On the global minimum time in the brachistochronic motion of Chaplygin sleigh

A. Obradovic[1] and R. Radulovic[1]

[1] Faculty of Mechanical Engineering University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35, e-mail: aobradovic@mas.bg.ac.rs, rradulovic@mas.bg.ac.rs

Abstract This paper presents a procedure for determining the global minimum time in the brachistochronic motion of Chaplygin sleigh [3,4] between two specified positions, with unchanged value of mechanical energy during motion. For this case, the problem is formulated as the simplest problem of optimal control theory that is solved by applying Pontryagin's Maximum Principle [1]. The corresponding two-point boundary value problem of the system of ordinary nonlinear differential equations is obtained that is necessary, in a general case, to solve numerically [2]. The numerical procedure is based on the shooting method, with the requirement for the assessment of the intervals in which the missing initial conditions can be found. The assessment is provided of the intervals of initial values of the conjugate variables, so that the TPBVP solution does not exist for sure outside those intervals. Graphic representation is given for corresponding surfaces in 3D space of the missing initial conditions, of which each surface corresponds to satisfying the missing conditions. A number of examples are provided for multiple solutions of the Maximum Principle, of which the global minimum is the one corresponding to the minimum time.

Key words: global minimum time, Chaplygin sleigh, Brachistochrone, Pontryagin's maximum principle.

1 Introduction

In this paper we present a procedure for determining the global minimum time in the brachistochronic motion of the Chaplygin sleigh [3, 4] between two specified positions in a horizontal plane, with unchanged value of mechanical energy during motion. Prior to deriving differential equations of motion, as well as for the needs of further considerations, two Cartesian coordinate reference systems have to be introduced. The immovable coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the horizontal plane of motion and the movable coordinate system $A\xi\eta\zeta$ that is stiffly attached to the knife edge, so that the coordinate plane $A\xi\eta$ coincides with the Oxy-plane, where the $A\xi$-axis coincides with the orientation of the edge (Fig. 1). Unit vectors of the movable coordinate system axes are $\vec{\lambda}, \vec{\mu}$ and $\vec{\nu}$, respectively. The configuration of the knife edge relative to the system Oxy is defined by a set of Lagrangian coordinates $\left(q^1, q^2, q^3\right)$, where $q^1 = x$ and $q^2 = y$ are Cartesian coordinates of the point A, while $q^3 = \varphi$ is the angle between the Ox-axis and the $A\xi$-axis. Further analysis involves the case when point A is not allowed to move in the direction perpendicular to the edge, causing the occurrence of horizontal reaction of the immovable surface $\vec{R} = R\vec{\mu}$. The motion of the edge is limited by an ideal nonholonomic homogeneous constraint

$$-\dot{x}\sin\varphi + \dot{y}\cos\varphi = 0. \tag{1}$$

The consequence of the imposed constrained motion is equal velocity $\vec{V}$ of point A of the edge to the direction of the axis $A\xi$, so the relation (1) can be expressed in the form

$$\dot{x} = V\sin\varphi, \quad \dot{y} = V\cos\varphi, \tag{2}$$

where $V = \vec{V}\cdot\vec{\lambda}$. In this paper, we consider the case when the center of mass of the edge, point C, is positioned on the $A\xi$-axis, i.e. $C \in A\xi$, at the distance $\overline{AC} = a$. The mass of the edge is m, whereas I_C is the moment of inertia around the principal central axis of inertia perpendicular to the Oxy-plane.

During the brachistochronic motion of the edge, the law of conservation of mechanical energy holds

$$\Phi\left(V, \dot{\varphi}\right) = V^2 + a^2k^2\dot{\varphi}^2 - \frac{2T_0}{m} = 0, \tag{3}$$

where

$$k^2 = 1 + \frac{I_C}{ma^2}, \tag{4}$$

and T_0 kinetic energy of the edge at initial time moment is $t_0 = 0$.

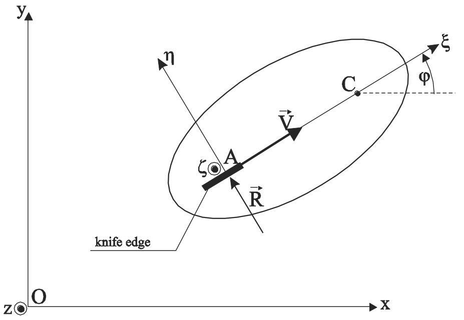


Fig. 1. Chaplygin knife edge.

2 Brachistochronic motion of the Chaplygin sleigh as the optimal control task

This Section considers the problem of the brachistochronic motion of the Chaplygin sleigh as a task of optimal control. Equations of state that describe the motion of the edge in state space can be defined in the form

$$\dot{x} = u_1 \cos\varphi, \quad \dot{y} = u_1 \sin\varphi, \quad \dot{\varphi} = u_2, \tag{5}$$

where controls u_1 and u_2 represent the velocity of point A of the edge and the angular velocity of the edge, respectively.

State coordinates x, y and φ are determined at the outset of motion:

$$t_0 = 0, \quad x(t_0) = 0, \, y(t_0) = 0, \, \varphi(t_0) = 0, \tag{6}$$

as well as state coordinates x, y and φ at the final position of the edge:

$$t = t_f, \quad x(t_f) = l, \, y(t_f) = l, \, \varphi(t_f) = \varphi_f. \tag{7}$$

The brachistochrone problem of the motion of the edge described by differential equations (5) consists in determining the controls u_1 and u_2 as well as their corresponding state coordinates x, y and φ, so that the edge starting from initial state (6) moves to the final state (7), with unchanged value of mechanical energy (3), for the minimum time. This can be expressed in the form of condition that the functional

$$I = \int_{t_0}^{t_f} dt, \tag{8}$$

in the interval $\left[t_0, t_f\right]$ has a minimum value. To solve the task of optimum control, defined by Pontryagin's Maximum Principle [1], Pontryagin's function is derived in the form as follows

$$H = \lambda_0 + \lambda_x u_1 \cos\varphi + \lambda_y u_1 \sin\varphi + \lambda_\varphi u_2 + \mu \Phi\left(u_1,\ u_2\right), \tag{9}$$

where $\lambda_0 = \text{const.} \le 0,\ \lambda_x,\ \lambda_y$ i λ_φ are the coordinates of conjugate vector, where it can be taken that $\lambda_0 = -1$, while μ is a multiplier corresponding to (3). Based on Pontryagin's function (9), the conjugate system of differential equations has the form

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_y = 0, \quad \dot{\lambda}_\varphi = u_1\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi\right). \tag{10}$$

If controls belong to an open set, like in this case, conditions for determining optimal control can be expressed in the form

$$\left(\frac{\partial H}{\partial u_i}\right)_{\boldsymbol{u}^{opt}} = 0, \quad \left(\frac{\partial^2 H}{\partial u_i \partial u_j}\right)_{\boldsymbol{u}^{opt}} u_i u_j \le 0, \quad \left(i,\ j\text{=1,2}\right). \tag{11}$$

Applying the Theorem 1 [1], it follows directly that the value of Pontryagin's function on the optimal trajectory equals zero for $\forall t \in \left[t_0,\ t_f\right]$, that is

$$-1 + \lambda_x u_1 \cos\varphi + \lambda_y u_1 \sin\varphi + \lambda_\varphi u_2 + \mu \Phi\left(u_1,\ u_2\right) = 0, \tag{12}$$

as well as that, based on (10), $\lambda_x = \text{const.}$ and $\lambda_y = \text{const.}$ for $\forall t \in \left[t_0,\ t_f\right]$. Now, based on (11) and (12), one obtains the value of the multiplier μ as well as the control function in the form as follows

$$\mu = -\frac{1}{4T_0}, \quad u_1 = \frac{2T_0}{m}\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right), \quad u_2 = \frac{2T_0}{m}\frac{1}{a^2 k^2}\lambda_\varphi. \tag{13}$$

Based on relation (12) determined at the initial time moment, as well as on (6), (11) and (13), one obtains the value of coordinate λ_φ at the initial time moment, whereas the conjugate vector coordinate λ_x can be expressed in the form

$$\lambda_\varphi(t_0) = \pm ak\sqrt{\frac{m}{2T_0} - \lambda_x^2}, \quad \lambda_x = \frac{mV_0}{2T_0}, \tag{14}$$

where V_0 is the velocity of point A at the initial time moment.

The fundamental and conjugate system of differential equations, based on (5), (10) and (13), can be given in the form as follows

$$\begin{aligned} &\dot{x} = \frac{2T_0}{m}\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right)\cos\varphi, \quad \dot{y} = \frac{2T_0}{m}\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right)\sin\varphi, \\ &\dot{\varphi} = \frac{2T_0}{m}\frac{1}{a^2k^2}\lambda_\varphi, \quad \dot{\lambda}_\varphi = \frac{2T_0}{m}\left(\lambda_x \cos\varphi + \lambda_y \sin\varphi\right)\left(\lambda_x \sin\varphi - \lambda_y \cos\varphi\right), \end{aligned} \tag{15}$$

whereas state coordinates x, y and φ, as well as the conjugate vector coordinate λ_φ, based on (6) and (14), are determined at the initial time moment

$$t_0 = 0, \quad x(t_0) = 0,\ y(t_0) = 0,\ \varphi(t_0) = 0,\ \lambda_\varphi(t_0) = \pm ak\sqrt{\frac{m}{2T_0} - \lambda_x^2}. \tag{16}$$

Numerical procedure for solving the corresponding two-point boundary value problem of the system of first-order ordinary nonlinear differential equations, based on (7), (15) and (16), is solved by the shooting method [2]. This method consists in determining the unknown coordinates of the conjugate vector λ_x and λ_y, as well as the minimum necessary time t_f, so that the knife edge starting from the initial state (6) moves to the final state (7).

3 Assessment of values for the conjugate vector coordinates in shooting method

The application of shooting method requires the assessment of the interval of parameters' values to be determined. Global assessment of the interval of values for the conjugate vector coordinate λ_x can be made based on (14)

$$-\sqrt{\frac{m}{2T_0}} \le \lambda_x \le \sqrt{\frac{m}{2T_0}}, \tag{17}$$

whereas the assessment of the value of the conjugate vector coordinate λ_y can be made based on the value of angle φ_f at the final position of the edge, as well as by using both (3) and (13), in the form

$$\left|\lambda_x \cos\varphi_f + \lambda_y \sin\varphi_f\right| \le \sqrt{\frac{m}{2T_0}}. \quad (18)$$

Based on the assessments of the interval of parameters' values that is determined, given in (17) and (18), it can be asserted that all solutions of corresponding two-point boundary value problem are definitely situated within the specified intervals, hence the global minimum time in brachistochronic motion of the Chaplygin sleigh. For the case of multiple solutions of the Maximum Principle, global minimum is the solution corresponding to minimum time. The solutions of obtained two-point boundary value problem can be represented by the following dependencies in the numerical form [5]

$$x_f = f_x\left(\lambda_x, \lambda_y, t_f\right), \quad y_f = f_y\left(\lambda_x, \lambda_y, t_f\right), \quad \varphi_f = f_\varphi\left(\lambda_x, \lambda_y, t_f\right). \quad (19)$$

Dependencies (19) can be graphically represented in three-dimensional $\lambda_x, \lambda_y, t_f$ -space, where the solutions of the system of nonlinear equations (19) are located at the cross-section of given surfaces.
Two-boundary value problem is solved for the following values of the parameters

$$m = 2\text{kg}, \quad l = 1\text{m}, \quad k = 1.5, \quad T_0 = 200\frac{\text{kgm}^2}{\text{s}^2}. \quad (20)$$

4. 1. Global minimum time at the final value of the angle of the knife edge $\varphi_f = \frac{\pi}{2}$

Based on (19) and (20), global assessment can be made of the interval of values for the conjugate vector coordinate λ_x

$$-0.0707 \le \lambda_x \le 0.0707, \quad (21)$$

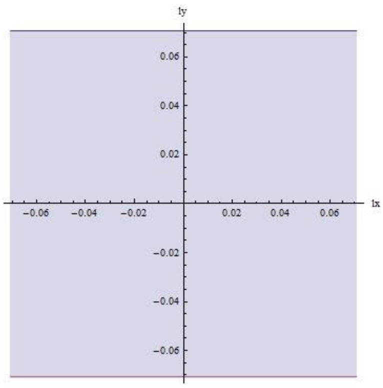
whereas the assessment of the interval of values for the conjugate vector coordinate λ_y can be made based on (18) and (20), as well as the final values of the angle of the knife edge $\varphi_f = \pi / 2$

$$-0.0707 \le \lambda_y \le 0.0707, \tag{22}$$

which is graphically represented in Fig. 2.

In accordance with (8), the time of brachistochronic motion of the edge as well as the conjugate vector coordinates for specified values of the parameters (20), are $t_f = 0.199832$ s, $\lambda_x = 0.0303507$s/m and $\lambda_y = 0.0303507$s/m.

Global minimum time in the brachistochronic motion of the edge can be given based on graphic representation of the solution of the system of nonlinear equations (19), as well as on the assessment of coordinates (21) and (22), as shown in Fig. 3.



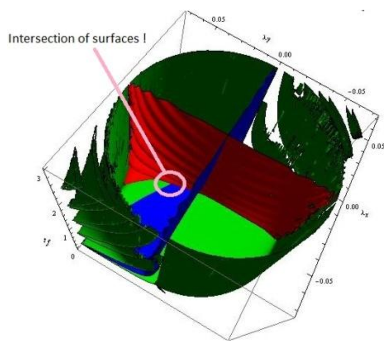


Fig. 2. Assessment of conjugate vector coordinates λ_x and λ_y at $\varphi_f = \pi / 2$.

Fig. 3. Cross-section of surfaces $x_f = f_x(\lambda_x, \lambda_y, t_f)$, $y_f = f_y(\lambda_x, \lambda_y, t_f)$ and $\varphi_f = f_\varphi(\lambda_x, \lambda_y, t_f)$ at $\varphi_f = \pi / 2$.

It is evident from Fig. 3 that the solution of corresponding two-point boundary value problem is unique, i.e. the cross-section of surfaces, corresponding to the fulfillment of the final position of the edge, determined by state coordinates x_f, y_f and φ_f is at a single point.

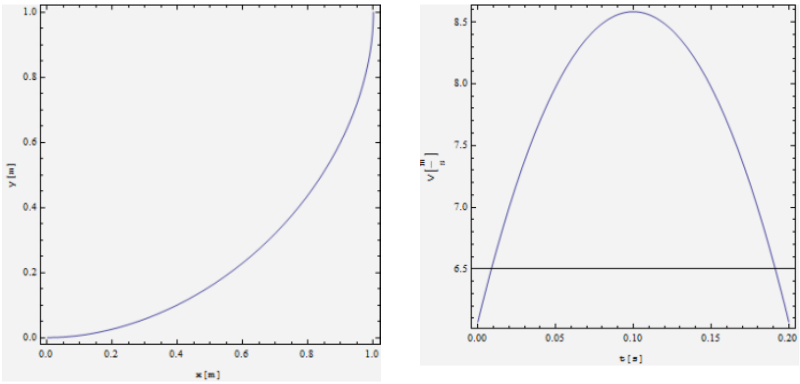


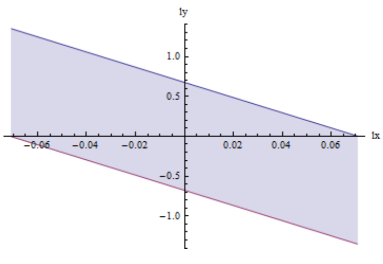
Fig. 4. Trajectory and velocity of point A of the knife edge at $\varphi_f = \pi/2$.

4. 2. Global minimum time at the final value of the angle of the knife edge $\varphi_f = \dfrac{\pi}{30}$

Based on (18), (20) and final value of the angle of the edge $\varphi_f = \pi/30$,an assessment can be made of the interval of values for the conjugate vector coordinate λ_y

$$-(0.6765 + 9.5144\lambda_x) \le \lambda_y \le 0.6765 - 9.5144\lambda_x, \tag{23}$$

whereas the assessment of coordinate λ_x is given in (21), as graphically represented in Fig. 5.



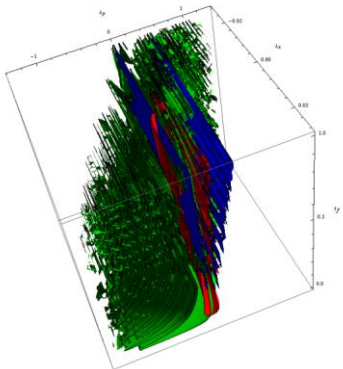


Fig. 5. Assessment of conjugate vector coordinates λ_x and λ_y at $\varphi_f = \pi/30$.

Fig. 6. Cross-section of surfaces $x_f = f_x(\lambda_x, \lambda_y, t_f)$, $y_f = f_y(\lambda_x, \lambda_y, t_f)$ and $\varphi_f = f_\varphi(\lambda_x, \lambda_y, t_f)$ at $\varphi_f = \pi/30$.

For this case, the solution of corresponding two-point boundary value problem is not unique. To find all possible solutions of the Maximum Principle, it is more convenient to use graphic analysis of the solution in 3D space of the missing parameters, as shown in Fig. 6

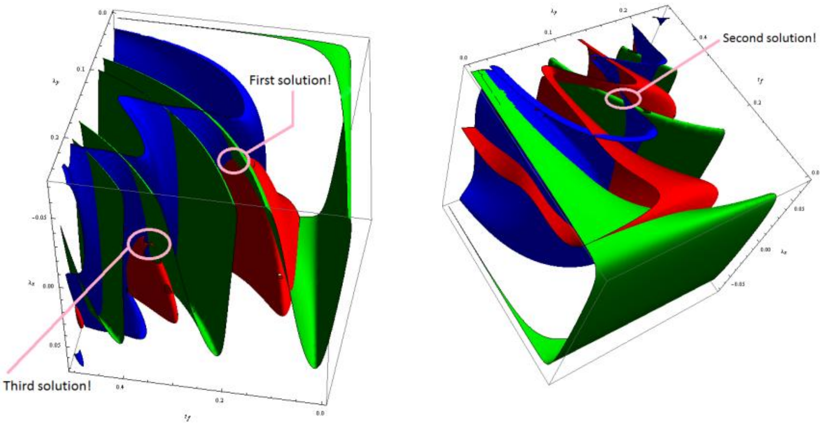


Fig. 7. Solutions at $\varphi_f = \pi / 30$ shown in Table 1.

Solutions of two-point boundary value problem at the final value of the angle of the knife edge $\varphi_f = \pi/30$, satisfying the Maximum Principle, in accordance with (8), are shown in Table 1.

Table 1.

Solutions	λ_x	λ_y	t_f
First solution	-0.042973	0.138484	0.229455
Second solution	0.0526389	0.151529	0.361141
Third solution	-0.0113855	0.198418	0.368861

Global minimum time in the brachistochronic motion of the knife edge, corresponding to the final value of the angle of the edge $\varphi_f = \pi / 30$, corresponds to the First solution presented in Table 1., i.e., $t_f = 0.229455\ \text{s}$.

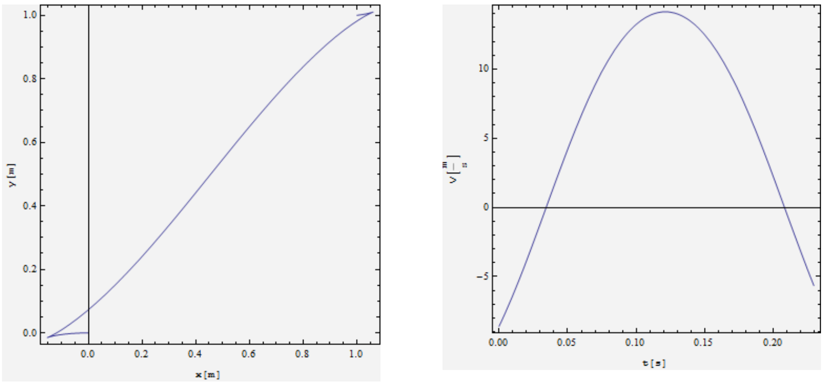


Fig. 8. Trajectory and velocity of point *A* of the knife edge at $\varphi_f = \pi/30$ corresponding to the First solution.

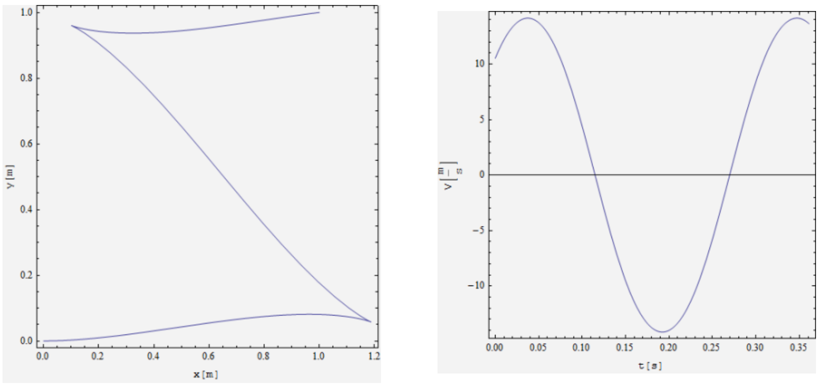


Fig. 9. Trajectory and velocity of point *A* of the knife edge at $\varphi_f = \pi/30$ corresponding to the Second solution.

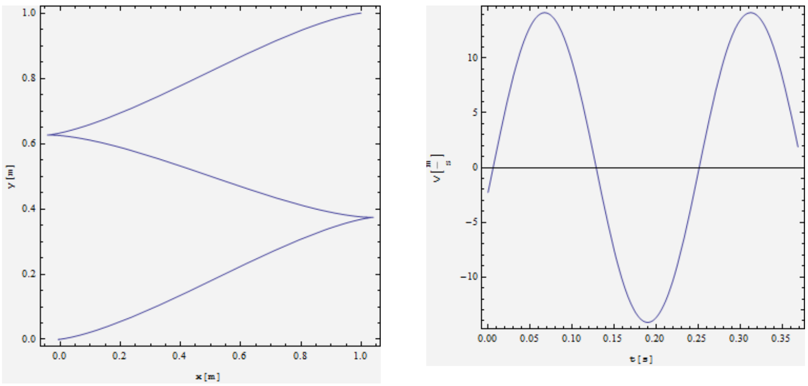


Fig. 10. Trajectory and velocity of point A of the knife edge at $\varphi_f = \pi/30$ corresponding to the Third solution.

4. 3. Global minimum time at the final value of the angle of the knife edge $\varphi_f = 0$

It should be taken into account that at the final position of the edge, corresponding to the value of the angle $\varphi_f = 0$, as well as (18), the assessment cannot be made of the interval of values for the conjugate vector coordinate λ_y, whereas the assessment of coordinate λ_x is given in (21).

For this case, the solution of corresponding two-point boundary value problem is not unique either, and can be given based on graphic representation of the solution of the system of nonlinear equations (19), as well as on the assessment of the coordinate (21), as shown in Fig. 11.

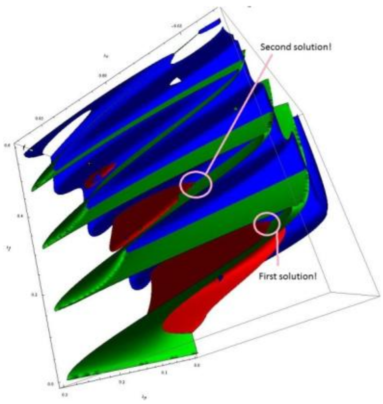


Fig. 11. Solutions at $\varphi_f = 0$ shown in Table 2.

The solutions of two-point boundary value problem at the final value of the angle of the edge $\varphi_f = 0$, satisfying the Maximum Principle, in accordance with (8), are shown in Table 2.

Table 2.

Solutions	λ_x	λ_y	t_f
First solution	-0.0391831	0.137402	0.239187
Second solution	-0.0111406	0.198342	0.379927

Global minimum time in the brachistochronic motion of the knife edge, corresponding to the final value of the angle of the edge $\varphi_f = 0$, corresponds to the First solution shown in Table 2., i.e., $t_f = 0.239187\ \text{s}$.

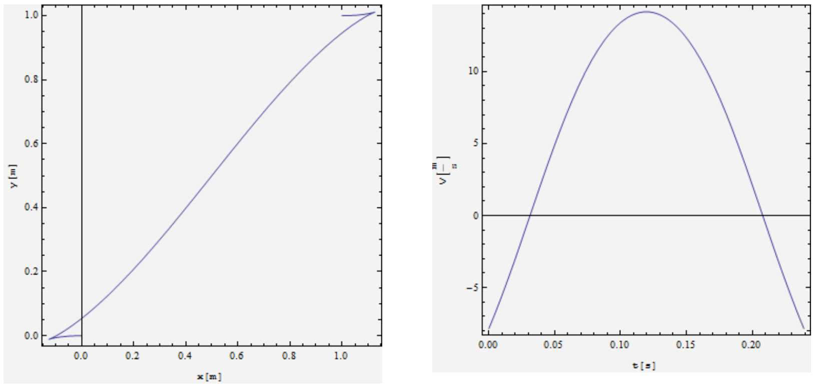


Fig. 12. Trajectory and velocity of point A of the knife edge at $\varphi_f = 0$ corresponding to the First solution.

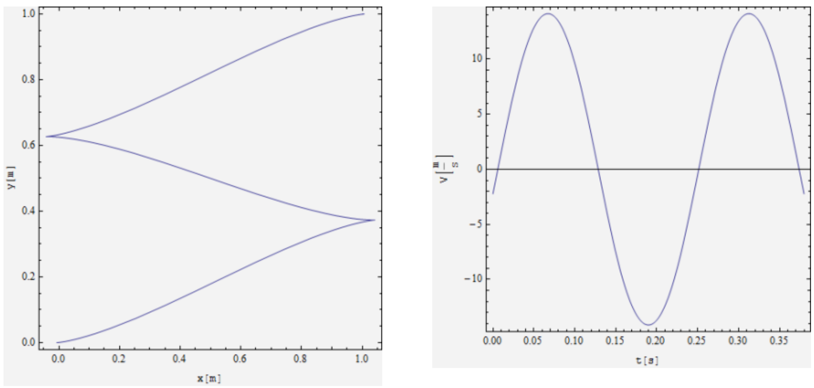


Fig. 13. Trajectory and velocity of point A of the knife edge at $\varphi_f = 0$ corresponding to the Second solution.

5 Conclusions

The paper presents a procedure for determining the global minimum time in the brachistochronic motion of the Chaplygin sleigh between two specified positions, with unchanged value of mechanical energy during motion. The formulated brachistochrone problem is solved, for this case, as the simplest possible task of optimal control by applying Pontryagin's Maximum Principle. Numerical procedure for solving the corresponding two-point boundary value problem is based on shooting method. The paper describes the assessment procedure for the values of unknown conjugate vector coordinates, so that the solution of obtained two-point boundary value problem is definitely situated within certain intervals. Graphic representation is given via corresponding surfaces in 3D space of the missing conditions, of which each surface corresponds to the fulfillment of one end condition. A number of examples are provided for multiple solutions of the Maximum Principle, of which global minimum is the one corresponding to the minimum time.

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