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Restrictions of Reactions of Internal and External Constraints of Optimally Controlled Systems of Bodies

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Abstract

The optimal controlled mechanical system is considered, where the values of constraint reactions and inner forces and momentums in certain bodies' sections appear as restrictive factors. The basic task is given in the paper. The task is optimal control of motion of system with n degree of freedom. By introducing "additional" coordinates, the new task of optimal control with presence of proper reaction with given restrictions is formed. The particular attention is given to the method which enables applying of symbolic programming.

Key words: optimal control, bodies system, constraints

Introduction

During operation of some mechanisms damaging consequences can appear due to great overload in constraint or some sections of mechanism's links. Due to that, plastic or elastic deformations can appear, and their sizes influence the precision of work where it is necessary. In controlled systems (manipulators, robots) it is possible to avoid those phenomena by change in controlling regime. Because of that, it is necessary that that overload is explicitly figured in controlled motion equations. By their constraining, the regime of optimal control is influenced, which is supposed to secure system from above mentioned damaging consequences. In paper [1], controlled system with analysis and constraining of reaction of portion of outside and inner constraint, is considered.

In this paper, beside the constraint reaction, the inner forces and momentums in sections of certain bodies' are considered. Rigid body, if needed, can be considered as a system of more bodies solidly linked in considered sections. Because of that, in further text of this paper, all constraint reactions among body and inner forces and momentums in body section will have common name - reactions. In first part of this paper the basic task of optimal control is given. In second part optimal control task in motion systems with "additional" coordinates with presence of one part of restricted reactions is given. The task is brought to the form which enables direct applying of maximum principle [2,3]. Systems with higher number of degrees of freedom (robots) demand computer applying for symbolic forming of controlled motion equation. In third part, it is shown for one class of mechanical systems on which way the system with reactions can be adjusted to program applying, which is given in paper [4].

1. The Basic Optimal Control Task

The state of mechanical system with n degrees of freedom is determined in 2n-dimensional phase V_{2n} space, by generalized coordinates $\overline{q}^i$ and generalized momentums $\overline{p}_i$. Hamilton's function [5] of scleronomic system represents mechanical energy and has the form:

$$\overline{H} = \overline{T} + \overline{V} = \frac{1}{2}\overline{a}^{ij}(\overline{q})\overline{p}_i\overline{p}_j + V(\overline{q}) \tag{1}$$

where $\overline{T}$ is kinetic energy, $\overline{V}$ is potential energy, and $\overline{a}^{ij}$ is contravariant tensor of configuration space R_n. Let the non-potential, beside the potential forces influence the system:

$$\overline{Q}_i = \overline{Q}_i(\overline{q},\overline{p},u) , \tag{2}$$

where $u = u(u_1,u_2,\dots,u_r)$ is control vector from space U_r. The behavior of such system is described in phase space V_{2n} by differential equations:

$$\dot{\overline{q}}^i = \frac{\partial \overline{H}}{\partial \overline{p}_i} , \quad \dot{\overline{p}}_i = -\frac{\partial \overline{H}}{\partial \overline{q}^i} + \overline{Q}_i(\overline{q},\overline{p},u) . \tag{3}$$

Let in general case allowable controls be restricted:

$$u \in G_u \subset U_r \tag{4}$$

and let the system be controllable, i.e. let among the allowable controls (4) exist such, which realize the goal that in accordance with the equations (3), the system from initial state:

$$\overline{A}_\sigma\left[\overline{q}(t_0),\overline{p}(t_0)\right] = 0 \ , \ \sigma = 1,2,\dots,a \le 2n \tag{5}$$

comes to final state:

$$\overline{B}_\theta\left[\overline{q}(t_1),\overline{p}(t_1)\right] = 0 \ , \ \theta = 1,2,\dots,b \le 2n \tag{6}$$

The criterion of optimality can be given in form:

$$\int_{t_0}^{t_1} f^0(\overline{q},\overline{p},u)\,dt \to \inf , \quad u \in G_u \tag{7}$$

By the relations (2) to (7) the basic task of optimal control is defined in form, which enables direct applying of maximum principle [2].

2. Optimal Control Task And Constraint Reaction

In previously defined optimal control task, the consequences which optimal control causes in system constraints and bodies themselves, are not considered.

The considering of reaction as restrictive factor of controlled motion will demand their explicit presence in differential equations. By partial or complete "freeing" from constraints and by " sectioning" of the body and introducing "additional" coordinates $\xi^{\nu}(\nu = 1,2,\ldots,l)$, configuration space of system is expanded to R_{n+l}. The number l of coordinates ξ^{ν} equals to number of considered reactions. In order system motion in space R_{n+l} to be equivalent to motion in space R_n the constraints should be introduced:

$$\varphi^{\nu}(\bar{q},\xi) = 0 \ , \ \nu = 1,2,\ldots,l \qquad (8)$$

Due to applying of symbolic programming [4] it is necessary to harmonize the order of indexes i and ν with body numeration in kinematic chain. Besides, it is useful to change the coordinates $\bar{q}^i, \xi^{\nu}$ with q^{α} $(\alpha = 1,2,\ldots,n+l)$ where the index α is also adjusted to mentioned numeration. The equations (8) get the form:

$$\varphi^{\nu}(q) = 0 \qquad (9)$$

On that way the state of system with "additional" coordinates is determined in phase space $V_{2(n+l)}$ by generalized coordinates q^{α} and generalized momentums p_{α} with constraint equations (9). Hamilton's function of such system has form:

$$H = T + V = \frac{1}{2}a^{\alpha\beta}(q)p_{\alpha}p_{\beta} + V(q) \ , \qquad (10)$$

$\alpha,\beta = 1,2,\ldots,n+l$,

and the behavior of system in phase space $V_{2(n+l)}$ is described by differential equations:

$$\dot{q}^{\alpha} = \frac{\partial H}{\partial p_{\alpha}}, \dot{p}_{\alpha} = -\frac{\partial H}{\partial q^{\alpha}} + Q_{\alpha}(q,p,u) + R_{\alpha} \qquad (11)$$

Quantities R_{α} represent reactions of constraint (9):

$$R_{\alpha} = \lambda_{\nu}\frac{\partial \varphi^{\nu}}{\partial q^{\alpha}} \qquad (12)$$

where λ_{ν} are undetermined multipliers. Supposing that all gradients of constraints (9) are independent:

$$rang\left[\frac{\partial \varphi^{\nu}}{\partial q^{\alpha}}\right] = l \ , \qquad (13)$$

the number of independent reactions R_{α} equals to number of constraints (9). If considered system represents kinematic chain, because of simpler applying of symbolic programming it is convenient for generalized coordinates q^{α} to be relative, i.e. determine motion of every body of kinematic chain in respect to previous body. On that way, the equations of constraints (8) can be brought to form:

$$\xi^{\nu} = \delta^{\nu}_{\alpha}q^{\alpha} = 0 \ , \ \left(\delta^{\nu}_{\alpha} = \begin{cases} 1, q^{\alpha} = \xi^{\nu} \\ 0, q^{\alpha} = \bar{q}^{i} \end{cases}\right) \qquad (14)$$

and proper reactions to form:

$$R_{\alpha} = \lambda_{\nu}\delta^{\nu}_{\alpha} \qquad (15)$$

In that case, the multipliers λ_{ν} represent independent constraint reactions from the space U_l.

The system described by equations (11) with constraints (9) or (14) is equivalent to system described by equations (3). That equivalency disappears by introducing further restrictions. Explicit presence of reactions (12) or (15) in equations (11) enables further restrictions to be introduced. Let the constraints be of form (14) and let there be the request that suitable reactions (15) undergo restrictions in control process:

$$\lambda \in G_{\lambda} \subset U_l \qquad (16)$$

The criterion of optimality (7) after introduction of variables q^{α}, p_{α} gets the form:

$$\int_{t_0}^{t_1} F^0(q,p,u)dt \to \inf \ , \quad u \in G_u, \lambda \in G_{\lambda} \ . \qquad (17)$$

The initial state (5) and final state (6) are determined in space $V_{2(n+l)}$ by manifolds:

$$A_{\sigma}[q(t_0),p(t_0)] = 0 \ , \sigma = 1,2,\ldots,a \le 2n \qquad (18)$$

$$B_{\theta}[q(t_1),p(t_1)] = 0 \ , \ \theta = 1,2,\ldots,b \le 2n. \qquad (19)$$

Thus formulated task should be transformed to form suitable for applying of optimal control theory. According to the Theorem 22 [2], constraints (14) should replace by equivalent restrictions. By differentiation equations of constraints (14), according to (11), the equations are achieved:

$$\frac{d\xi^{\nu}}{dt} = 0 \ \Rightarrow \ \delta^{\nu}_{\alpha}\frac{\partial H}{\partial p_{\alpha}} = 0 \qquad (20)$$

$$\frac{d^2\xi^{\nu}}{dt^2} = 0 \Rightarrow$$

$$\Phi^{\nu}(q,p,u,\lambda) = \delta^{\nu}_{\alpha}[\frac{\partial^2 H}{\partial p_{\alpha}\partial q^{\beta}}\frac{\partial H}{\partial p_{\beta}} + \qquad (21)$$

$$\frac{\partial^2 H}{\partial p_{\alpha}\partial p_{\beta}}(-\frac{\partial H}{\partial q^{\beta}} + Q_{\beta}(q,p,u) + \lambda_{\mu}\delta^{\mu}_{\beta})] = 0,$$

where the quantities δ^{ν}_{α} are determined in (15). In order the relations (21) to be equivalent to constraints (14) it is enough to add conditions:

$$\delta^{\nu}_{\alpha}q^{\alpha}(t_0) = 0, \left(\delta^{\nu}_{\alpha}\frac{\partial H}{\partial p_{\alpha}}\right)_{t_0} = 0 \ . \qquad (22)$$

to initial conditions (18).

On that way, the optimal control task with restricted reactions, defined by relations (11), (17), (18), (19), (21) and (22) and restrictions (4) and (16) , is transformed into form for direct applying of Theorem 22 [2], which gives necessary conditions for optimal solution determining.

The reactions λ_ν , together with quantities u_m in this task represent control function. Linear dependence of λ_ν in equations (11) and (21) implicate the possibility of singular solution existence [6,7,8]. If in whole time interval $[t_0, t_1]$ all optimal values λ^*_ν are singular, i.e. inside of area G_λ , the optimal solutions u^*_m and the corresponding motion is identical to the solution of basic task of optimality which is formulated in the second part of this paper. On the other hand, if at least one sub-interval $[t', t''] \subset [t_0, t_1]$ exists in which optimal value λ^* is at the boundary of the area G_λ . optimal control u^* and corresponding motion differ from the solution of basic optimality task.

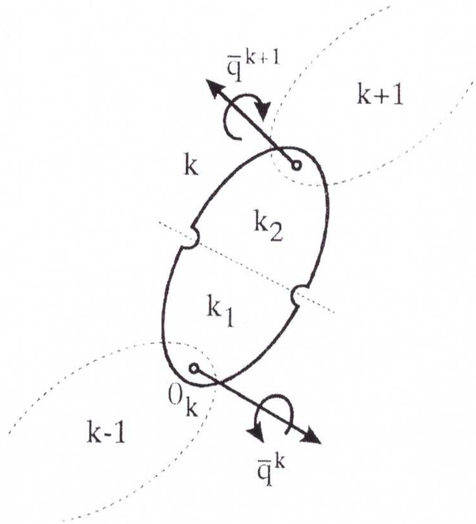

Fig.1

3. Symbolic Programming of System with Reactions

In paper [4] the procedure of differential equations forming of robotic system motion by using symbolic programming is given. Program for open loop kinematic chain with kinematic pairs of fifth class is shown in detail. Input quantities of program are: the quantities which define geometry of system and bodies mass and moments of inertia.

Configuration of open loop kinematic chain with pairs of fifth class is determined by coordinates $\bar{q}^i$ that determine position of body i in relation to body i-1 . The important role in symbolic programming have Rodrigo's matrix of transformation [9] :

$$[\bar{A}_{i-1,i}] = [I] +$$

$$\delta(i)\Big((1-\cos\bar{q}^i)[e_i^{d(i)}]^2 + \sin\bar{q}^i[e_i^{d(i)}]\Big) \qquad (23)$$

where [I] is identity matrix, and:

$$[e_i^{d(i)}] = \begin{bmatrix} 0 & -e_{iz} & e_{iy} \\ e_{iz} & 0 & -e_{ix} \\ -e_{iy} & e_{ix} & 0 \end{bmatrix} \qquad (24)$$

The quantity $\delta(i)$ defines the type of constraints between bodies i and i-1 . In case of translation $\delta(i) = 0$, and in case of rotation $\delta(i) = 1$. Vectors $\vec{e}_i$ determine direction of translation or rotation axes.

The application of program [4] is restricted to possibility that in every section only one reaction in defined direction is considered. Let there be the request that reaction in only one section of body k is considered (Fig.1).

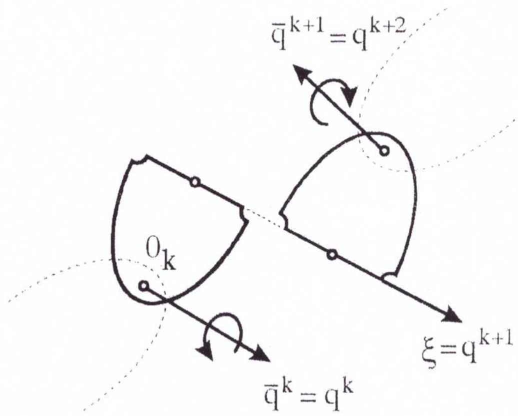

Fig.2

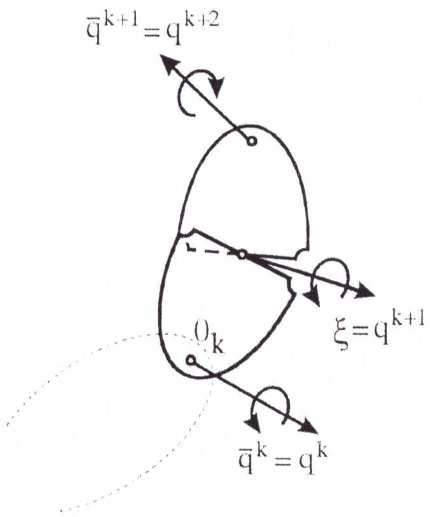

Fig.3

In that sense body k should be considered as a system of two bodies k_1 and k_2 that are rigidly interconnected. By interchanging of the constraint with kinematic pair of fifth class and by introducing the coordinate ξ that defines the position of the body k_2 in relation to body k_1, the configuration space is expanded to R_{n+1}. If in the section the force is considered, the translation of the body k_2 in relation to body k_1 is introduced in the given direction of force (Fig.2). If the momentum of flexion (or torsion) is considered, the rotation around the proper axes is introduced (Fig.3).

The configuration of such system is determined by coordinates q^α whose values are:

$$q^\alpha = \bar{q}^i \quad , \alpha = i(i \le k)$$
$$q^{k+1} = \xi , \tag{25}$$
$$q^\alpha = \bar{q}^i \quad , \alpha = i+1(i > k) .$$

In that sense, new numeration of bodies should be done. Rodrigo's matrix of transformation for bodies k_1 and k_2 are:

$$\left[\bar{A}_{k_1,k}\right], \left[\bar{A}_{k_2,k_1}\right] \tag{26}$$

where the matrix of body k_1 equals the matrix of body k :

$$\left[\bar{A}_{k_1,k}\right] = \left[\bar{A}_{k-1,k}\right] \tag{27}$$

To existing matrices only the matrix of body k_2 should be added, so that we have n+1 matrices for symbolic programming:

$$\left[A_{\alpha-1,\alpha}\right] = \begin{cases} \left[\bar{A}_{i-i,i}\right], i \le k \\ \left[\bar{A}_{k_1,k_2}\right], \alpha = k+1 \\ \left[\bar{A}_{i,i+1}\right], i > k \end{cases} \tag{28}$$

Instead of input data for body k , the data on bodies k_1 and k_2 should be separately put in program. On this way, by applying of symbolic programming the differential equations similar to form (11) can be formed. There appears one reaction R of constraint $q^{k+1} = 0$ ($\xi = 0$).

The restriction (16) have the form:

$$|R| \le R_{\max} \tag{29}$$

where $R_{\max}$ is in advance given quantity.

The further procedure is totally based on methods of mathematical theory of optimal control.

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