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OPTIMAL CONTROL AND ATTENUATION OF THE OSCILLATING SYSTEMS

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Abstract: *Among oscillating motion optimization problems, attenuation problems are specially aparted in the purpose of protection of particular system elements from the dynamic influences, caused by extensive accelerations or impacts. That is very important for the weight transports, which are sensitive on influences like that, or for proper operation of devices and instruments connected to mobile parts of the system. Basic aim of the attenuation system theory is the decrease of the undesired actions. Various problems of the attenuation system optimization were discussed in numerous papers and monographs dedicated to this area. Generalization of the optimal attenuation problem is presented in this paper from the point of view of analytical mechanic and theory of optimal control. Two coordinate systems were introduced: system with absolute coordinates and system with relative coordinates which, together with relative velocities, determine states of attenuation elements. Corresponding system of differential equations is being formed dependent of external influence character (dynamic or kinematic). Different criteria of optimality is considered according to the postulated attenuation goal. General mathematical model is transformed in such manner that it is suitable for application of optimal control methods. Solution procedure, presented in this paper, was applied on some concrete examples, closely connected to the practical problems.*

Key words: *mechanical system, optimal attenuation, impact, absorber*

1. INTRODUCTION

Great number of technical objects or their parts is frequently exposed to harmful mechanical influences (a great accelerations, impacts, oscillations, etc.). This can cause undesirable consequences such as: damage of sensitive materials during the transport, disturbance of work and reduction of accuracy precision instruments, noise, etc.. For protection, their rigid constraints are, whenever possible, replaced with the system of shock absorbers. The way of installation and type of shock absorber are caused with restrictions and criteria of optimality. Contemporary researches of attenuation problems are in that directions. Numerous references in monographs [4,5] are proof that sudden development of theory of optimality induced at the beginning of seventies intensive researches in that area. The level of the up-to-date computer technique allows researches to direct towards general type of system model. Attenuation of one general mechanical system is discussed in this paper. Basic optimal attenuation problems are classified here, in aim to simplify the choise of their solution method.

2. ATTENUATED SYSTEMS. DIFFERENTIAL EQUATIONS OF MOTION

In attenuation problems, mechanical system can be considered as set of two subsystems: basic system O and attenuated system A. Let the position of basic system O be determined by coordinates z^i $(i = 1,2,\dots,r)$ in regard to some inertial system in space X^r, and relative position of attenuated system in space X^s be determined by coordinates x^α $(\alpha = 1,2,\dots,s)$ in regard to system related with basic system O. Kinetic energy T of that system is represented by the sum of basic system kinetic energy T^0 and attenuated system kinetic energy T^A, i.e. :

$$T = T^O + T^A \,,\quad T^O = \frac{1}{2} a_{ij}^{O} \dot{z}^i \dot{z}^j \,,$$
$$T^A = \frac{1}{2} a_{ij}^{A} \dot{z}^i \dot{z}^j + b_{i\alpha} \dot{z}^i \dot{x}^\alpha + \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad (1.1)$$

where $a_{ij}^{O}(z), a_{ij}^{A}(z,x), b_{i\alpha}(z,x)$ and $g_{\alpha\beta}(x)$ are elements of covariant metric tensor :

$$a_{ab} = \begin{Bmatrix} a_{ij} & b_{i\beta} \\ b_{\alpha j} & g_{\alpha\beta} \end{Bmatrix} \qquad \begin{pmatrix} a_{ij} = a_{ij}^{O} + a_{ij}^{A} \\ (a,b = 1,2,\dots,n = r+s) \end{pmatrix} \qquad (1.2)$$

in space $X^n = X^r \cup X^s$. Let us presume that the entire system is placed in the field of potential forces, whose potential energy is $\Pi = \Pi(z,x)$. Let, in addition to potential forces, unpotential forces F_i act upon basic system O. Let the system be related to system O by shock absorbers A_ν (Fig.1) :

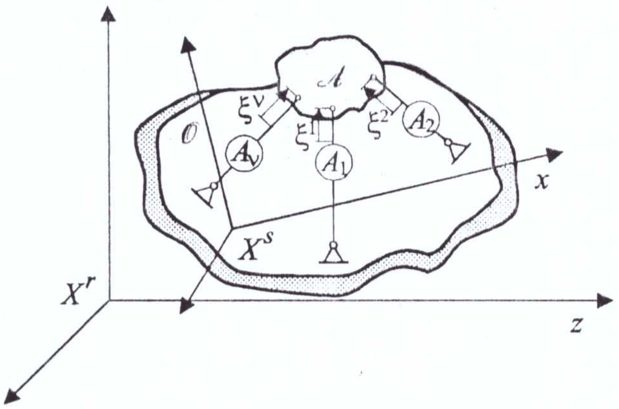

Fig.1

so that the work of forces on virtual displacements of the system is :

$$\delta A = \left(-\frac{\partial \Pi}{\partial z^i} + F_i \right) \delta z^i + \left(-\frac{\partial \Pi}{\partial x^\alpha} + Q_\alpha \right) \delta x^\alpha \quad (1.3)$$

where $Q_\alpha(x,\dot{x})$ are generalized forces exerted by shock absorbers upon the system.Differential equations of motion of the system, regarding on (1.1) and (1.3),have form :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}^i}\right) - \frac{\partial T}{\partial z^i} = -\frac{\partial \Pi}{\partial z^i} + F_i \quad (i = 1,2,\dots,r)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} = -\frac{\partial \Pi}{\partial x^\alpha} + Q_\alpha \quad (\alpha = 1,2,\dots,s) \qquad (1.4)$$

In cases when the action of system upon system O is negligible, first r equations of the system (1.4) are significantly simplified, i.e. have form :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^O}{\partial \dot{z}^i}\right) - \frac{\partial T^O}{\partial z^i} = \frac{\partial \Pi}{\partial z^i} + F_i \qquad (1.5)$$

and represents the system of differential equations of motion of basic system, which can be solved separately, if forces $F_i(t)$ are known. In some cases equations of motion of the system in phase space with generalized coordinates z^i, x^α and generalized momentums p_i, p_α have convenient form for optimization problem solution. By introducing Hamilton`s function $H(z,x,\overline{p},p)$ which represents mechanical energy of scleronomic systems, differential equations of motion have form :

$$\dot{z}^i = \frac{\partial H}{\partial p_i}, \quad \dot{\overline{p}}_i = -\frac{\partial H}{\partial z^i} + F_i$$
$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha} + Q_\alpha \qquad (1.6)$$

In attenuation problem analysis it is necessary to consider following cases :

a) Motion $z^i(t)$ of basic system is known (kinematics type action [5]). In that case it is enough to take in consideration s equation of system (1.4) which are transformed into the system of unautonomic equations by introducing $z^i(t), \dot{z}(t), \ddot{z}(t)$.

b) Forces $F_i(t)$ which act upon the basic system O are known (external dynamic type action). In this case it is necessary to consider entire system of equations (1.4). If the influence of system upon basic system O can be neglected, one can obtain solution $z^i = z^i(t)$ by solving system (1.5) with initial conditions $z^i(t_0) = z_0^i, \dot{z}^i(t_0) = \dot{z}_0^i$. In that case dynamic type action is reduced into a kinematics type action.

c) There is no complete information about motion of basic system and external forces action. In that case it is necessary to determine set U_F of possible external actions ($F_i \in U_F$) and to solve attenuation problems on that set. By analogy this considerations can also be applied on system of equations (1.6).

2. CHARACTERISTICS OF ABSORBERS

Forces which absorbers exert upon the system should restrict relative displacements of system and to reduce the influence of great accelerations of basic system. In that sence it is necessary absorbers to have both restitutional and damping characteristics. If ξ^{ν} is relative displacement in direction of absorber A_{ν} action (Fig.1), then the force of their action has form :

$$K_{\nu} = K_{\nu}^{e}\left(\xi^{(\nu)}\right) + K_{\nu}^{w}\left(\dot{\xi}^{(\nu)}\right), \quad \nu = 1,2,\ldots,m \quad (2.1)$$

where: $K^{e}(\xi)$ is restitutional force, $K^{w}(\dot{\xi})$ is damping force and m is number of absorbers. Forces $K_{\nu}(\xi^{(\nu)},\dot{\xi}^{(\nu)})$ are called characteristic of absorbers. Their determination represents basic goal of attenuation problems. It is important to note that each characteristic depends only on its own displacement and its own velocity. By introducing a following relations :

$$\xi^{\nu} = \xi^{\nu}\left(x^{1}, x^{2}, \ldots, x^{s}\right) \quad (2.2)$$

and their differentiation, relative velocities in absorber action direction are obtained :

$$\dot{\xi}^{\nu} = \frac{\partial \xi^{\nu}}{\partial x^{\alpha}}\dot{x}^{\alpha}. \quad (2.3)$$

Generalized forces Q_{α} in equations (1.4) are obtained from the tensor transformation law, i.e.

$$Q_{\alpha}(x,\dot{x}) = K_{\nu}\left[\xi^{\nu}(x),\dot{\xi}^{\nu}(x,\dot{x})\right]\frac{\partial \xi^{\nu}}{\partial x^{\alpha}}. \quad (2.4)$$

These quantities represent generalized characteristic of absorbers. Unlike chara-cteristics K_{ν} (2.4) depend on absorber position in system X^{s} The number of independent absorbers is determined by the number of independent quantities ξ^{ν}.

If all absorbers are independent then $m \le s$ and following condition is satisfied :

$$\text{rang}\left\{\frac{\partial \xi^{\nu}}{\partial x^{\alpha}}\right\} = m. \quad (2.5)$$

The attenuation is partial if $m < s$, and is total if $m = s$. If we presume that all absorbers are independent and that there is total attenuation, then quantities ξ^{α} can represent independent coordinates of relative position of system A. On the basis (2.5) there exist transformations, which are inverse to transformations (2.2) and (2.3) :

$$x^{\alpha} = x^{\alpha}\left(\xi^{1},\xi^{2},\ldots,\xi^{s}\right), \quad \dot{x}^{\alpha} = \frac{\partial x^{\alpha}}{\partial \xi^{\beta}}\dot{\xi}^{\beta} \quad (2.6)$$

which cause following forms of kinetic energy (1.1) and potential energy $\Pi(z,x)$:

$$\begin{aligned}\bar{T}\left(z,\dot{z},\xi,\dot{\xi}\right) &= T^{o}(z,\dot{z}) + \bar{T}^{A}\left(z,\dot{z},\xi,\dot{\xi}\right)\\ \bar{\Pi}(z,\xi) &= \Pi\left(z,x(\xi)\right)\end{aligned} \quad (2.7)$$

Equations (1.4) are transformed into : (2.8)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{T}}{\partial \dot{z}^{i}}\right) - \frac{\partial \bar{T}}{\partial z^{i}} = -\frac{\partial \bar{\Pi}}{\partial z^{i}} + F_{i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{T}}{\partial \dot{\xi}^{\alpha}}\right) - \frac{\partial \bar{T}}{\partial \xi^{\alpha}} = -\frac{\partial \bar{\Pi}}{\partial x^{\alpha}} + K_{\alpha}\left(\xi^{(\alpha)},\dot{\xi}^{(\alpha)}\right)$$

Similarly, by introducing generalized coordinates ξ^{α} and corresponding momentums equations (1.6) can be also transformed. If, we take in consideration that forces K_{α} depend only on one coordinate ξ^{α} and one velocity $\dot{\xi}^{\alpha}$, equations (2.8) are very convenient. However, transformations (2.2) frequently have transcendental form which represents very difficult problem when terms for kinetic and potential energy are to be formed. If the difference from approximate solution can be tolerated, in some cases, that problem can be simplified by certain geometric approxio-mations.

3. BASIC PROBLEMS OF OPTIMAL ATTENUATION. OPTIMALITY CRITERIA

Quantities which give some estimation behavior of attenuated system (maximum relative displacement and maximum absolute accelerations of attenuated system) are discussed in optimal attenuation considerations. Their values depend on external actions F_{i} and absorber characteristics Q_{α} , and in that sence they represent functionals of these values. Let, in general case, functions $d_{1}^{\lambda}[x(t)] \ge 0$ $(\lambda = 1,2,\ldots,p \le s)$ represent certain characteristics of relative displacements, and functions $d_{2,\mu}[Q(x(t),\dot{x}(t))] \ge 0 \; (\mu = 1,2,\ldots,q \le s)$ represent characteristics of absolute accelerations of system A. Then we consider functionals in following form :

$$\begin{aligned}J_{1}^{\lambda}(Q,F) &= \max_{t\in[t_{o},\infty]} d_{1}^{\lambda}\left[x(t)\right]\\ J_{2,\mu}(Q,F) &= \max_{t\in[t_{o},\infty]} d_{2,\mu}\left[Q(x(t)),\dot{x}(t)\right].\end{aligned} \quad (3.1)$$

Functionals (3.1) have local maximum character, because of functions $x(t)$ and $\dot{x}(t)$ continuity. Among them, those with maximum values should be considered. In cases when norm $\|x\|$

represents characteristic of relative displacement of system A, and norm $\|Q\|$ represents characteristic of absolute acceleration of system A, functionals have following form :

$$
\begin{aligned}
J_1(Q,F) &= \max_{t\in[t_o,\infty]} \|x(t)\| \\
J_2(Q,F) &= \max_{t\in[t_o,\infty]} \|Q(x(t)),\dot{x}(t)\|.
\end{aligned}
\tag{3.2}
$$

It should be noted, that when the system of external forces act upon system (commonly it is weight) its absolute acceleration depends not only on characteristic Q of absorber. Various problems of optimal attenuation are allowed by definition of functional (3.1). Let the motion of system be described by some of mentioned equations systems $((1.4)\vee(1.6)\vee(2.8))$ with given initial conditions :

$$
\begin{aligned}
& t_o,\ z^i(t_o)=z_o^i,\ \dot{z}^i(t_o)=\dot{z}_o^i, \\
& x^\alpha(t_o)=x_o^\alpha,\ \dot{x}^\alpha(t_o)=\dot{x}_o^\alpha
\end{aligned}
\tag{3.3}
$$

Let the set of allowable functions $Q_\alpha(x,\dot{x})$ be noted by U , i.e. $Q_\alpha(x,\dot{x})\in U$. Without loss of generality, on the example of functional (3.2), following problems can be formulated :

1. Among functions $Q_\alpha(x,\dot{x})\in U$ determine such optimal values Q_α^o , that is :

$$
\begin{aligned}
& J_1\left(Q^o,F\right)=\min_{Q\in U} J_1(Q,F), \\
& J_2\left(Q^o,F\right)\le D_2.
\end{aligned}
\tag{3.4}
$$

Quantity D_2 is known and it restricts maximum accelerations of system A. Criterion of optimality (functional J_1 minimum), minimize relative displacements of system A. It can be used for gabarit system O optimization.

2. If system displacements are restricted with given basic system gabarits, the optimality problem has following form :

$$
\begin{aligned}
& J_2\left(Q^o,F\right)=\min_{Q\in U} J_2(Q,F), \\
& J_1\left(Q^o,F\right)\le D_1.
\end{aligned}
\tag{3.5}
$$

where D_1 is given.

Optimality problems 1 and 2 have sence only if external forces are kinematics or dynamic type. In cases where external forces are undetermined, where $F_i\in U_F$, following problems can be formulated :

3.

$$
\begin{aligned}
& \max_{F\in U_F} J_1\left(Q^o,F\right)=\min_{Q\in U}\max_{F\in U_F} J_1(Q,F), \\
& \max_{F\in U_F} J_2\left(Q^o,F\right)\le D_2.
\end{aligned}
\tag{3.6}
$$

4.

$$
\begin{aligned}
& \max_{F\in U_F} J_2\left(Q^o,F\right)=\min_{Q\in U}\max_{F\in U_F} J_2(Q,F), \\
& \max_{F\in U_F} J_1\left(Q^o,F\right)\le D_1.
\end{aligned}
\tag{3.7}
$$

Problems 3 and 4 represent generalization of problems 1 and 2.

Attenuation problem, when characteristics of absorber and their configuration is known, can be considered as special case. Motion of basic system can influence motion of system A. In some transportation problems, following problem can be formulated :

5. Let the basic system have initial state $t_0,z^i(t_0),\dot{z}^i(t_0)$ and final state $t_1,z^i(t_1),\dot{z}^i(t_1)$. States should be changed by action of forces $F_i\in U_F$ with restricted behavior of system and optimality conditions [1,2,3] :

$$
J_o=g\left[z(t_1),\dot{z}(t_1)\right]+\int_{t_o}^{t_1} f_o(z,\dot{z},F,t)dt \tag{3.8}
$$

This problem can be formulated in following form :

$$
\begin{aligned}
& J_o\left(F^o\right)=\min_{F\in U_F} J_o(F), \\
& J_1\left(Q,F^o\right)\le D_1, J_2\left(Q,F^o\right)\le D_2
\end{aligned}
\tag{3.9}
$$

Functional (3.8) form, in practical problems, depends on optimality demands (minimum time of motion, minimum consumption of energy, etc.). Problems (1-5) represent basic forms in optimal attenuation problems, and some additional conditions can be added. It should be noted, that in general case of (3.1), following problems can also be formulated :one of functionals can be minimized and others should be restricted, or more functionals can be minimized and others should be restricted.

4. METHODS OF SOLUTION OF OPTIMAL ATTENUATION PROBLEMS

Difficulty and variety of given problems do not allow general method of their solution to be formed. Problems 1 and 2 in some cases can be solved by using maximum principle [2], or as classic extremal problems. Difficulties in their solutions are caused by complexity of systems with more degrees of freedom. That is why systems with one degree of freedom of attenuated body are considered in most references. Problems 3 and 4 belong the type of game theory problems, although there are other methods of solution [4]. Problem 5 represents problem of optimal controls. They can be solved by using methods based on maximum principle.

5. IMPACT AND OPTIMAL CHARACTERISTIC OF ABSORBER

Let, during the motion, basic system be exposed to action of external forces in following way :

$$F_i = \begin{cases} 0 & ,t < t^* \\ F_i(t) & ,t^* \le t \le t^* + \tau \\ 0 & ,t > t^* + \tau \end{cases} \tag{5.1}$$

and impulses be :

$$I_i = \int_{t^*}^{t^*+\tau} F_i(t) dt \tag{5.2}$$

with finite values for $\tau \to 0$. By integration of equations (1.4) in interval $[t^*, t^* + \tau]$, due to finity of function Q_α and derivatives of kinetic and potential energy, for $\tau \to 0$, it follows :

$$\begin{aligned} \left(\frac{\partial T}{\partial \dot z^i}\right)_{t^*+\tau} - \left(\frac{\partial T}{\partial \dot z^i}\right)_{t^*} &= I_i \\ \left(\frac{\partial T}{\partial \dot x^\alpha}\right)_{t^*+\tau} - \left(\frac{\partial T}{\partial \dot x^\alpha}\right)_{t^*} &= 0 \end{aligned} \tag{5.3}$$

If immediately before the impact, state of basic system was $z^i(t^*), \dot z^i(t^*)$, and system was in relative equilibrium, i.e. $x^\alpha(t^*) = 0$, $\dot x^\alpha(t^*) = 0$, only system velocities were changed in time $\tau \to 0$, i.e. :

$$\begin{aligned} &z^i\left(t^* + \tau\right) = z^i\left(t^*\right), \quad x^\alpha\left(t^* + \tau\right) = 0 \\ &\dot z^i\left(t^* + \tau\right) \ne \dot z^i\left(t^*\right), \quad \dot x^\alpha\left(t^* + \tau\right) \ne 0 \end{aligned} \tag{5.4}$$

Velocities $\dot z^i(t^* + \tau)$ and $\dot x^\alpha(t^* + \tau)$ are determined from equations (5.3). If we take that $t^* + \tau = t_0$, for further system motion following equations are considered :

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot z^i}\right) - \frac{\partial T}{\partial z^i} = -\frac{\partial \Pi}{\partial z^i} \qquad , \quad t \ge t_0 \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot x^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} = -\frac{\partial \Pi}{\partial x^\alpha} + Q_\alpha(x, \dot x) \end{aligned} \tag{5.5}$$

with initial conditions :

$$\begin{aligned} &t_0, \; z^i(t_0) = z_0^i, \; \dot z^i(t_0) = \dot z_0^i, \\ &x^i(t_0) = x_0^i, \; \dot x^i(t_0) = \dot x_0^i \end{aligned} \tag{5.6}$$

System A, by acting of impulse, gets relative velocity $\dot x_0^\alpha$. Therefore it is advisable that absorbers have elastic and dumping properties (2.1). In that case motion of system can be oscillating, and estimations of relative displacement and absolute acceleration of system have more than one local maximum. If those functions have maximum in instants t_i, then J is corresponding functional :

$$J = \max\left[(J)_{t_1}, (J)_{t_2}, \ldots\right] \quad t_1 < t_2 < \ldots . \tag{5.7}$$

In numerous practical problems, due to dissipative action of absorber, it is functional $(J)_{t_1}$. When all functionals are determined, corresponding instants should be analyzed. Among them it should be chosen one which is corresponding to maximum value of functional. It represents the boundary of time interval in which optimal characteristic of absorber should be determined.

Let us illustrate impact attenuation with following example (fig.2) :

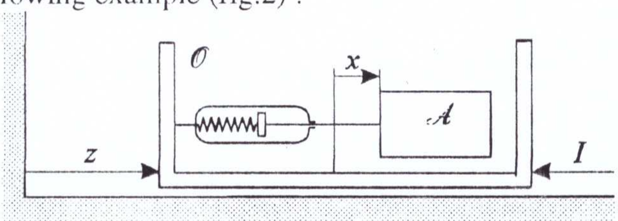

Fig.2

where z is coordinate of body O, x is coordinate of body A, I is impulse of impact, M is mass of body O and m is mass of body A. In instant of impact body was at relative rest $x^{\alpha}=0$ and body O was moving. After impact body gets relative velocity $\dot{x}_0 = I/M$. Differential equations of motion of system after impact are:

$$\begin{aligned} &M\ddot{z} + m(\ddot{z} + \ddot{x}) = 0 \\ &m(\ddot{z} + \ddot{x}) = Q(x, \dot{x}). \end{aligned} \tag{5.8}$$

Let the characteristic of absorber be linear, i.e. :

$$Q(x,\dot{x}) = -cx - b\dot{x}, \tag{5.9}$$

Then determination of characteristics is reduced on determination of constant parameters b and c. Let :

$$\begin{aligned} &J_1(Q) = \max_{t\in[0,\infty)} |x(t)|, \\ &J_2(Q) = \max_{t\in[0,\infty)} |cx + b\dot{x}| \end{aligned} \tag{5.10}$$

Let $t_1 < t_2 < t_3 < \ldots$ are instants in which function $|x(t)|$ has local maximum, then, due to energy dissipation, $|x(t_1)| > |x(t_2)| > \ldots$, and $J_1 = |x(t_1)|$. Let in instant $\tau' \geq t_1$ function $|Q(x,\dot{x})| = |cx + b\dot{x}|$ have local maximum. It exists such instant $t' \in [0, t_1]$ that is $|x(t')| = |x(\tau')|$. Due to dissipation, it is $|\dot{x}(t')| > |\dot{x}(\tau')|$, and :

$$|cx(t') + b\dot{x}(t')| > |cx(\tau') + b\dot{x}(\tau')|, \tag{5.11}$$

It means that there exists instant $t^* \in [t_0, t_1]$ in which function $|Q(x,\dot{x})|$ has maximum, i.e. :

$$J_2 = \left|cx(t^*) + b\dot{x}(t^*)\right|, \tag{5.12}$$

In that way, optimal attenuation problem is considered in interval $[t_0, t_1]$. On base (5.8) and (5.9) we have differential equation :

$$\ddot{x} = -\frac{M+m}{mM}(cx + b\dot{x}) \tag{5.13}$$

with boundary conditions :

$$t_o = 0,\ x(t_o) = 0,\ \dot{x}(t_o) = \frac{1}{M}I,\ \dot{x}(t_1) = 0. \tag{5.14}$$

Quantities $x(t)$ and $\dot{x}(t)$ are positive into interval $[0, t_1]$ $(\dot{x}(t_1) = 0)$, and functionals (5.10) have following form :

$$J_1(c,b) = x(t_1),\ J_2(c,b) = \max_{t\in[0,t_1]} (cx + b\dot{x}), \tag{5.15}$$

In this case following problems can be formulated :

$$1.\ x(t_1) \underset{c,b}{\to} \min, \quad \max_{t\in[0,t_1]} (cx + b\dot{x}) \leq D_2 \tag{5.16}$$

$$2.\ \max_{t\in[0,t_1]} (cx + b\dot{x}) \underset{c,b}{\to} \min, \quad x(t_1) \leq D_1 \tag{5.17}$$

In monograph [5] problem 2 of this example is solved, and following optimal solutions are obtained :

$$c_o \approx 0.361\frac{x_o^2}{D_1^2},\ b_o \approx 0.481\frac{\dot{x}_o}{D_1}. \tag{5.18}$$

Same solutions are obtained by authors of this paper, by solving given example as optimal control problem with constant parameters [6], but with condition $\frac{M+m}{Mm} = 1$. Therefore, solutions (5.18) should be multiplied with factor $\frac{M+m}{Mm}$.

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