0.057798	–0.005680	–0.005680
2nd solution (M_2)	0.091917	0.011108	–0.003956
3rd solution (M_3)	0.091917	–0.003956	0.011108
4th solution (M_4)	0.096656	0.006239	–0.004027
5th solution (M_5)	0.096656	–0.004027	0.006239
6th solution (M_6)	0.125569	0.009218	0.009218
7th solution (M_7)	0.133524	0.004878	0.009556
8th solution (M_8)	0.133524	0.009556	0.004878
9th solution (M_9)	0.156655	0.004235	0.004235

Table 2.1 Numerical solutions to TPBVP

Figure 2.3 shows the trajectories of particle M for specified boundary values given in Table 2.1. Based on specified values it can be inferred that global minimum time for the brachistochronic motion of particle M in space corresponds to the 1st solution (M_1 presented in Figure 2.2) and amounts to $t_f = 0.057798\,\mathrm{s}$.

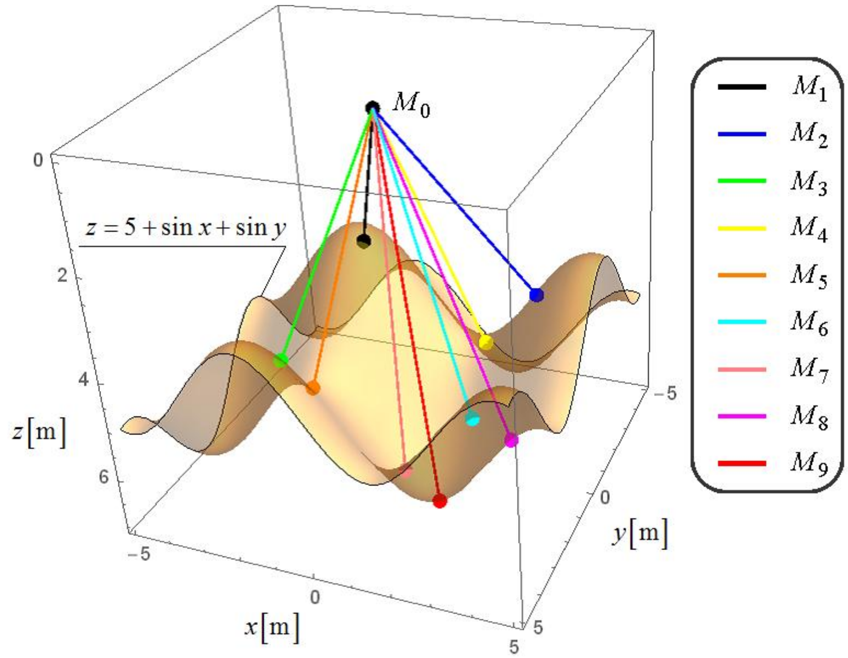

Figure 2.3 Trajectories of particle M corresponding respectively to the solutions $M_i \left(i=\overline{1,9}\right)$

Figure 2.4 shows graphical representation of finite equations of motion $x = x(t), y = y(t)$ and $z = z(t)$ of particle M corresponding to the 1st solution M_1, and also optimal controls $u_x = u_x(t), u_y = u_y(t)$ and $u_z = u_z(t)$.

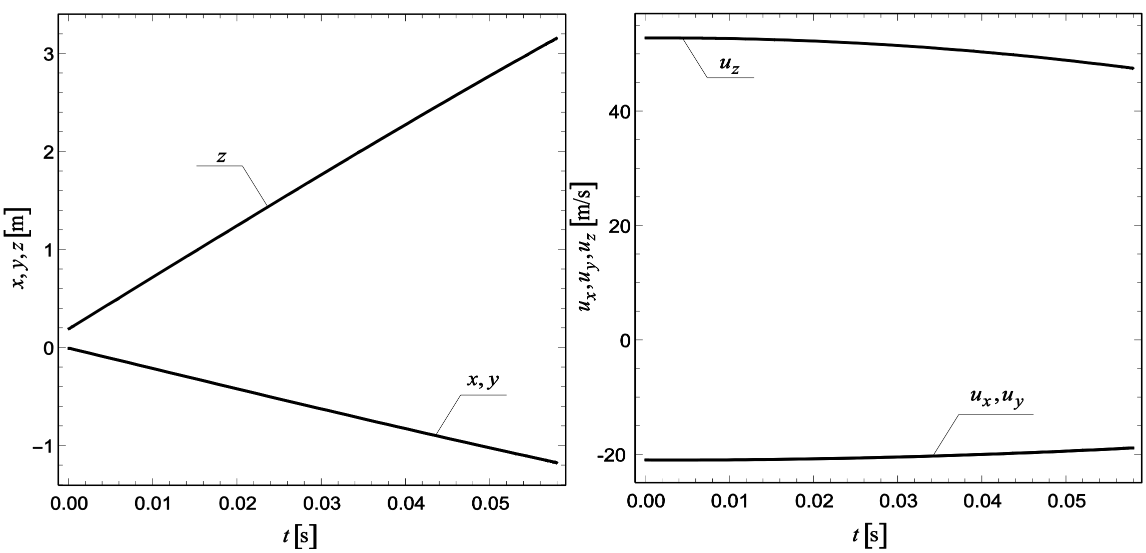

Figure 2.4 Finite equations of motion $x = x(t), y = y(t)$ and $z = z(t)$ and optimal controls $u_x = u_x(t), u_y = u_y(t)$ and $u_z = u_z(t)$ of particle M corresponding to the first solution (M_1)

Conclusions

The paper considers the brachistochronic motion of a particle moving in the space of arbitrary field of potential forces. The brachistochrone problem is formulated as an optimal control task by applying Pontryagin's maximum principle [3; 4; 5; 6; 7]. The paper provides the procedure for the global estimation of initial values of the conjugate vector coordinates, so that the solutions for the obtained TPBVP are certainly located within specified intervals. Numerical procedure for solving TPBVP is based on the method of shooting [12]. Using given estimations for initial values of the conjugate vector coordinates, the global minimum time for the brachistochronic motion of the particle considered is given.

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