CONSTRAINT REACTIONS IN OPTIMAL CONTROL OF MECHANICAL SYSTEMS

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This paper is dedicated to the establishment of a general procedure of forming the optimal control problem of variable-mass nonholonomic rheonomic mechanical systems, where reactions of constraints are present in differential equations of motion. Dimensions and structure of a configuration space depend on the number of reactions of constraints that are the subject of our interest, i.e. only the reactions whose magnitudes are subjected to limitations are considered. In this paper, the procedure enables the direct application of Pontryagin's maximum principle for the systems with limited phase state. Attention is particularly focused on discussing various modes of realizing the control by combining the active control forces and subsequent imposition of ideal holonomic mechanical constraints. Brachistochronic motions play an important role in this type of problems, because in them the control of motion can be realized exclusively with ideal constraints. The paper provides three examples of this method application, which are related to the realization of the brachistochronic motion of mechanical systems.

1. Introduction

In the theoretical field of optimal control, the motion optimization of systems with limited phase state has been discussed in detail [1-5], but the mathematical models of those limitations do not show their physical essence. The basic problem of the determination of the optimal control which provides the motion along the optimal trajectory according to the restrictions imposed (theorems 22-25, [1]) is being solved. The immediate applications of such solving procedure of mechanical system control problems are sensible if the phase limitations belong to some subjective requirements (insurance against undesirable behaviour of the systems, for example). But, if the motion of mechanical system is limited by material constraints, then, independent of the control, the constraint occurs which, according to the procedure mentioned, would remain "hidden" in the solutions for optimal control. Another important fact, which is not considered in the mathematical theory of optimal control, is the difference between holonomic and nonholonomic mechanical constraints, even though some authors refer to holonomic constraints as all phase limitations.

The constraint reactions in practical problems present the load of the system with useful or harmful consequences, and thus it is necessary to have a possibility of influencing their behaviour during the motion control process. Therefore, in this paper, the mathematical model of a mechanical system will be written in such a way so as to provide the explicit presence of constraint reactions. Thereby, if the consideration of all constraint reactions is unnecessary, the structure and dimension of the configuration space will depend upon which and how many of them are the object of our interest.

This paper is dedicated to the establishment of a general procedure of setting the problem of optimal control of mechanical systems in whose differential equations a section of constraint reactions figure. The choice of those reactions of constraints, in addition to their limitations required, can be also conditioned by the need for a simpler manner of solving the problem. Due to this fact, the reactions of all nonholonomic constraints and a chosen section of holonomic constraints will figure in the equations of mechanical systems. The procedure applied in this paper enables the direct application of Pontryagin's maximum principle, therefore the optimal control problem solving procedure itself will not be discussed herein.

The procedure is based on methodology presented in [6] and extension of a general procedure used in [7-9], with special reference to the problems involving limited internal reactions of constraints between some bodies in the system or even the force in the intersections of bodies themselves. The solution is found then for a number of examples related to the control by the help of mechanical constraints at various optimality criteria [10-13], however, the cases with limited reactions of constraints are not considered.

This paper considers further generalization to the rheonomic systems and variable-mass systems. In Section 2 the optimal control problem is set for the systems with mechanical constraints and Section 3 shows the procedure for solving the problem by applying Pontryagin's maximum principle. Section 4 is dedicated to the discussion of various modes of realizing the control by combining active control forces and subsequent imposition of ideal holonomic mechanical systems.

Brachistochronic motions play an important role in this type of problems, because in them the control of motion can be also often realized exclusively by ideal constraints, i.e. without the action of active control forces. The latest results, where the application of mentioned method enabled this paper's authors and their collaborators to solve a number of optimal control problems, are presented in [14-22], while the results of other authors are given in references [23-58]. A more detailed presentation of other authors' results is also found in [14-22]. We briefly emphasize the fact that in the cited references by other authors there are no considerations of the problems concerning variable-mass systems and, except for [45], rheonomic systems have not been considered. It is noticeable that nonholonomic mechanical constraints are present only in a smaller number of papers [48-56], which naturally belong to the dynamics of mechanical systems, while other papers dealt mainly with the optimal motion of a particle. This paper provides three examples illustrating a general method developed herein.

Section 5 shows the realization of control for the case of a nonholonomic rheonomic mechanical system [15]. Two modes are presented, control by active forces and without the action of active forces.

A general procedure of determining brachistochronic motions for the case of a variable-mass system is presented in [16], and the example from that paper is shown in Section 6. We have demonstrated the procedure of realizing the control exclusively by the help of constraints realized in this case by the rolling without slipping of a moving centrode on the fixed one.

Section 7 refers to the case when the reaction of nonholonomic constraint is limited in the problem of brachistochronic motion of the Chaplygin sleigh. In optimal motion the reaction of the blade is positioned on its boundary in one section of the motion interval. Here too the motion is realized by imposition of an additional ideal holonomic constraint.

2. Problem statement

The controlled motion of a rheonomic mechanical system with restrictions of a certain number of reactions of ideal mechanical constraints is considered, so their presence should be ensured in differential equations of the motion of a system. Those holonomic constraints whose reactions are not subject to restrictions should be eliminated from consideration, which is achieved by establishing the configuration space on those constraints. The number of dimensions n of such space equals the difference between the total number of the system coordinates and the number of eliminated constraints, respectively the sum of the number of DOFs of the system and the total number of constraints whose reactions are considered. In that space the configuration of the mechanical system is determined by generalized coordinates $\overline{q}=(q^1,q^2,\ldots,q^n)$. Let, in a general case, the system also contain ℓ variable-mass particles

$$m_\rho = m_\rho(t), \quad \rho=1,\ldots,\ell \tag{1}$$

whose particles are expelled $(\dot{m}_\rho<0)$ or gained $(\dot{m}_\rho>0)$ to the system by relative velocities

$$\vec{v}_\rho^{rel}=\vec{v}_\rho^{rel}(\overline{q},\dot{\overline{q}},t)\ . \tag{2}$$

Hereafter it will be considered that (1) and (2) are known functions.
The kinetic energy of the rheonomic system has the form

$$T=\frac{1}{2}a_{ij}(\overline{q},t)\dot{q}^i\dot{q}^j+a_i(\overline{q},t)\dot{q}^i+a_0(\overline{q},t)\ , \qquad i,j=1,...,n\ . \tag{3}$$

Let the system move in the field of known potential forces with potential energy

$$\Pi=\Pi(\overline{q},t), \tag{4}$$

and let the system be also acted upon by arbitrary known nonpotential forces, whose generalized forces are

$$Q_i^w=Q_i^w(\overline{q},\dot{\overline{q}},t), \qquad i=1,\ldots,n. \tag{5}$$

In the configuration space there figure the mechanical constraints that are intentionally left to consider their reactions. Let the motion of the system be limited by r ideal independent nonholonomic rheonomic mechanical constraints

$$b^\alpha\left(\overline{q},t\right)+b_i^\alpha(\overline{q},t)\dot{q}^i=0, \quad \alpha=1,...,r, \tag{6}$$

and p ideal independent holonomic rheonomic mechanical constraints

$$\theta^\beta(\overline{q},t)=0, \quad \beta=1,...,p. \tag{7}$$

on the basis of which the conditions for velocities are obtained

$$\frac{\partial\theta^\beta}{\partial q^i}\dot{q}^i+\frac{\partial\theta^\beta}{\partial t}=0. \tag{8}$$

The change in the mass of the system causes the occurrence of reaction forces $\vec{F}_\rho^{\mathrm{var}}$ that in the system of generalized coordinates have the form

$$Q_i^{\mathrm{var}}(\overline{q},\dot{\overline{q}},t)=\sum_{\rho=1}^{\ell}\vec{F}_\rho^{\mathrm{var}}\frac{\partial\vec{r}_\rho}{\partial q^i}=\sum_{\rho=1}^{\ell}\dot{m}_\rho\vec{v}_\rho^{rel}\frac{\partial\vec{r}_\rho}{\partial q^i}\ . \tag{9}$$

In addition to mentioned forces and restrictions, the system is subjected to the action of k active control forces $\vec{F}_\delta$, that is

$$Q_i^c = \sum_{\delta=1}^{k} \vec{F}_\delta \frac{\partial \vec{r}_\delta}{\partial q^i} . \tag{10}$$

Differential equations of motion of such variable-mass system in the form of Lagrange's equations have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = -\frac{\partial \Pi}{\partial q^i} + Q_i^w + \tilde{Q}_i^{\text{var}} + R_i + Q_i^c \tag{11}$$

where generalized forces

$$\tilde{Q}_i^{\text{var}}(\overline{q}, \dot{\overline{q}}, t) = \sum_{\rho=1}^{\ell} \dot{m}_\rho \vec{v}_\rho \frac{\partial \vec{r}_\rho}{\partial q^i} + Q_i^{\text{var}} . \tag{12}$$

The expressions occurring here with Q_i^{var} are a consequence of the form of differential equations (11). They are annulled by the same expressions on the left-hand side of those equations, when differentiation with respect to time is done.

Generalized reactions of constraints R_i have the form

$$R_i = b_i^\alpha \nu_\alpha + \frac{\partial \theta^\beta}{\partial q^i} \mu_\beta \tag{13}$$

where ν_α and μ_β represent multipliers of mechanical constraints (6) and (7). However, as the directions of ideal constraint reactions $\vec{N}_\gamma$ are defined by the constraints themselves, the generalized reactions can be expressed in the form

$$R_i = d_i^\gamma(\overline{q}, t) N_\gamma, \ \gamma = 1, ..., p + r \tag{14}$$

where $d_i^\gamma(\overline{q}, t)$ are known functions that define the directions of ideal constraint reactions. The quantities N_γ represent the projections of each reaction to its eigendirection in particular. Such a manner of introducing the ideal constraint reactions into the equations of motion is convenient, because all limitations of constraint reactions can be imposed only by their projections N_γ .

Such procedure can be also applied to control forces $\vec{F}_\delta$, especially in those systems where the directions of control forces are defined by the system configuration. Then

$$Q_i^c = c_i^\delta(\overline{q}, \dot{\overline{q}}, t) F_\delta, \ \delta = 1, ..., k , \tag{15}$$

where $c_i^\delta(\overline{q}, \dot{\overline{q}}, t)$ are known functions that define the directions of control forces $\vec{F}_\delta$ in the configuration space. The quantities F_δ are the projections of forces to those directions.

Further, we can introduce the vectors $\overline{F} = (F_1, F_2, \ldots, F_k)$ and $\overline{N} = (N_1, N_2, \ldots, N_{p+r})$ from corresponding vector spaces V^k and V^{p+r}. The boundaries of those vectors determine the sets U_F and U_N from corresponding spaces such that:

$$\overline{F} \in U_F \subset V^k \ , \ \overline{N} \in U_N \subset V^{p+r} . \tag{16}$$

In a general case, these sets can be constant or variable. The sets

$$U_N: \quad N_\gamma^* \le N_\gamma \le N_\gamma^{**} \tag{17}$$

and

$$U_F: \quad F_\delta^* \le F_\delta \le F_\delta^{**}, \tag{18}$$

where the boundaries, designated by $()^*$ and $()^{**}$, are constant represent an example of constant sets.

If, in addition to the constraints (6) and (7), following limitations are imposed on the system

$$\psi^\omega(\overline{q}, \dot{\overline{q}}, t) = 0, \quad \omega = 1,...,s \quad , \quad p + r + s < n, \tag{19}$$

which do not represent mechanical constraints, the structure of (11) will not be changed. This fact indicates a crucial difference between mechanical constraints and limitations (19). They only represent the required integrals of differential equations (11).

Let the optimality of the system motion in the interval $\left[t_0, t_1\right]$ be required by the condition

$$J = \int_{t_0}^{t_1} F^0(\overline{q}, \dot{\overline{q}}, \overline{F}, \overline{N}, t)dt \underset{\substack{\overline{F}\in U_F \\ \overline{N}\in U_N}}{\rightarrow} \inf \ . \tag{20}$$

The state of the system on the interval ends is presented with manifold

$$\theta^\eta(\overline{q}(t_0), \dot{\overline{q}}(t_0), t_0, \overline{q}(t_1), \dot{\overline{q}}(t_1), t_1) = 0, \quad \eta = 1,..., z \le 4n, \tag{21}$$

with given t_0 and, in a general case, undetermined t_1.

Let the system be controllable, i.e. let among the allowable forces $\overline{F} = \overline{F}(t)$ exist such that satisfy forces restrictions (16), and based on which we can obtain the solutions of differential equations (11) that satisfy end conditions (21) and equations of constraints (6), (7) and (19).

Solving the problem of optimal control of a variable-mass mechanical system, whose differential equations are (11), consists in determining the control forces $\overline{F} = \overline{F}(t)$ and the system motions corresponding to them, so that the condition of optimality (20) is satisfied. At the same time, the conditions (6), (7), (16), (19), and (21) are satisfied.

3. Solution of the optimal control task

The previously defined problem should be formulated in a form convenient for the solution by applying the method of the optimal control theory. For that purpose, let us introduce n dimensional control vector $\overline{u} = (\overline{F}, \overline{N})$ and $2n+1$ dimensional state vector $\overline{x} = (\overline{q}, \dot{\overline{q}}, q^{n+1})$ where rheonomic coordinate q^{n+1} satisfies

$$\dot{q}^{n+1} = 1, \quad q^{n+1}(t_0) = t_0 . \tag{22}$$

Differential equations (11) can be written in a contravariant form, and on the basis of this, we can also obtain state equations that are linear for controls

$$\dot{x}^i = x^{n+i}, \quad \dot{x}^{n+i} = f^i(\overline{x}) + f^{ij}(\overline{x})u_j, \quad \dot{x}^{2n+1} = 1, \quad i,j,m = 1,...,n, \tag{23}$$

where

$$f^i = a^{ij}(-\dot{a}_{jm}\dot{q}^m - \dot{a}_j + \frac{\partial T}{\partial q^j} - \frac{\partial \Pi}{\partial q^j} + Q_j^w + \tilde{Q}_j^{\text{var}}), \; f^{ij} = \begin{cases} a^{im}c_m^j, j = 1,...,k \\ a^{im}d_m^{j-k}, j = k+1,...,n \end{cases}. \tag{24}$$

Restrictions to state variables (6),(8) and (19) can be then integrated

$$\varphi^{\varepsilon}(\overline{x}) = 0, \quad \varepsilon = 1,..., p+r+s \;. \tag{25}$$

Differentiating them with respect to time, in accordance with differential equations (23), we obtain equations

$$\Phi^{\varepsilon}(\overline{x},\overline{u}) = 0, \tag{26}$$

where initial condition (22) should be added to end conditions (21) as well as conditions (6), (7), (8) and (19) at initial time moment

$$b^{\alpha}\left(\overline{q}(t_0),t_0\right) + b_i^{\alpha}(\overline{q}(t_0),t_0)\dot{q}^i(t_0) = 0,$$

$$\theta^{\beta}(\overline{q}(t_0),t_0) = 0, \qquad (\frac{\partial\theta^{\beta}}{\partial q^i}\dot{q}^i + \frac{\partial\theta^{\beta}}{\partial t})\Big|_{\overline{x}=\overline{x}(t_0)} = 0, \tag{27}$$

$$\psi^{\omega}(\overline{q}(t_0),\dot{\overline{q}}(t_0),t_0) = 0.$$

All mentioned conditions on the interval boundaries can be represented in the form

$$\Theta^{\tau}(\overline{x}(t_0),\overline{x}(t_1)) = 0, \quad \tau = 1,...,z+2p+r+s+1 \;. \tag{28}$$

The vector of allowable controls $\overline{u}$ from the vector space $V^n = V^k \times V^{p+r}$, considering (16), belongs to the set U, that is

$$\overline{u} \in U \subset V^n \;. \tag{29}$$

Additionally, the allowable controls must also satisfy the relations (26). The allowable controls on the interval $\left[t_0,t_1\right]$are, in a general case, limited functions, piecewise continuous, with a finite number of discontinuities.
The optimality criterion (20) obtains the form

$$J = \int_{t_0}^{t_1} f^0(\overline{x},\overline{u})dt \underset{\overline{u}\in U}{\rightarrow} \inf \;. \tag{30}$$

The relations (23), (25), (28), (29) and (30), with a specified initial time moment t_0, define fully the problem of optimal control in the form convenient for immediate application of Pontryagin's maximum principle. In this regard, based on (23) and (30), the function is formed

$$H = \lambda_0 f^0(\overline{x},\overline{u}) + \lambda_i x^{n+i} + \lambda_{n+i}(f^i(\overline{x}) + f^{ij}(\overline{x})u_j) + \lambda_{2n+1} \tag{31}$$

and, based on it and the relations (26), the conjugate system of equations

$$\lambda_0 = const \le 0, \quad \dot{\lambda}_g = -\frac{\partial H}{\partial x^g} + \kappa_{\varepsilon}\frac{\partial \Phi^{\varepsilon}}{\partial x^g}, \; g = 1,...,2n+1 \;, \tag{32}$$

where κ_{ε} are multipliers corresponding to the relations (26). If among the allowable controls (29) there exists optimal control $\overline{u}_{opt} = \overline{u}_{opt}(t)$, the solutions $\overline{x}_{opt} = \overline{x}_{opt}(t)$ and

$\bar{\lambda}_{opt} = \bar{\lambda}_{opt}(t)$ correspond to it, so that the maximum principle generates the conditions on the optimal trajectory

$$H(\bar{x}_{opt}, \bar{\lambda}_{opt}, \bar{u}_{opt}) = \sup_{\bar{u} \in U} H(\bar{x}_{opt}, \bar{\lambda}_{opt}, \bar{u}) \; . \tag{33}$$

The relations following immediately from this condition represent only the necessary conditions for extremal solutions $\bar{u}_{ext} = \bar{u}_{ext}(\bar{x}, \bar{\lambda})$, among which optimal ones should be sought.

Note that in differential equations (23) and relations (26) the controls figure linearly. Then, if the subintegral function $f^0(\bar{x}, \bar{u})$ of the functional (30) is also linearly dependent of control or does not depend of it at all, there may occur singular controls on some subintervals.

We give an example of a case when allowable controls belong to a constant set

$$U : C_j^* \le u_j \le C_j^{**} \tag{34}$$

and when minimization of the interval $[t_0, t_1]$ is conditioned, i.e. when $f^0(\bar{x}, \bar{u}) = 1$. It follows then from (33) that

$$u_{j,ext} = \begin{cases} C_j^* , & \lambda_{n+i} f^{ij}(\bar{x}) < 0 \\ u_{\text{j,singular}}(\bar{x}, \bar{\lambda}), & \lambda_{n+i} f^{ij}(\bar{x}) = 0 \\ C_j^{**}, & \lambda_{n+i} f^{ij}(\bar{x}) > 0 \end{cases} , \tag{35}$$

where a special problem is that we do not know in advance the structure of the control for time subintervals, and singular controls are determined by the help of a special theory of singular optimal controls [59]. The theory also makes available the Kelley conditions [60] on singular intervals as well as conditions for the conjugation of singular and nonsingular sections [61].

Note that there is a possibility that some controls u_i do not have imposed constraints (29), i.e. they belong to an open set. Let such controls be k_1. In that case, taking into account that the functions $\Phi^{\varepsilon} = \Phi^{\varepsilon}(\bar{x}, \bar{u})$ linearly depend here of the control, it is always from k_1 of relations $\Phi^{\varepsilon}(\bar{x}, \bar{u}) = 0$ that such controls can be expressed via other controls, and thus obtained dependency is also linear. In differential equations (23), then $j = 1, ..., n - k_1$ and mentioned k_1 relations $\Phi^{\varepsilon}(\bar{x}, \bar{u}) = 0$ are excluded from relations (26) and (32).

End conditions (28) are completed by corresponding transversality conditions [1], so that their total number equals the total number of differential equations, $4n + 2$, where, in cases when t_1 is not specified, the condition $H(\bar{x}(t_0), \bar{\lambda}(t_0)) = 0$ or $H(\bar{x}(t_1), \bar{\lambda}(t_1)) = 0$ is available.

Substituting extremal control $\bar{u}_{ext} = \bar{u}_{ext}(\bar{x}, \bar{\lambda})$ into a basic system (20) and conjugate system (28), we obtain the two-point boundary value problem of the system of ordinary nonlinear differential equations. The difficulties in its numerical solving are a limiting factor in the application of the Maximum principle in optimal control of mechanical systems, and each successfully solved problem of this type certainly deserves attention. The lack of general theorems on the existence of TPBVP solutions makes us most often seek

solutions without such theorems, and a special problem are also multiple solutions among which optimal ones too have to be sought.
After numerical computations [62] we obtain solutions in a numerical form

$$\overline{x}_{ext} = \overline{x}_{ext}(t),\ \ \overline{\lambda}_{ext} = \overline{\lambda}_{ext}(t),\ \ \overline{u}_{ext} = \overline{u}_{ext}(t)\ , \tag{36}$$

therefore the procedure of determining extremal controls is completed.

4. Realization of optimal control in systems with mechanical constraints

On the basis of solution (36) , in addition to the law of change of k control forces, we also obtained the laws of change in $p+r$ reactions of ideal constraints

$$F_\delta = F_\delta(t)\ ,\ \ N_\gamma = N_\gamma(t)\ \ , \tag{37}$$

so it is evident here that the realization of optimal motion is achieved by the action of active control forces. Note that this paper considers the case when the number of active control forces equals the number of DOFs of a mechanical system, which is the most common case in motion control problems, although the methodology used in this paper is applicable to other cases as well.
If the number of active control forces is increased, some of them can be arbitrarily taken to satisfy extremal solutions (36). It is even possible to have the case when new active forces $F_\gamma = u_{k+\gamma}(t)$ are introduced, which would lead to all reactions of holonomic and nonholonomic mechanical constraints being equal to zero ($\nu_\alpha = 0, \mu_\beta = 0$).
Reasoning in another direction is also of interest, and it is whether new ideal mechanical constraints can be subsequently imposed on the system instead of active control forces, in accordance with (36), so that motion control can be realized only by ideal constraints without any action of active forces. In that case, the independent newly introduced holonomic scleronomic constraints

$$\phi^\pi(\overline{q}) = 0\ \ ,\ \ \pi = 1,...,k-1 \tag{38}$$

must be such that their reactions substitute the action of active control forces

$$Q_i^c = c_i^\delta u_\delta = \rho_\pi \frac{\partial \phi^\pi}{\partial q^i} \tag{39}$$

and lead to the conditions critical for further consideration

$$c_i^\delta u_\delta \dot{q}^i = 0. \tag{40}$$

They indicate that the power $Q_i^c \dot{q}^i$ of control forces equals zero, wherefrom there follows the restriction to the class of problems where it is possible to control constraints only. This class of problems also includes the problem of brachistochronic motion as a special type of time minimization problem ($f^0 = 1$). The conditions (40) represent part of the conditions $\Phi^\varepsilon(\overline{x},\overline{u}) = 0$, where

$$t_0 = 0 \ , q^i(t_0) = q_0^i, T(\overline{q}_0, \dot{\overline{q}}_0, t_0) + \Pi(\overline{q}_0, t_0) = E_0, \quad q^i(t_1) = q_1^i . \qquad (41)$$

In real problems of the mechanical systems, rigid bodies, in particular, the simplest way is to impose guides which ensure that the trajectories of the corresponding number of points are in accordance with the previously determined brachistochronic motion. This can be realized in an infinite number of ways, where the determination of reactions of the constraints and multipliers alone is not of particular interest. The choice of point can also be imposed by the simplest design solution.

In other types of optimality problems, where on the optimal trajectory the power $Q_i^c \dot{q}^i$ of control forces is different from zero, it is possible by introducing new ideal constraints to reduce the number of k active control forces by the number identical to the number of newly introduced independent mechanical constraints.

5. Example 1: Brachistochronic motion of a nonholonomic rheonomic mechanical system [15]

On a flat horizontal block of ice (plane π) caught in the sea current, which starts to move translatory in a straight line with constant acceleration a (see Fig. 1), a sleigh is moving so that it remains in contact with π at three points as shown in Fig.1. The first two points, on the supports N_2 and N_3, can slide in arbitrary directions on plane π. The third point, (M), is the point of contact of the vertical blade S, connected with support N_1, of the sleigh with the plane π. This point cannot move in the direction perpendicular to the flat surface of the blade. The configuration of the sleigh in relation to plane π defines the set of Lagrange coordinates (q^1, q^2, q^3), where $q^1 = x_C$, $q^3 = y_C$, are Cartesian coordinates of point M, while $q^2 = \varphi$ represents the angle made by the axis Ox and the straight line which is the intersection of the blade's flat surface with plane π. The mass of the sleigh is m and radius of gyration ρ for the central axis of inertia perpendicular to the plane of motion π. Also, acting in the centre C of the mass is the viscous friction which is the linear function of the mass centre's absolute velocity, with the constant coefficient of proportionality b. Points C and M are on a joint vertical straight line. Define the brachistochronic motion of the sleigh. Determine the control force $\vec{F} = \vec{F}(t)$ to be acted with at point A ($\overline{CA} = 2m$) of the sleigh so as to realize the mentioned brachistochronic motion. Show that the control force can be realized by the imposition on the sleigh a new bilateral ideal holonomic constraint identical with the trajectory of the point A, obtained from finite equations of brachistochronic motion (on an iceberg a keyway is made, whose axis is the trajectory of the point A and a key, tied to the sleigh, is sliding along the keyway; or: along a smooth wire bent along the trajectory of the point A a small-sized ring is sliding, tied to the point A of the sleigh).

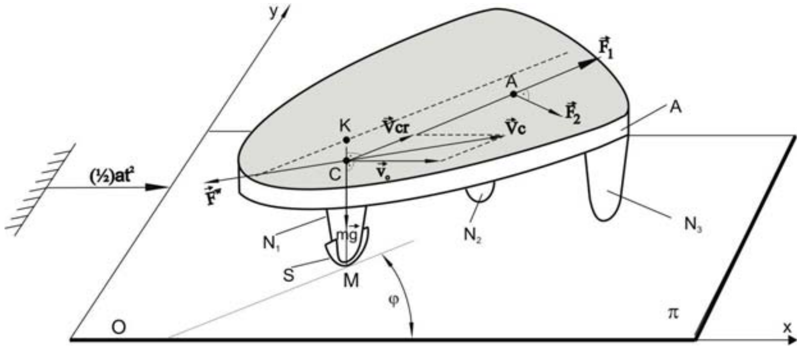

Figure 1. The sleigh of the iceberg.

The kinetic energy, potential energy and generalized dissipative forces read

$$T = \frac{m}{2}((\dot{q}^1)^2 + \rho^2(\dot{q}^2)^2 + (\dot{q}^3)^2) + mat\dot{q}^1 + \frac{ma^2t^2}{2}, \quad (42)$$
$$\Pi = 0,\ Q_1^w = -b(\dot{q}^1 + at),\ Q_2^w = 0,\quad Q_3^w = -b\dot{q}^3,$$

where the nonholonomic constraint equation (6) is

$$\dot{q}^3 - (\tan q^2)\dot{q}^1 = 0. \quad (43)$$

In this example there are not limitations of constraint reactions at the contact point M of the blade on the plane, nor are there constraints of active control forces $F_1 = u_1$ and $F_2 = u_2$, i.e. there are no limitations (29), so that the reaction of nonholonomic constraint can be eliminated from further procedure by using one of the conditions (26), where the power of active control forces equals zero, and the relation (40) has the form

$$(u_1 \cos q^2 + u_2 \sin q^2)\dot{q}^1 - \overline{CA}u_2\dot{q}^2 + (u_1 \sin q^2 - u_2 \cos q^2)\dot{q}^3 = 0\,. \quad (44)$$

End conditions (28) are

$$t_0 = 0, q^1(t_0) = 0, q^2(t_0) = \theta_0, q^3(t_0) = 0, q^1(t_1) = 3L, q^2(t_1) = \theta_1, q^3(t_1) = L,$$
$$\frac{m}{2}((\dot{q}^1(t_0))^2 + \rho^2(\dot{q}^2(t_0))^2 + (\dot{q}^3(t_0))^2) + ma\,{}_0\dot{q}^1(t_0) + \frac{ma^2t_0{}^2}{2} = E_0, \quad (45)$$
$$\dot{q}^3(t_0) - \tan q^2(t_0))\dot{q}^1(t_0) = 0\ .$$

The two-point boundary value problem of the system of 14 differential equations of the first kind with an unknown terminal time moment was solved by the shooting method for the following numerical values of the parameters

$$\frac{2E_0}{m} = 175\ \frac{m^2}{s^2},\ a = 1\frac{m}{s^2},\ \rho = 1m,\ L = 3m,\ \frac{b}{m} = 1\ \ s^{-1},\ \theta_0 = 0,\ \theta_1 = \frac{\pi}{4}. \quad (46)$$

Fig.2 shows diagrams of active control forces and Fig.3 trajectories of points A and C.

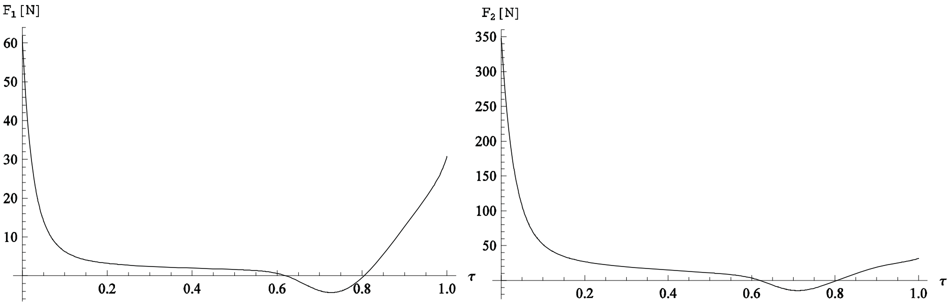

Figure 2. Diagrams $F_1 = F_1(\tau)$ and $F_2 = F_2(\tau)$, $\tau = t / t_1$

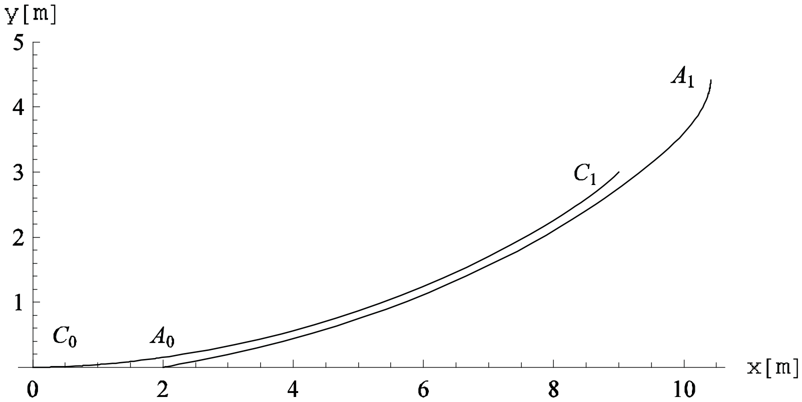

Figure 3. Trajectories of points *A* and *C*

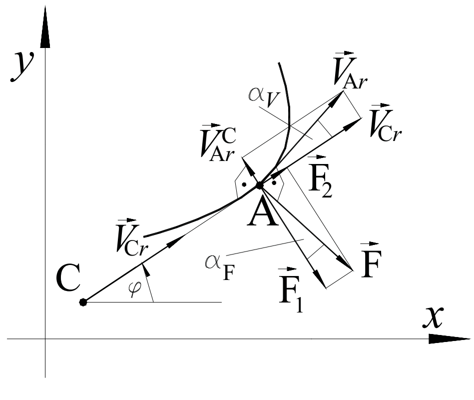

Figure 4. Control forces at point *A*

Fig.4 shows control forces at point *A*, where

$$\tan\alpha_V = \frac{V_{Ar}^C}{V_{Cr}} = \frac{\overline{CA}\,\dot{q}^2}{\sqrt{(\dot{q}^1)^2 + (\dot{q}^3)^2}}, \tag{47}$$

and in [15] a proof has been derived that $\alpha_V = \alpha_F$ and $\vec{F}\cdot\vec{V}_{Ar} = 0$.

The second way to realize the control of motion is a subsequent imposition to the system an additional ideal holonomic constraint (it is obvious here that Bernoulli's idea on the

realization of control forces by means of an ideal constraint has been utilized - the reaction of that constraint will substitute the action of the mentioned active control force $\vec{F}$).
Also, it has been shown that if an additional constraint is written in the form

$$\phi^1(\overline{q}) = y_A(x_A(q^1,q^2)) - q^3 - \overline{CA}\sin q^2 \ , \ \ x_A(q^1,q^2) = q^1 + \overline{CA}\cos q^2 \ , \tag{48}$$

its multiplier $\rho_1 = F\cos(q^2 + \alpha_F)$ actually represents negative projection of the reaction of new constraint on axis O_y .

6. Example 2: Brachistochronic motion of a variable-mass mechanical system [16]

The rod AB has a mass m, length $2L$ and radii of gyration $i_C = L$ about an axis perpendicular to the rod and passing through the mass center C of the rod ($\overline{AC} = \overline{BC} = L$). The rod moves in a uniform gravitational field in a vertical plane coinciding with the fixed plane Oxy (see Fig.1) where y represents the vertical axis directed upwards and x is the horizontal axis of a Cartesian coordinate system. The local Cartesian coordinate system $C\xi\eta$ has its origin at point C and it is attached to the rod in the manner shown in Fig.1. In the moment $t_0 = 0$, the direction of the rod coincides with axis x (position A_0B_0 in Fig.1).

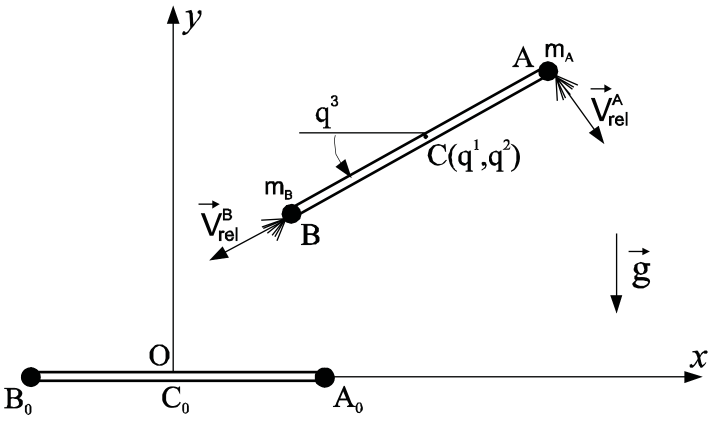

Figure 5. Variable-mass system

In Fig.1, the quantities q^1 and q^2 are Cartesian coordinates of the mass center C , i.e., $x_C = q^1$ and $y_C = q^2$, and q^3 is the rotational angle of the rod. At both ends of the rod there are two variable-mass particles A and B , whose masses change according to the following law

$$m_A(t) = m_B(t) = m - kt \tag{49}$$

where k is a specified positive constant. The expelled masses have the relative velocities $\vec{v}_{rel}^A$ and $\vec{v}_{rel}^B$ of the constant magnitude

$$v_{rel}^A = v_{rel}^B = v \tag{50}$$

where v is a specified positive constant. During motion, the velocity $\vec{v}_{rel}^{B}$ has the direction of the rod, while the velocity $\vec{v}_{rel}^{A}$ lies in the plane Oxy and has the direction perpendicular to the rod.
It is needed to determine the brachistochronic motion of the system described as well as its realization without the action of active control forces. The initial and terminal conditions are specified as:

$$
\begin{aligned}
&t_0 = 0\,,\;\; q^1(t_0) = q^2(t_0) = q^3(t_0) = 0\,,\\
&\frac{3m}{2}\left[\left(\dot{q}^1(t_0)\right)^2 + \left(\dot{q}^2(t_0)\right)^2 + L^2\left(\dot{q}^3(t_0)\right)^2\right] = E_0,\\
&q^1(t_f) = q^2(t_f) = L,\, q^3(t_f) = \frac{\pi}{2}.
\end{aligned}
\tag{51}
$$

The differential equations of motion in the expanded form read

$$
\begin{aligned}
&(3m-2kt)\ddot{q}^1 = kv\left(\cos q^3 - \sin q^3\right) + Q_1^c\,,\\
&(3m-2kt)\ddot{q}^2 = kv\left(\cos q^3 + \sin q^3\right) - (3m-2kt)g + Q_2^c\,,\\
&(3m-2kt)L^2\ddot{q}^3 = kvL + Q_3^c\,,
\end{aligned}
\tag{52}
$$

where $\vec{g} = -g\vec{j}$, g is the acceleration of gravity.

The two-point boundary value problem is solved for the following values of the parameters:

$$
L = 1m,\, m = 1kg,\, E_0 = 30\frac{kg\,m^2}{s^2},\, v = 1\frac{m}{s},\;\; k = 1\frac{kg}{s},\, g = 9.81\frac{m}{s^2}, \tag{53}
$$

In the example considered, the realization of generalized control forces Q_i^c has been achieved via reactions of constraints using fixed and moving centrodes of the rod . The moving centrode represents a curve attached to the rod which traces the instantaneous center of zero velocity of the rod. The fixed centrode represents a curve fixed in the plane Oxy which traces the instantaneous center of zero velocity of the rod. The motion of the rod AB is equivalent to the rolling without slipping of the moving centrode on the fixed one. The parametric equations of the fixed centrode are :

$$
x = q^1 - \frac{\dot{q}^2}{\dot{q}^3}, \qquad y = q^2 + \frac{\dot{q}^1}{\dot{q}^3}, \tag{54}
$$

and of the moving one are:

$$
\xi = \frac{1}{\dot{q}^3}\left(\dot{q}^1 \sin q^3 - \dot{q}^2 \cos q^3\right), \quad \eta = \frac{1}{\dot{q}^3}\left(\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3\right). \tag{55}
$$

The fixed centrode and the moving centrode for the positions of the rod corresponding to the time instances $t_0 = 0$ and t_f as well as the trajectories of points A , B , and C are shown in Fig.6.

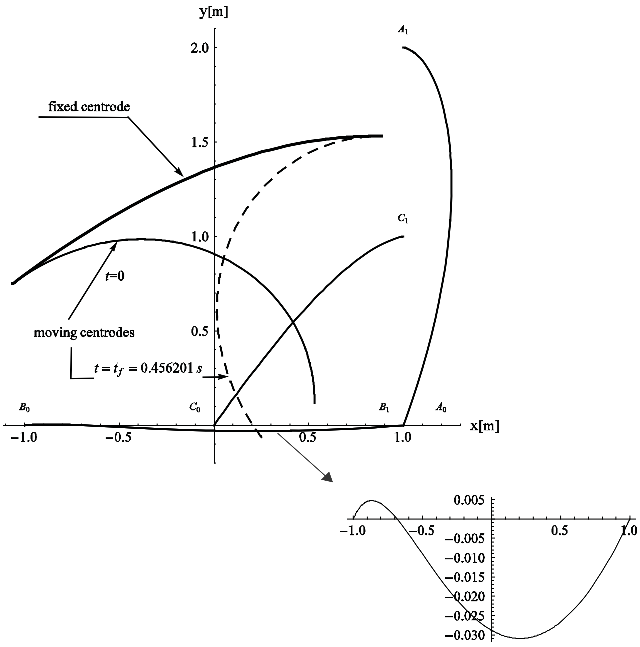

Figure 6. Brachistochronic motion of the variable-mass system

7. Example 3: Brachistochronic motion of the Chaplygin sleigh with restricted reactions of nonholonomic constraints [17]

In the example considered, the realization of time motion minimization for Chaplygin sleigh is achieved via the subsequent imposition of ideal constraint at the mass center *C*, Fig.7, where the blade is positioned at point A and has the direction of the straight line *AC*.

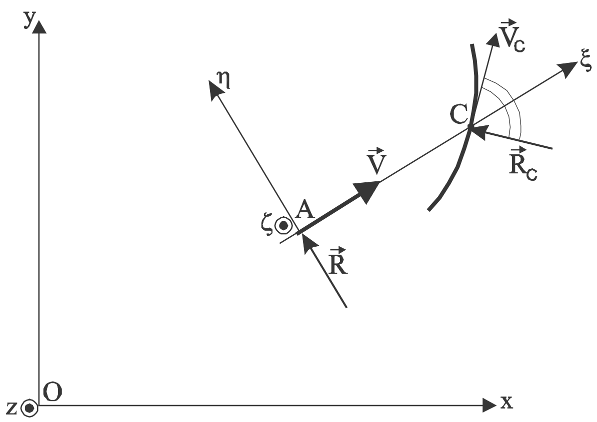

Figure 7. The Chaplygin sleigh

Differential equations of the simplified optimal control problem read

$$\dot{x} = V\cos\varphi,\ \dot{y} = V\sin\varphi,\ \dot{\varphi} = \omega,\ \dot{\omega} = u,\ \dot{V} = -\frac{\omega}{V}a^2k^2u,$$
$$k^2 = 1 + \frac{J_c}{ma^2},\ \ a = AC, \tag{56}$$

where the control variable u represents the angular acceleration of the sleigh, m is the mass of the sleigh and J_c is the central principal moment of inertia. Note that the last differential equation was obtained by differentiating the kinetic energy with respect to time t. Angular acceleration and the reaction of nonholonomic constraint are linked by the relation $J_c u = -aR$, and here the reaction of nonholonomic constraint is limited $|R| \le N_b$, so that the control is restricted by relations

$$-\frac{aN_b}{J_c} \le u \le \frac{aN_b}{J_c}. \tag{57}$$

Let the values of state variables x, y, φ and kinetic energy of the sleigh be specified at the beginning of motion

$$t_0 = 0\ \ x(t_0) = 0\ \ y(t_0) = 0\ \ \varphi(t_0) = 0\ \ V^2(t_0) + a^2k^2\omega^2(t_0) = \frac{2T_0}{m}, \tag{58}$$

as well as the values of state variables x, y, φ at the terminal position of the sleigh

$$t = t_1,\ x(t_1) = a,\ y(t_1) = a,\ \varphi(t_1) = \varphi_1 \tag{59}$$

where t_1 is free.

The brachistochronic motion problem of the Chaplygin sleigh consists in determining of the control u and the state variables x, y, φ, ω, v corresponding to them, so that the sleigh starting from initial position with given initial kinetic energy T_0 reaches the terminal position for the minimum time t_1.

The problem was solved for the following values of the parameters $m = 2kg$, $a = 1m$, $T_0 = 200kgm^2/s^3$, $k = 2$, $\varphi_f = \pi/2$ and various values of N_b. Fig.8 shows the dependency of optimal control on angle φ for the case when the restriction of constraint reaction is $N_b = 30\,kg\,m/s^2$. For $N_b < (N_b)_{cr} = 17.35104kgm/s^2$, the control is of the bang-bang type.

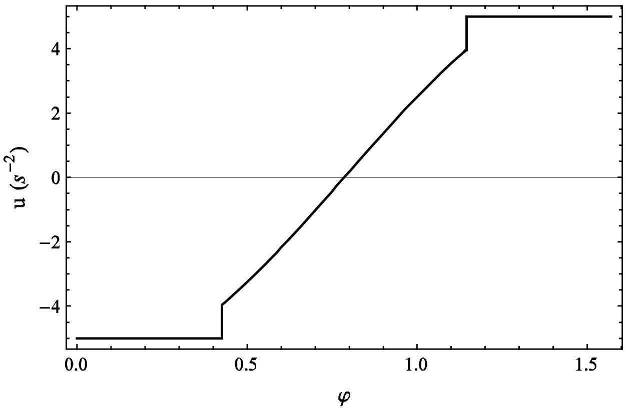

Figure 8. Optimal control $u = u(\varphi)$ for $N_b = 30\,kg\,m\,/\,s^2$

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