REALIZING BRACHISTOCHRONIC MOTION OF A VARIABLE MASS BODY BY CENTRODES

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Abstract:

The paper considers realization of the brachistochronic motion of a mechanical system, composed of free body and variable mass material points, by means of an ideal constraint in the form of the centrodes. It is assumed that the system performs planar motion in an arbitrary field of forces and that it has three degrees of freedom. In addition, the laws of the time-rate of mass variation of the material points, as well as relative velocities of the expelled particles, respectively, are known. Constraint reactions of the centrodes are expressed in the function of the generalized forces. Applying Pontryagin's maximum principle and singular optimal control theory, the problem of brachistochronic motion is solved as a two-point boundary value problem (TPBVP). The considerations are illustrated via an example, where it is examined how the change in the initial energy of the system affects the normal reaction of the connection and thus the coefficient of rolling friction.

Key words: brachistochrone, variable mass, mechanical system, Pontryagin's maximum principle, optimal control

1. Formulation of the problem

Consider planar motion of the mechanical system composed of free body and 2 variable mass material points. The example shows a disk with radius R and mass M and two variable mass points A and B on the periphery of the disc, as indicated by Fig. 1. The system moves in a vertical plane. In step 1, for the needs of further considerations, two Cartesian coordinate reference systems must be introduced. The first, a stationary coordinate system $Oxyz$, whose coordinate plane Oxy coincides with the vertical plane of motion, and the second, a non-stationary coordinate system $C\xi\eta\zeta$, whose coordinate origin is attached to center of the disk, the coordinate plane $C\xi\eta$ coinciding with the plane Oxy. In addidtion, the axis of the non-stationary coordinate system $C\xi$ is defined by the direction AB. Unit vectors of the non-stationary coordinate system axes are $\vec{\lambda}$, $\vec{\mu}$ and $\vec{\nu}$, respectively. The configuration of the considered system is defined by a set of Lagrangian coordinates $\boldsymbol{q} = \left(q^1, q^2, q^3\right)^T$, where $q^1 \triangleq x$ and $q^2 \triangleq y$ are Cartesian coordinates of center of the disk, $q^3 \triangleq \varphi$ is the angle between the axes Ox and $C\xi$, which are geometrically independent, and based on them the mechanical system position is unambiguously determined. Changes in masses of the points A and B are specified in the following form:

$$m_A(t) = m_B(t) = m(t) = m_0 - kt, \tag{1}$$

where m_0 is mass of the points A and B at the initial instant of time, and k is defined positive constant. Without loss of generality, the magnitudes of relative velocities of the particles' expelling from the points A and B are constant and mutually equal:

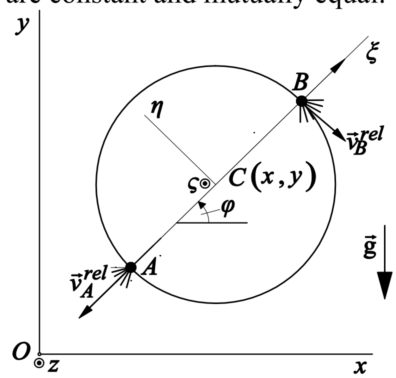

Figure 1. Variable mass mechanical system.

$$v_A^{rel} = v_B^{rel} = v_r, \tag{2}$$

where v_r, is a defined positive constant, and $\vec{v}_A^{rel} = -v_r\vec{\lambda}$ and $\vec{v}_B^{rel} = -v_r\vec{\mu}$. The kinetic energy of mechanical system is a homogeneous quadratic form of generalized velocities [1, 2]:

$$T = \frac{1}{2} a_{ij}\dot{q}^i\dot{q}^j, \qquad i,j = 1,2,3, \tag{3}$$

where $a_{ij} = a_{ij}(\boldsymbol{q}, t)$ are the covariant coordinates of metric tensor of the function of generalized coordinates and time t. The kinetic energy of the system, according to (3), is written in the following form:

$$T = \frac{1}{2}\left((M+2m)\left(\dot{q}^1\right)^2 + (M+2m)\left(\dot{q}^2\right)^2 + (M+2m)R^2\left(\dot{q}^3\right)^2 \right). \tag{4}$$

Also, the well known Einsten summation convention is deployed in the paper, where the indices have a range of values as follows: $i,j,k,r,\delta = 1,2,3;\ \alpha,\ \beta = 1,2.$ It can be considered that the studied mechanical system is moving in a field of known potential forces, whose potential energy equals:

$$\Pi = \Pi(\boldsymbol{q}, t), \tag{5}$$

and that the system is acted on by known arbitrary nonpotential forces, so that the generalized forces are:

$$Q_i^w = Q_i^w(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{6}$$

The differential equations of motion for the considered system, as a function of kinematically independent coordinates, are written in contravariant form [1, 2, 3]:

$$\ddot{q}^\delta + \Gamma_{kr}^\delta \dot{q}^k \dot{q}^r = -a^{i\delta}\dot{a}_{ij}\dot{q}^j + Q^\delta, \tag{7}$$

where:

$$Q^\delta = a^{i\delta} Q_i, \tag{8}$$

are the generalized forces in contravariant form, $a^{i\delta}$ are the contravariant coordinates of metric tensor and Γ_{kr}^δ are Christoffel symbols of the second kind. The generalized forces corresponding to geometrically independent coordinates can be represented, in a general case, in the form as follows [4, 5]:

$$Q_i(\boldsymbol{q},\dot{\boldsymbol{q}},t) = -\frac{\partial \Pi}{\partial q^i} + Q_i{}^w + Q_i{}^{\text{var}} + Q_i{}^c. \tag{9}$$

The generalized reaction forces that develop due to expelling of the particles, respectively, can be written as [4, 5]:

$$Q_i^{\text{var}}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \sum_{l=1}^{2} \dot{m}_l \left(\vec{v}_l + \vec{v}_l^{rel}\right) \cdot \frac{\partial \vec{r}_l}{\partial q^i}, \tag{10}$$

while at the same time $Q_i^c = Q_i^c(t)$ are generalized control forces, whose total power during brachistochronic motion equals zero:

$$Q_i^c \dot{q}^i = 0. \tag{11}$$

Let the values of generalized coordinates be specified, as well as the value of mechanical energy of the mechanical system at the initial instant of time:

$$\begin{aligned} &t_0 = 0, \quad q^1(t_0) = 0,\ q^2(t_0) = 0,\ q^3(t_0) = 0, \\ &T\left(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, t_0\right) + \Pi\left(\boldsymbol{q}_0, t_0\right) = \\ &\frac{1}{2}\left(\left(M + 2m(t_0)\right)\left(\dot{q}^1(t_0)\right)^2 + \left(M + 2m(t_0)\right)\left(\dot{q}^2(t_0)\right)^2 + \left(M + 2m(t_0)\right)R^2\left(\dot{q}^3(t_0)\right)^2\right) = E_0, \end{aligned} \tag{12}$$

and also the values of generalized coordinates corresponding to the final position of the system:

$$q^1(t_f) = 0.7a,\ q^2(t_f) = 0,\ q^3(t_f) = \pi/2. \tag{13}$$

where $E_0 \in \mathbb{R}$ and $t_f \in \mathbb{R}$. The problem of brachistochronic motion of a variable mass mechanical system, whose differential equations of motion are given in the form (7), consists of defining the generalized control forces $Q_i^c = Q_i^c(t)$, and corresponding equations of the system motion $q^i = q^i(t)$, so that the system moves in the minimum time t_f from the initial state defined by (12), to the final position defined by (13).

2. Brachistochrone problem as an optimal control task

The presented brachistochrone problem can be formulated as a task of optimal control by introducing controls u_i:

$$u_i = Q_i^C \tag{14}$$

where Q_i^C are generalized control forces. Taking into account (11), certain control can be expressed in the function of others:

$$u_3 = -\frac{u_\alpha \dot{q}^\alpha}{\dot{q}^3}. \tag{15}$$

The normal form of first-order differential equations, known in the optimal control theory as the state equations, taking into account (7), (8), (9), (14) and (15), can be written by incorporating the rheonomic coordinate $q^4 \triangleq t$ in the following manner:

$$\begin{aligned} &\dot{q}^i = f_i(\boldsymbol{q}, \boldsymbol{p}, q^4, \boldsymbol{u}) \equiv p^i, \\ &\dot{q}^4 = f_4(\boldsymbol{q}, \boldsymbol{p}, q^4, \boldsymbol{u}) \equiv 1, \\ &\dot{p}^\delta = f_\delta(\boldsymbol{q}, \boldsymbol{p}, q^4, \boldsymbol{u}) \equiv c^\delta(\boldsymbol{q}, \boldsymbol{p}, q^4) + d^{\alpha\delta}(\boldsymbol{q}, \boldsymbol{p}, q^4) u_\alpha, \end{aligned} \tag{16}$$

where:

$$c^{\delta}=-a^{i\delta}\dot{a}_{ij}\dot{q}^{j}-\Gamma^{\delta}_{kr}\dot{q}^{k}\dot{q}^{r}+a^{i\delta}\left(-\frac{\partial\Pi}{\partial q^{i}}+Q_{i}^{w}+Q_{i}^{\mathrm{var}}\right),$$
$$d^{\alpha\delta}=\left(a^{\alpha\delta}-a^{3\delta}\frac{\dot{q}^{\alpha}}{\dot{q}^{3}}\right). \tag{17}$$

Now, based on (7),(14), (15) and (16) differential equations of motion of the system can be constructed:

$$\begin{aligned}
&\dot{q}^{1}=p^{1},\\
&\dot{q}^{2}=p^{2},\\
&\dot{q}^{3}=p^{3},\\
&\dot{q}^{4}=1,\\
&\dot{p}^{1}=\left(\dot{m}v_{r}\left(\sin q^{3}-\cos q^{3}\right)+u_{1}\right)\Big/\left(M+2m\right),\\
&\dot{p}^{2}=\left(-\dot{m}v_{r}\left(\sin q^{3}+\cos q^{3}\right)-\left(M+2m\right)g+u_{2}\right)\Big/\left(M+2m\right),\\
&\dot{p}^{3}=\left(-\dot{m}v_{r}/R-\left(u_{1}p^{1}+u_{2}p^{2}\right)\Big/p^{3}\right)\Big/\left(M+2m\right).
\end{aligned} \tag{18}$$

where $\vec{g}=-g\vec{j}$, and g is the acceleration due to gravity.

The brachistochrone problem of the considered mechanical system motion described by the state equations (18), consists of defining the optimal controls u_{α} and corresponding optimal trajectories in state space $q^{i}(t)$, so that the mechanical system moves from the initial state defined by (12) to the final position (13), in the minimum time, which can be expressed using conditions for the functional [6]:

$$J\left(\boldsymbol{q},\dot{\boldsymbol{q}},q^{4},u\right)=\int_{0}^{t_f}dt, \tag{19}$$

over the interval [0, t_f] it has minimum value. In order to solve the problem of optimal control by applying Pontryagin's maximum principle [7], the Hamiltonian is created of the Hamilton-Pontryagin form [6]:

$$H\left(\boldsymbol{q},\boldsymbol{p},q^{4},\boldsymbol{u},\boldsymbol{\lambda},\boldsymbol{v}\right)=-1+\lambda_{i}p^{i}+\lambda_{4}+\nu_{\delta}(c^{\delta}+d^{\alpha\delta}u_{\alpha}), \tag{20}$$

where $\boldsymbol{\lambda}=\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)^{T}$, $\boldsymbol{\nu}=\left(\nu_1,\nu_2,\nu_3\right)^{T}$, whereas $\lambda_i(\cdot):\left[0,t_f\right]\to\mathbb{R}$, $\lambda_4(\cdot):\left[0,t_f\right]\to\mathbb{R}$ and $\nu_\delta(\cdot):\left[0,t_f\right]\to\mathbb{R}$ are costate variables, so that the costate system of differential equations has the form:

$$\begin{aligned}
&\dot{\lambda}_{i}=-\frac{\partial H}{\partial q^{i}}=-\nu_{\delta}\left(\frac{\partial c^{\delta}}{\partial q^{i}}+\frac{\partial d^{\alpha\delta}}{\partial q^{i}}u_{\alpha}\right),\\
&\dot{\lambda}_{4}=-\frac{\partial H}{\partial q^{4}}=-\nu_{\alpha}\left(\frac{\partial c^{\delta}}{\partial q^{4}}+\frac{\partial d^{\alpha\delta}}{\partial q^{4}}u_{\alpha}\right),\\
&\dot{\nu}_{i}=-\frac{\partial H}{\partial p^{i}}=-\lambda_{i}-\nu_{\alpha}\left(\frac{\partial c^{\delta}}{\partial p^{i}}+\frac{\partial d^{\alpha\delta}}{\partial p^{i}}u_{\alpha}\right).
\end{aligned} \tag{21}$$

Based on (20), it can be written:

$$H\left(\boldsymbol{q},\boldsymbol{V},q^4,u,\boldsymbol{\lambda},\boldsymbol{\nu}\right)=H_0+H^{\alpha}u_{\alpha}, \tag{22}$$

where:

$$\begin{aligned} &H_0=-1+\lambda_i p^i+\lambda_4+\nu_{\delta}c^{\delta},\\ &H^{\alpha}=\nu_{\delta}d^{\alpha\delta}. \end{aligned} \tag{23}$$

Since controls figures linearly in the state equations, such case of controls are known in the optimal control theory as singular controls. Then, the necessary optimality condition of Pontryagin's maximum principle is of the form as follows [8]:

$$\frac{\partial H}{\partial u_{\alpha}}=H^{\alpha}=0, \tag{24}$$

from where singular optimal controls u_α cannot be explicitly defined. Hence, it is required that H_α be identically equal to zero alongside the optimal trajectory of state. Singular optimal controls u_α is defined by further differentiation with respect to time (24) taking into account (16) and (21):

$$\frac{d^k}{dt^k}\left[\frac{\partial H}{\partial u_{\alpha}}\right]=0,\quad k=0,1,2,... \tag{25}$$

In defining the relations (25) the Poisson bracket formalism will be applied [9]:

$$\dot{H}^{\alpha}=\left\{H,H^{\alpha}\right\}=\left\{H^{\alpha},H_0\right\}+\left\{H^{\alpha},H^{\beta}\right\}u_{\beta}=0. \tag{26}$$

Taking into account (24) as well as the fact that for multidimensional singular controls along an optimal trajectory it holds $\left\{H^{\alpha},\, H^{\beta}\right\}=0$ [9] further differentiation of (26) yields:

$$\left\{\left\{H^{\alpha},H_0\right\},H_0\right\}+\left\{\left\{H^{\alpha},H_0\right\},H^{\beta}\right\}u_{\beta}=0. \tag{27}$$

In addition, the boundary conditions can be represented in the following form:

$$\left(\lambda_i\Delta q^i+\lambda_4\Delta q^4+\nu_i\Delta p^i\right)\Big|_0^{t_f}=0, \tag{28}$$

$$\left(H\Delta t\right)\Big|_0^{t_f}=0, \tag{29}$$

where $\Delta(\cdot)$ represents the noncontemporaneous variation [1, 2] of the quantity $(\cdot)$. Based on conditions (24) the costate variables ν_α can be expressed as a function of the costate variable ν_3:

$$\nu_{\alpha}=\nu_{\alpha}\left(\boldsymbol{q},\boldsymbol{p},q^4,\nu_3\right). \tag{30}$$

Now, from equations (30) taking into account (23) and (30) it can be expressed:

$$\lambda_{\alpha}=\lambda_{\alpha}(\boldsymbol{q},\boldsymbol{p},q^4,\lambda_3,\nu_3). \tag{31}$$

Since the initial position of the mechanical system according to (12) is defined, it follows:

$$\Delta t(t_0)=0,\qquad \Delta q^i(t_0)=0\,,\qquad \Delta q^4(t_0)=0. \tag{32}$$

If (32) is taken into account and the operator of asynchronous variation is applied to the initial energy of the system (12) it can be obtained:

$$a_{ij}(t_0)p^i(t_0)\Delta p^j(t_0)=0, \tag{33}$$

and lastly, after substituting (32) and (24) into (28), it is obtained:

$$\nu_j(t_0)\Delta p^j(t_0)=\frac{\nu_{\delta}(t_0)a^{3\delta}(t_0)}{p^3(t_0)}a_{ij}(t_0)p^i(t_0)\Delta p^j(t_0)=0. \tag{34}$$

Based on (32), (33) and (34), it is obvious that the transversality conditions (28) and (29) in the initial configuration of the system are satisfied. In the final configuration (13) of the mechanical system the time is not known, and based on it, the transversality condition results from (29):

$$H\left(t_f\right)=0, \tag{35}$$

and as quantities $p^i(t_f)$ and $q^4(t_f)$ are not a priori defined ($\Delta p^i(t_f)\neq 0, \Delta q^4(t_f)\neq 0$), the next transversality conditions are obtained from (28)

$$\nu_i(t_f)=0,\ \lambda_4(t_f)=0. \tag{36}$$

Using (18) it can be defined (20) and (21), as well as all other needed quantities so as to solve the formulated problem. Based on (20), (31), (35) and (36) the following dependence can be established in analytical form:

$$\lambda_3(t_f)=p^3(t_f)\Big/\left(\left(p^1(t_f)\right)^2+\left(p^2(t_f)\right)^2+\left(p^3(t_f)\right)^2\right). \tag{37}$$

If considerations are restricted to the first order singular controls, where $\left\{\left\{H^{\alpha},H_0\right\},H^{\beta}\right\}\neq 0$, using (27), (30) and (31), singular controls u_β can be represented in the form as follows:

$$u_\beta=u_\beta(\boldsymbol{q},\boldsymbol{p},q^4,\lambda_3,\nu_3). \tag{38}$$

Also, the Kelley necessary conditions for the first order singular control is given in the form [9]:

$$-\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial u_\beta}\left(\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial u_\alpha}\right]\right)\leq 0, \tag{39}$$

Applying the Poisson brackets, this condition is fulfilled if the matrix $K=\left[\partial/\partial u_\beta\left(d^2(\partial H/\partial u_\alpha)/dt^2\right)\right]$ is positive definite [10].

Substituting (38) into (18) and calculated (21) and taking into account (30) and (31), yields a two-point boundary value problem (TPBVP) with nine first-order nonlinear normal form differential equations. Due to nonlinearity, in a general case, it is necessary to apply the appropriate numerical method [11]. In this paper, the shooting method will be deployed. The shooting method is most suitable to perform in this case by the backward numerical integration choosing four values $p^i(t_f), t_f$, which will ensure fulfillment of the same number of initial conditions (12). The value $\lambda_3(t_f)$ was defined via (37). Taking into account (13) and (36) the remaining values in the final position are known. Solution of the problem was found for the following parameters:

$$\begin{aligned}&a=1\text{m},\ k=0.4\frac{1}{\text{s}},\ v_r=10\frac{\text{m}}{\text{s}},\ m_0=10\text{kg},\\&M=20\text{kg},\ R=0.5m,\ g=9.80665m/s^2.\end{aligned} \tag{40}$$

The numerical procedure gives solutions for the system of differential equations of motion, as well as for the costate system in numerical form (see [12]):

$$q^1(t),\ q^2(t),\ q^3(t),\ p^1(t),\ p^2(t),\ p^3(t), \lambda_3(t),\ \nu_3(t), \tag{41}$$

and the time of brachistochronic motion t_f.

Table 1 displays values of the missing boundary values for different values of the initial energy of the system. It is evident that as the initial value of the energy increases the time of brachistochronic motion t_f decreases, horizontal projection of the disk center final velocity increases, while vertical projection of the disk center final velocity firstly increases and then decreases. Also, the disk final angular velocity increases with increasing initial value of the energy of the system. Figure 2 displays graphs of controls for corresponding values from Table 1, where graphs of controls u_1 are shown with full lines, and graphs of controls u_2 are shown with dashed lines.

	E_0 [J]	t_f[s]	p_f^1 [m/s]	p_f^2 [m/s]	p_f^3 [rad/s]
1	10	0.8721	0.142251	1.020697	0.308147
2	30	0.774232	0.280072	1.359575	0.614382
3	60	0.664431	0.520366	1.681863	1.151118
4	90	0.570351	0.806867	1.84264	1.79308
5	150	0.424045	1.43232	1.772866	3.197759
6	201	0.352064	1.8575	1.584221	4.153642
7	203.322	0.349594	1.874221	1.576176	4.191238

Table 1. Numerical solutions of TPBVP for different values of the initial energy of the system.

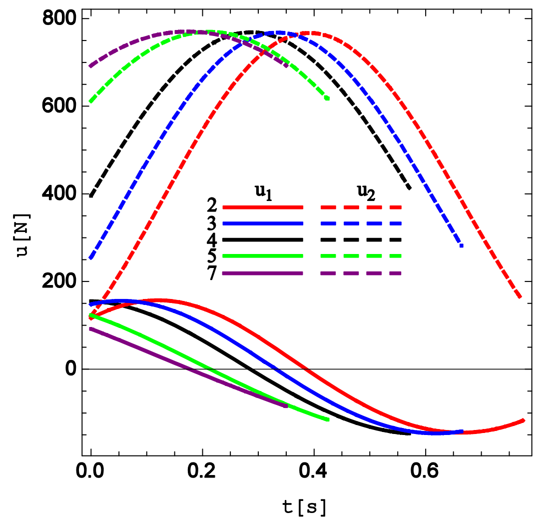

Figure 2. Graphs of controls $u_1(t)$ and $u_2(t)$.

In this example, the possibility of realization of brachistochronic motion of a variable mass mechanical system by centrodes - rolling without slipping roulette on a base, will be examined. In this way, the generalized control forces are achieved via reactions of constraint. The moving centrode (roulette) represents a curve attached to the disk which traces the instantaneous center of zero velocity of the disk. The fixed centrode (base) represents a curve fixed in the plane of the motion which traces the instantaneous center of zero velocity of the disk in fixed coordinate system. Thus, the motion of the system is equivalent to the rolling without slipping of the moving centrode on the fixed one (see [13]). Based on a previously determined brachistochronic motion, it is possible to determine fixed and moving centrodes, as well as normal and tangential component of constraint reaction, unequivocally. Based on them, in order for this motion to be possible, the condition for the Coulomb friction coefficient must also be determined. This way, the brachistochronic motion is realized without active forces' influence, which is in accordance with the elementary brachistochrone problem of a particle in a vertical plane.. The parametric equations of the fixed centrode are [13]:

$$x_b(t) = q^1(t) - \frac{p^2(t)}{p^3(t)},$$
$$y_b(t) = q^2(t) + \frac{p^1(t)}{p^3(t)}, \tag{42}$$

and those of the moving one are [13]:

$$\xi_r(t) = \left(p^1(t)\sin q^3(t) - p^2(t)\cos q^3(t)\right) \Big/ p^3(t),$$
$$\eta_r(t) = \left(p^1(t)\cos q^3(t) + p^2(t)\sin q^3(t)\right) \Big/ p^3(t). \qquad (43)$$

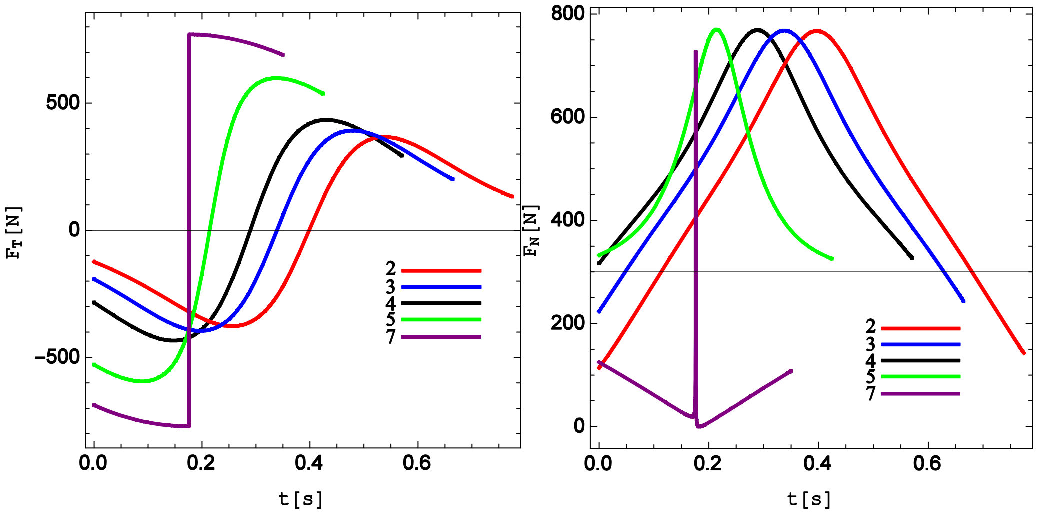

Figure 3. Graphs of fixed centrode reactions of constraint F_T and F_N.

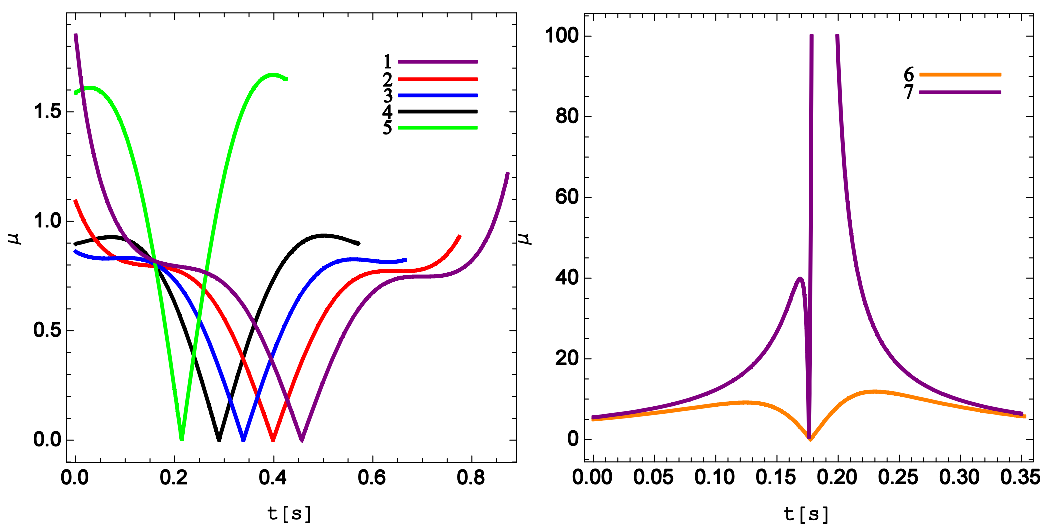

Figure 4. Graphs of rolling friction coefficient.

The reactions of constraint can now be determined as follows:

$$F_T(t) = \left(u_1(t)\dot{x}_b(t) + u_2(t)\dot{y}_b(t)\right) / \left|\vec{v}_b\right|,$$
$$F_N(t) = \left(u_1(t)\dot{y}_b(t) - u_2(t)\dot{x}_b(t)\right) / \left|\vec{v}_b\right|, \qquad (44)$$

where $\left|\vec{v}_b\right|$ is base speed intensity at connection point.

Figure 3 shows a comparative graphic representation of constraint reactions corresponding to the numbers from Table 1. It should be noted that when the initial energy of the system increases, the normal reaction of the fixed centroide decreases. It is clearly shown in the diagrams that for a the initial energy E_0=203,322 [J] at one point of the motion, the normal constarint reaction reaches a value equal to zero. Since, it is considered unilateral constraint, at this point, the system is detached from the constarint. So the initial energy of the system must be smaller than this value. In order to solve this problem centroids must be realized with small toothing, so that instead of rolling without slipping, the constraint is achieved in the form of gears.

Figure 4 shows a comparative graphic representation of rolling friction coefficients. It is clearly seen that the coefficient of rolling friction increases with the reduction of the normal constraint reaction, and for its zero value tends to infinity. From here, you can see the minimum rolling friction coefficient that is needed to make motion for the given initial energy possible.

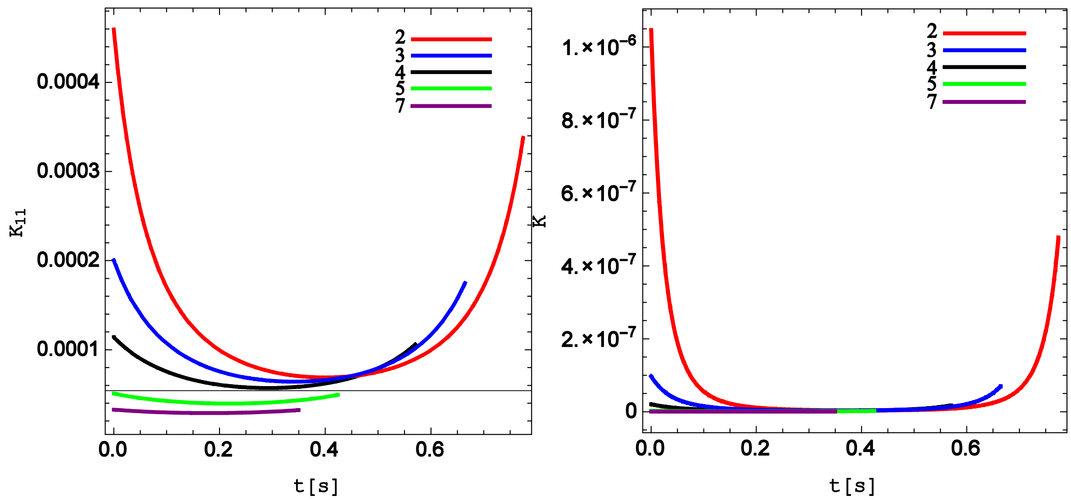

Figure 5. Evidence of Kelley's optimality conditions.

Figure 5 shows the fulfillment of Kelley's optimality condition.

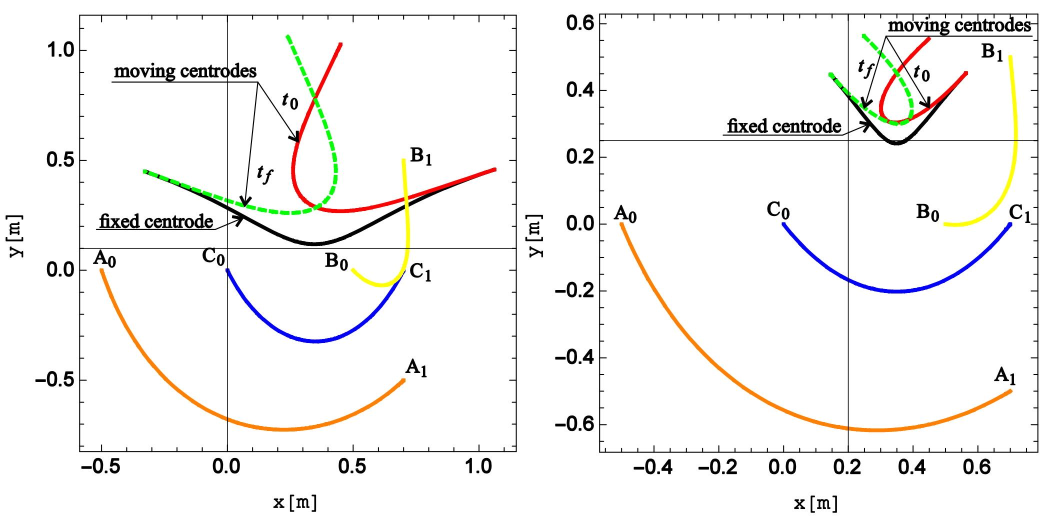

Figure 6. The centrodes and the trajectories of points A, B and C for different values of the initial energy of the system - E_0=90 J and E_0=150 J.

Figure 6. shows the trajectories of variable mass points A and B, disk center C, as well as centrodes graphs, for the initial energy values E_0=90 J and E_0=150 J. Graphs of moving centrode are given for initial and final position of the system.

3. Conclusions

The present work has solved the problem of realizing brachistochronic planar motion of a variable mass mechanical system by means of an ideal constraint in the form of the centrodes.

Considerations presented in this work rely on the work [3] and thus are a kind of continuation of mentioned study. Brachistochrone problem is formulated as an optimal control task, where constraint reactions of the centrodes are expressed in the function of the generalized forces. Paper showed how increase of the initial energy of the system affects decrease of normal constraint reaction. In this way, the initial energy of the system can be increased to a certain limit for which the normal constraint reaction at one point equals zero, so for higher values of the initial energy the constraint must be carried out in the form of small gears. Further research can go towards limiting the rolling friction coefficient, whereby non-singular controls occur.

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