DETERMINATION OF CONSTANT PARAMETERS OF OPTIMALLY CONTROLLED SYSTEMS

Josif Vuković, Aleksandar Obradović**
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PROBLEM STATEMENT

We consider a system the state of which in a space X is defined by an n-dimension vector $x=(x^1, x^2, ..., x^n)$. The vector equation of the controlled motion of such a system has the form of

$$\dot{x} = f(x, c, u), \quad x \in X, \quad u \in U, \quad c \in W \tag{1}$$

where: $u=(u_1, u_2, ..., u_r)$ is an r-dimension control vector, $c=(c^1, c^2, ..., c^m)$ an m-dimension vector of constant parameters of the system, U a domain of permitted controls and W a domain of permitted parameters. We will assume that the vector function f(x, c, u) and its derivatives in terms of all variables are defined and continuous in U×W. Let the initial state of the system in the multiplicity be

$$\varphi_\sigma\left[x(t_o), c\right] = 0, \quad \sigma = \overline{1, p}; \quad \text{rank}\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial c}\right) = p \tag{2}$$

and the final state in the multiplicity

$$\varphi_\rho\left[x(t_1), c\right] = 0, \quad \rho = \overline{1, q}, \quad \text{rank}\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial c}\right) = q \tag{3}$$

Let us set up the task to determine controls $u^* \in U$, and constant parameters $c^* \in W$ that, along the trajectory of the system which is moving from the state (2) into the state (3), the functional

$$J = g\left[x(t_1), c\right] + \int_{t_0}^{t_1} f^o(x, c, u)dt \tag{4}$$

attains a minimal value.

THE SOLUTION OF THE PROBLEM

We will apply Pontryagin's maximum principle [2]. The task, however has to be brought to such a form that the theorem of the maximum principle is directly applicable. To this end we propose to introduce new variables:

$$x^o = \int_{t_0}^{t} f^o(x, c, u)d\tau, \quad x^{n+v} = c^v,$$

$$x^{n+m+1} = \int_{t_o}^{t} u_o d\tau \tag{5}$$

where:

$$x^{n+m+1}(t_o) = 0, \quad x^{n+m+1}(t_1) - g\left[x(t_1), c\right] = 0. \tag{6}$$

Thus we obtain vectors: $\tilde{x} = (x^o, x^i, x^{n+v}, x^{n+m+1})$, $(i = \overline{1, n})$, $(v = \overline{1, m})$ and $\tilde{u} = (u_o, u_\alpha)$, $(\alpha = \overline{1, r})$. On the basis of (5) and (6) we can write these equations:

$$\dot{x}^o = \tilde{f}^o(\tilde{x}, \tilde{u}) = f^o(\tilde{x}, u) + u_o,$$
$$\dot{x}^{n+v} = 0, \quad \dot{x}^{n+m+1} = u_o \tag{7}$$

which, together with (1), give the system

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}), \quad \tilde{x} \in \tilde{X}, \quad \tilde{u} \in \tilde{U}. \tag{8}$$

Therefore, we have reduced the problem to the consideration of the motion of the system in an n+m+2-dimension space $\tilde{X}$ under the influence of an r+1-dimension control vector $\tilde{u} \in \tilde{U}$.

The initial and final states of such a system in the space $\tilde{X}$, in view of (2), (3) and (6), are located in the multiplicities:

$$\tilde{\varphi}\left[\tilde{x}(t_o)\right] = \left\{\varphi_\sigma\left[\tilde{x}(t_o)\right] = 0, \ x^{n+m+1}(t_o) = 0\right\} \tag{9}$$

$$\tilde{\psi}\left[\tilde{x}(t_1)\right] = \left\{\psi_\rho\left[\tilde{x}(t_1)\right] = 0, \ x^{n+m+1}(t_1) - g\left[\tilde{x}(t_1)\right] = 0\right\}. \tag{10}$$

The functional (4), referring to (5) and (6) has the form of

$$J = \int_{t_0}^{t_1} \tilde{f}^o(\tilde{x}, \tilde{u})dt. \tag{11}$$

The form of the equations (8), conditions (9), (10) and the functional (11) permits a direct application of the maximum principle.

Pontryagin's function has the form of

$$\tilde{\mathcal{H}} = \tilde{\lambda} \cdot \tilde{f}(\tilde{x}, \tilde{u}), \tag{12}$$

where $\tilde{\lambda} = (\lambda_o, \lambda_i, \lambda_{n+v}, \lambda_{n+m+1})$ is an n+m+2-dimension vector.

According to the theorem of the maximum principle it is necessary that there should exist such a vector func-

* Faculty of Mechanical Engineering, University of Belgrade, Yugoslavia

tion $\tilde{\lambda}(t)$ different from zero, the solution of the equations

$$\dot{\tilde{\lambda}} = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{x}} \quad (13)$$

as to satisfy the following conditions:

a) $\tilde{\mathcal{H}}(\tilde{x}^*, \tilde{u}^*, \tilde{\lambda}^*) = \sup_{\tilde{u} \in U} \tilde{\mathcal{H}}(\tilde{x}^*, \tilde{u}, \tilde{\lambda}^*)$, (14)

b) $\lambda_o = \text{const.} \leq 0$, (15)

c) conditions of transversality in the initial and final states are satisfied, i.e.:

$$(\tilde{\lambda}\delta\tilde{x})_{t_o} = 0, \quad \left(\frac{\partial\tilde{\varphi}}{\partial\tilde{x}}\delta\tilde{x}\right)_{t_o} = 0 \quad (16)$$

$$(\tilde{\lambda}\delta\tilde{x})_{t_1} = 0, \quad \left(\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\delta\tilde{x}\right)_{t_1} = 0. \quad (17)$$

By introducing the function

$$\mathcal{H} = \lambda_o f^o(x, c, u) + \lambda_i f^i(x, c, u), \quad (18)$$

we can write the function (12) in a developed form:

$$\tilde{\mathcal{H}} = \mathcal{H} + \lambda_o u_o + \lambda_{n+m+1} u_o. \quad (19)$$

Now, the equations (13) obtain the form:

$$\dot{\lambda}_o = 0, \quad \dot{\lambda}_i = -\frac{\partial \mathcal{H}}{\partial x^i},$$

$$\dot{\lambda}_{n+v} = -\frac{\partial \mathcal{H}}{\partial c^v}, \quad \dot{\lambda}_{n+m+1} = 0. \quad (20)$$

The term u_o is not limited ($-\infty < u_o < \infty$), thus, from (14) we have

$$\frac{\partial\tilde{\mathcal{H}}}{\partial u_o} = 0,$$

$$\mathcal{H}(x^*, c^*, u^*, \lambda^*) = \sup_{u \in U} \mathcal{H}(x^*, c^*, u, \lambda^*). \quad (21)$$

Referring to (19), from (21) we have

$$\lambda_{n+m+1} = -\lambda_o = \text{const.} \geq 0. \quad (22)$$

The conditions of transversality(16) and (17), bearing in mind (22), have the form of:

$$(\lambda_i\delta x^i + \lambda_{n+v}\delta c^v)_{t_o} = 0$$

$$\left(\frac{\partial\varphi_\sigma}{\partial x^i}\delta x^i + \frac{\partial\varphi_\sigma}{\partial c^v}\delta c^v\right)_{t_o} = 0. \quad (23)$$

$$\left(\left(\lambda_i - \lambda_o\frac{\partial g}{\partial x^i}\right)\delta x^i + \left(\lambda_{n+v} - \lambda_o\frac{\partial g}{\partial c^v}\right)\delta c^v\right)_{t_1} = 0.$$

$$\left(\frac{\partial\psi_\rho}{\partial x^i}\delta x^i + \frac{\partial\psi_\rho}{\partial c^v}\delta c^v\right)_{t_1} = 0 \quad (24)$$

In our further consideration we shall omit the first and the last equation of the system (13). From the equations (13) we obtain m relationships:

$$\int_{t_o}^{t_1} \frac{\partial \mathcal{H}}{\partial c^v} dt = \lambda_{n+v}(t_o) - \lambda_{n+v}(t_1) \quad (25)$$

By using the results obtained we can formulate the following theorem: If $u^*(t)$, c^* are the optimal solutions of the problem and $x^*(t)$ the corresponding solution of the equations (1), then there must exist a function $\mathcal{H}(\lambda, x, c, u)$ and the vector λ different from zero, the solution of the equations:

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} \quad (26)$$

so that

a) $\mathcal{H}(x^*, c^*, u^*, \lambda^*) = \sup_{u \in U} \mathcal{H}(x^*, c^*, u, \lambda^*)$. (27)

b) $\lambda_o = \text{const.} \leq 0$ (28)

c) conditions of transversality (23) and (24) are satisfied.

d) $\int_{t_o}^{t_1} \frac{\partial \mathcal{H}}{\partial c^v} dt = \lambda_{n+v}(t_o) - \lambda_{n+v}(t_1)$. (29)

This theorem is different from the theorem of the principle of maximum, the difference being the condition (d) which provides m additional relationship for the determination of parameters c^*.

Now, attention should be drawn to the following special cases.

I Let the parameters be limited by the relationships $F\mu(c) \leq 0$, ($\mu = \overline{1, s \leq m}$) and let the optimal solution c^* be such that

$$F_k(c^*) = 0, \quad F_j(c^*) < 0, \quad (k = \overline{1, \eta}, j = \overline{\eta + 1, s}). \quad (30)$$

Then, the corresponding part of the relationships in (29) is substitutes by the equations from (30).

II If initial and final conditions do not depend on the parameters c, then the relationship (29) obtain the form of

$$\int_{t_o}^{t_1} \frac{\partial \mathcal{H}}{\partial c^v} dt = -\lambda_o\left(\frac{\partial g}{\partial c^v}\right)_{t_1}. \quad (31)$$

III If the initial and final conditions and the terminal part of the functional (4) do not depend on the parameters c, then the relationships (29) obtain the form of

$$\int_{t_o}^{t_1} \frac{\partial \mathcal{H}}{\partial c^v} dt = 0 \quad (32)$$

EXAMPLE

Weight having mass m is lifted by applying the moment $M = uR$ ($|u| \leq F = \text{const}$) and counterweight m_1. By disregarding inertia of all remaining moveable parts in the

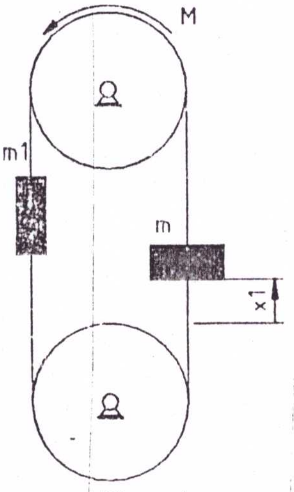

Figure 1.

system, solve the optimal problem by finding the parameter m_1 and control u, in respect to the condition that time of lifting the weight to the height h should be minimal.

Having in mind all this, the optimal problem is defined:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{(m_1 - m)g + u}{m_1 + m} \tag{33}$$

$$t_o = 0, \quad x_1(t_o) = x_2(t_o) = 0; \quad x_1(t_1) = h, \quad x_2(t_1) = 0 \tag{34}$$

$$\int_0^{t_1} dt \to \inf_u, \quad |u| \le F. \tag{35}$$

Owing to (18), (33) and (35) we have:

$$\mathcal{H} = \lambda_o + \lambda_1 x_2 + \lambda_2 \frac{(m_1 - m)g + u}{m_1 + m} \tag{36}$$

where based on (26):

$$\lambda_1 = c_1, \quad \lambda_2 = -c_1 t + c_2. \tag{37}$$

condition (27) allows us to find the optimal control:

$$u^* = F \operatorname{sign} \lambda_2$$

or, with (37):

$$u^* = \begin{cases} F & \forall\ t \in [0, \tau) \\ -F & \forall\ t \in [\tau, t_1]. \end{cases} \tag{38}$$

where, $\lambda_2(\tau) = 0$, or:

$$\lambda_2 = c_1(\tau - t) \quad \forall\ t \in [0, t_1]. \tag{39}$$

Substituting (38) in differential equations (33), from general solutions, and conditions (34), and the conditions of continuity:

$$x_1(\tau^-) = x_1(\tau^+), \quad x_2(\tau^-) = x_2(\tau^+).$$

we get:

$$[(m_1 - m)g - F]t_1 + 2F\tau = 0 \tag{40}$$

$$[(m_1 - m)g - F]t_1^2 + 4F\tau t_1 - 2F\tau^2 = 2h(m_1 + m) \tag{41}$$

Based on (31), we get:

$$\int_{t_o}^{t_1} \frac{\partial \mathcal{H}}{\partial m_1} dt = 0 \Rightarrow \int_{t_o}^{t_1} \lambda_2 \frac{2mg - u}{(m_1 + m)^2} dt = 0$$

or, from (38) and (39):

$$(2mg - F)\tau^2 - (2mg + F)(t_1 - \tau)^2 = 0. \tag{42}$$

From equations (40), (41) and (42), we arrive at:

$$\tau = \sqrt{\frac{h}{g}(2mg + F)}, \quad t_1 = \sqrt{\frac{h}{g}}(\sqrt{2mg + F} + \sqrt{2mg - F}$$

$$m_1 = \frac{1}{g}\sqrt{4m^2g^2 - F^2} - m.$$

From these solutions we can conclude, that for $F \ge mg\sqrt{3}$, placement of counterwieght has no significance.

THEORETICAL EXAMPLE

Consider a controlled motion of a holonomic, scleronomic mechanical system which is described in a phase space V_{2n} by the equations:

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + Q_\alpha^N(q, p) + u_\alpha, \tag{43}$$

where: q^α - generalised coordinates, p_α - generalised impulses, $H = T + V$ - Hamilton's function, T - kinetic energy, V - potential energy, $Q\alpha^N$ - non-potential generalised force, and u_α - generalised force of control. Let the initial and the final state of the system be:

$$\varphi_\sigma = [q(t_o), p(t_o)] = 0, \quad \psi_\rho[q(t_1), p(t_1)] = 0.$$

We will set the following problem. Let the intensity of the control force be constant, i.e.

$$a^{\alpha\beta} u_\alpha u_\beta - c^2 = 0 \tag{44}$$

where $a^{\alpha\beta}$ is a contravariant metric tensor of the configuration space R^n. Let us determine the force of minimum intensity under the action of which the mechanical system will pass from the initial to the final state. Therefore, the functional which is to be minimized has the form of

$$J = c. \tag{45}$$

Pontryagin's function, referring to (43) and (44) has the form of

$$\mathcal{H} = \lambda_\alpha \frac{\partial H}{\partial p_\alpha} + v^\alpha\left(-\frac{\partial H}{\partial q^\alpha} + Q_\alpha^N + u_\alpha\right) + \\ + \mu(a^{\alpha\beta} u_\alpha u_\beta - c^\alpha), \tag{46}$$

where $\mu(t)$ is an underterminated multiplier, and the vector λ_α, v^α is the solution of the equations:

$$\dot{\lambda}_\alpha = -\frac{\partial \mathcal{H}}{\partial q^\alpha}, \quad \dot{v}^\alpha = -\frac{\partial \mathcal{H}}{\partial p_\alpha}. \tag{47}$$

On the basis of (25) we have

$$u^*_\alpha = \frac{a_{\alpha\beta} v^\beta}{\| v \|}, \quad [\| v \| = (a_{\alpha\beta} v^\alpha v^\beta)^{1/2}],$$

and from (31) we have

$$\int_{t_0}^{t_1} \| v \| dt = 1.$$

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DETERMINATION OF CONSTANT PARAMETERS OF THE OPTIMALLY CONTROLLED SYSTEMS

J. Vuković, A. Obradović

The problem of the optimality of the controlled motion was considered by making a suitable selection of some of its constant parameters. The space of the system vectors of states, and of the vectors of control was expanded by introducing the space of vectors of the problem is based upon well-known methods of optimisation.

Under consideration is also the most general case where the parameters are limited, and where they appear in the given initial and final conditions of motion of the system. Additional conditions for the determination of unknown parameters were obtained. These conditions were considered especially for some classes of mechanical systems, the motions of which were described in the phase space V_{2n}.

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ODREDJIVANJE KONSTANTNIH PARAMETARA OPTIMALNO UPRAVLJANIH SISTEMA

J. Vuković, A. Obradović

Razmatra se problem optimalnosti upravljanog kretanja sistema pogodnim izborom nekih njegovih konstantnih parametara. Prostor vektora stanja sistema i vektora upravljanja proširen je prostorom vektora parametara koje treba odrediti. Rešavanje problema zasnovano je na poznatim metodima optimizacije. Razmatra se najopštiji slučaj kada su parametri ograničeni i kada oni figurišu u zadatim početnim i krajnjim uslovima kretanja sistema.

Dobijeni su dopunski uslovi za odredjivanje nepoznatih parametara. Ovi uslovi su posebno razmatrani za neke klase mehaničkih sistema čije kretanje je opisano u faznom prostoru V_{2n}.

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