7th International Congress of Serbian Society of Mechanics
Sremski Karlovci, Serbia, June 24-26, 2019

PRESSURE DISTRIBUTION IN MICROTUBES WITH VARIABLE CROSS SECTION

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Abstract:

A steady compressible isothermal slip gas flow in a microtube is analyzed for low Reynolds numbers. The microtube is with varying cross-section, which from the standpoint of geometry gives three considered cases: convergent microtube, divergent microtube and microtube with constant radius. The gas flow is caused by the pressure difference between the inlet and the outlet cross-section. The solution for pressure and mass flow rate is obtained by macroscopic approach from Navier-Stokes equations with the second order velocity slip boundary condition. Analysis of the order of dimensionless terms in governing equations is possible with the usage of exact relation between Reynolds, Mach and Knudsen numbers. The solution procedure for this flow model is based on perturbation approach, where significant variables are assumed in the form of perturbation series. The first approximation represents the solution for the continuum flow conditions, and the second approximation incorporates the effect of gas rarefaction. The obtained solution is compared with verified results of numerical experiment found in literature and good agreement is achieved. Hence, the reliability of obtained solution and presented method is confirmed.

Key words: slip gas flow, microtube, rarefied gas flow, isothermal, compressible

1. Introduction

Microtubes belong to the class of mechanical parts of microdimension, which are today most often used as an integral part of the micro-electromechanical systems (MEMS). MEMS are consisted of electrical and mechanical components, with characteristic dimensions between 1μm and 1 mm.

Through history, man has been producing tools for everyday life. In the beginning dimensions were the order of the human size and later bigger and bigger. In 1947 the first transistor was built. Along with Richard Feynman's talk "There's Plenty of Room at the Bottom", at a meeting of the American Physical Society in 1959 [1], the growth and development of micro technology was popularized.

Today MEMS devices find increasing application in various fields of industry and medicine. MEMS are often used in gas environments, therefore, the study of gas flow in elements of MEMS is very up-to-date.

In the gas flow in microtubes ratio of the molecular mean free path λ and characteristic dimension is not negligible. As a result, the effect of rarefied gas occurs. The measure of gas dilution is the Knudsen number (*Kn*). According to the values of the Knudsen number, we differ: continuum flow ($Kn \leq 0.001$), slip flow ($0.001 \leq Kn \leq 0.1$), transition regime ($0.1 \leq Kn \leq 10$) and free molecular flow ($Kn > 10$) [1]. The case of slip gas flow is considered in this paper.

The gas flow behavior in long microtubes were investigated in [2]-[5]. Moreover, the similar approach for microtube with constant radius is developed in [6], where velocity contours are also considered.

In this paper the gas flow in microtubes with variable cross section is analyzed. In section 2, the governing equations and necessary assumptions for obtaining solution of characteristic variables is presented. An analysis of the obtained results for the pressure field and the mass flow field is presented in section 3. The conclusion with the future research plan is given in section 4.

2. Problem description

The problem of gas flow through a long microtube of a variable cross-section is considered. The microtube radius varies depending on the longitudinal coordinate $\tilde{z}$. The three cases of the tube geometry are considered: convergent, divergent and microtube with constant cross-section.

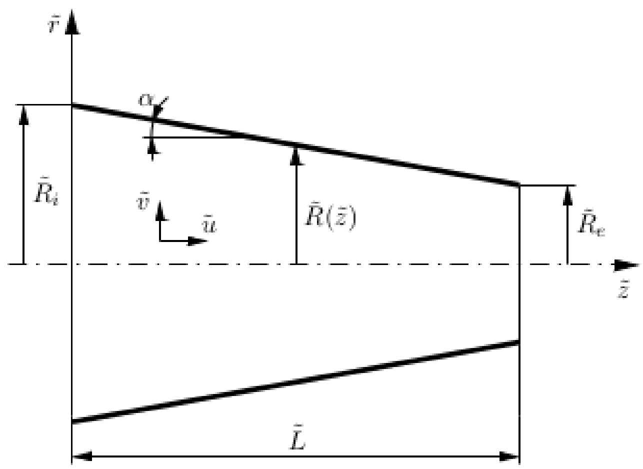

Fig. 1. Geometry of a microtube.

For the solving systems of equations, it is necessary to define radius in every point of the microtube wall. In order to consider the different cases of microtube geometry, it is necessary to define a unique law of radius change. Radius of the tube $\tilde{R}$ changes linearly with longitudinal coordinate $\tilde{z}$:

$$\tilde{R}(z) = \tilde{R}_i - \tilde{z}(\tilde{R}_i - 1), \qquad (1)$$

where all dimensional variables are marked with ~, while all non-dimensional variables in further text will be without ~.

The gas flow in the microtube occurs due to the pressure difference at the inlet and the outlet cross-section. The considered flow is also compressible, isothermal, stationary and axisymmetric. The microtube geometry indicates the use of a cylindrical coordinate system (Fig.1.). The gas flows at low values of Mach number and can be considered as highly subsonic.

2.1 Governing equations

The governing equations for stationary, isothermal, axisymmetric and compressible flow in a long microtube are consisted of continuity equation, momentum equation and equation of state for an ideal gas. The governing equations expressed in cylindrical coordinates, i.e. continuity equation (2), streamwise momentum equation (3), radial momentum equation (4) and equation of state for an ideal gas (5) are:

$$\frac{1}{\tilde{r}}\frac{\partial(\tilde{\rho}\tilde{r}\tilde{v})}{\partial\tilde{r}}+\frac{\partial(\tilde{\rho}\tilde{u})}{\partial\tilde{z}}=0\,, \tag{2}$$

$$\tilde{\rho}\left(\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{z}}\right)=\tilde{\mu}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}+\frac{\partial^2\tilde{v}}{\partial\tilde{r}\partial\tilde{z}}\right)-\frac{\partial\tilde{p}}{\partial\tilde{z}}-\frac{2}{3}\tilde{\mu}\left(\frac{\partial}{\partial\tilde{z}}\frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\tilde{v})}{\partial\tilde{r}}\right)+\frac{4}{3}\tilde{\mu}\frac{\partial^2\tilde{u}}{\partial\tilde{z}^2}+\frac{\tilde{\mu}}{\tilde{r}}\left(\frac{\partial\tilde{u}}{\partial\tilde{r}}+\frac{\partial\tilde{v}}{\partial\tilde{z}}\right) \tag{3}$$

$$\tilde{\rho}\left(\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{r}}+\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{z}}\right)=-\frac{\partial\tilde{p}}{\partial\tilde{r}}+2\tilde{\mu}\frac{\partial^2\tilde{v}}{\partial\tilde{r}^2}-\frac{2}{3}\tilde{\mu}\left(\frac{\partial}{\partial\tilde{r}}\left(\frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\tilde{v})}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{z}}\right)+\tilde{\mu}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{z}}+\frac{\partial^2\tilde{v}}{\partial\tilde{z}^2}\right)$$
$$+2\frac{\tilde{\mu}}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{r}}-\tilde{\mu}\frac{\tilde{v}}{\tilde{r}^2}\,, \tag{4}$$

$$\tilde{p}=\tilde{\rho}R_g\tilde{T} \tag{5}$$

Here $\tilde{\mu}$ is dynamic viscosity, $\tilde{p}$ is the pressure, $\tilde{\rho}$ is the density, $\tilde{T}$ the temperature, and gas constant is R_g. Longitudinal component of velocity is $\tilde{u}$, while radial velocity component is $\tilde{v}$.

In order to analyze and solve these equations they are transformed into dimensionless form and several assumptions are introduced.

2.2 Dimensionless variables and assumptions

Dimensionless equations can be obtained by introducing the following dimensionless variables: longitudinal velocity *u*, radial velocity *v*, pressure *p*, temperature *T*, radial coordinate *r*, longitudinal coordinate *z* and dynamic viscosity *μ*,

$$u=\frac{\tilde{u}}{\tilde{u}_e}\,,\ v=\frac{\tilde{v}}{\tilde{u}_e}\,,\ p=\frac{\tilde{p}}{\tilde{p}_e}\,,\ T=\frac{\tilde{T}}{\tilde{T}_e}\,,\ r=\frac{\tilde{r}}{\tilde{R}_e}\,,\ z=\frac{\tilde{z}}{\tilde{L}}\,,\ \mu=\frac{\tilde{\mu}}{\tilde{\mu}_e}\,. \tag{6}$$

Equation (6) introduces the consideration of the variables according to the reference cross-section. In order to satisfy the condition of subsonic flow, it is necessary that characteristic non-dimensional numbers have values in a certain range. As the pressure in the microtube changes from the inlet to the outlet cross section, it follows that Knudsen and Mach numbers change from the inlet to the outlet. The necessary condition is that these non-dimensional numbers are not allowed to exceed critical values - the values which determine the flow regime. As the maximum Mach number occurs at the outlet cross section, Mach number will not exceed critical value if its value is fixed at the outlet cross section. Because of this condition, the outlet cross section is chosen for the reference cross section. All variables in the reference cross section are indexed with "*e*".

As a next step, equations obtained with dimensionless quantities must be transformed into a simpler form, which can provide a solution for the field of pressure and mass flow. The assumptions are necessary to transform equations into a simpler form. Assumptions are based on the characteristics of a flow. In our problem the considered flow is carried out at the small values of the Mach and Knudsen numbers. Based on this condition, it is assumed that values of the Mach and Knudsen number at the outlet cross section, Ma_e and Kn_e, are equal to the product of the small parameter ε and the corresponding constants γ and η:

$$\gamma\varepsilon^m=\kappa Ma_e^2\,, \tag{7}$$

$$\eta\varepsilon^n=Kn_e\,. \tag{8}$$

The small parameter ε, with the condition ε <<1, is defined as the ratio of the diameter at the outlet cross section and the length of the microtube:

$$\varepsilon = \frac{2\tilde{R}_e}{\tilde{L}} . \tag{9}$$

From the condition that the values Ma_e and Kn_e are small, it can be concluded that the exponents of the small parameter ε in equation (3) and (4) must be positive, that is: $m > 0$, $n > 0$.

As the flow occurs at low values of Reynolds numbers, it is assumed that the ratio of dimensionless numbers Ma_e and Re_e, at the microtube exit is of the ε order of magnitude:

$$\beta\varepsilon = \frac{\kappa Ma_e^2}{Re_e} . \tag{10}$$

Here Mach and Reynolds numbers at the outlet cross section are defined by:

$$Ma_e = \frac{\tilde{u}_e}{\sqrt{\kappa \frac{\tilde{p}_e}{\tilde{\rho}_e}}} , \tag{11}$$

$$\mathrm{Re}_e = \frac{\tilde{\rho}_e \tilde{u}_e 2\tilde{R}_e}{\tilde{\mu}_e} . \tag{12}$$

The Knudsen number at the microtube outlet cross section is:

$$Kn_e = \frac{\lambda_e}{2R_e} , \tag{13}$$

and the dependency between Knudsen, Mach and Reynolds number can be obtained as:

$$Kn_e = \sqrt{\frac{\pi\kappa}{2}} \frac{Ma_e}{Re_e} , \tag{14}$$

as well as the dependence between the parameters γ, β and η:

$$\gamma = \frac{\beta^2 \pi}{2\eta^2} . \tag{15}$$

The microtube geometry varies slowly with the growth of the longitudinal coordinate, therefore the slope angle of the microtube wall is small. This leads to the conclusion that the radial component of the velocity (v) is much smaller than the longitudinal velocity component (u). Accordingly, it is assumed that the radial velocity component can be expressed as:

$$\tilde{v} = \varepsilon \tilde{V} . \tag{16}$$

2.3 Dimensionless governing equations

According to previously defined dimensionless numbers and assumptions, the dimensionless form of the system of equations is obtained:

$$\frac{2}{r} \frac{\partial (rpV)}{\partial r} + \frac{\partial (\rho u)}{\partial z} = 0 \tag{17}$$

$$z: \frac{\partial p}{\partial z} = 4\beta \frac{\partial^2 u}{\partial r^2} + \frac{4\beta}{r} \frac{\partial u}{\partial r} \tag{18}$$

$$r: \frac{\partial p}{\partial r} = 0 \tag{19}$$

$$p = \rho \tag{20}$$

Velocity slip boundary condition in dimensionless form, necessary for the definition of the velocity at the wall, is the second order boundary condition:

$$u\,|_{r=R} = \frac{2-\sigma_v}{\sigma_v}\left(-2Kn_e\,\frac{1}{p}\frac{\partial u}{\partial r} - 2Kn_e^2\,\frac{1}{p^2}\frac{\partial u^2}{\partial r^2}\right)|_{r=R}\,. \tag{21}$$

As a next step, it is necessary to express characteristic variables in the form of perturbation series:

$$p = p_0 + Kn_e p_{1/4}\,, \tag{22}$$

$$u = u_0 + Kn_e u_{1/4}\,, \tag{23}$$

$$V = V_0 + Kn_e V_{1/4}\,, \tag{24}$$

where the values indexed with “0” mark the first approximation values, and the values with index “1/4” are the second approximations. It is possible from the dimensionless system of equations, on the basis of perturbation method, to obtain system of equation for every approximation. The equation system is consisted of governing equations together with boundary and axisymmetric condition for every approximation:

- the first approximation

$$\frac{\partial p_0}{\partial z} = 4\beta\frac{\partial^2 u_0}{\partial r^2} + \frac{4\beta}{r}\frac{\partial u_0}{\partial r} \tag{25}$$

$$\frac{\partial p_0}{\partial r} = 0 \tag{26}$$

$$\int_0^{R(z)} 2 p_0 u_0 r \mathrm{d}r = 1 \tag{27}$$

$$r = R(z): u_0 = 0 \tag{28}$$

$$r = 0: \frac{\partial u_0}{\partial r} = 0 \tag{29}$$

- the second approximation

$$\frac{\partial p_{1/4}}{\partial z} = 4\beta\frac{\partial^2 u_{1/4}}{\partial r^2} + \frac{4\beta}{r}\frac{\partial u_{1/4}}{\partial r} \tag{30}$$

$$\frac{\partial p_{1/4}}{\partial r} = 0 \tag{31}$$

$$\int_0^{R(z)} 2(p_0 u_{1/4} + p_{1/4} u_0) r \mathrm{d}r = 0 \tag{32}$$

$$r = R(z): u_{1/4} = -\frac{2-\sigma_v}{\sigma_v}\frac{2}{p_0}\frac{\partial u_0}{\partial r} \tag{33}$$

$$r = 0: \frac{\partial u_{1/4}}{\partial r} = 0 \tag{34}$$

It is possible to solve the system of equations and get the pressure field and mass flow field. Therefore, the differential equations for the pressure field are:

$$p_0 p_0' = -\frac{32\beta}{R^4} \tag{35}$$

$$\left(p_0 p_{1/4}\right)' = -\frac{2-\sigma_v}{\sigma_v}\frac{8}{R}p_0' \tag{36}$$

On the basis of the pressure field, defined with equations (35)-(36), it is possible to obtain the mass flow field.

3. Results

The results for the gas flow at small Reynolds numbers through a divergent, convergent and microtube with constant radius are presented in this section. Since solution the mass flow rate of microtube with constant cross-section exist in the literature [7,8], comparison of this solution with the results obtained in this paper is also showed.

The first approximation for all variables represents the solution for the continuum flow conditions, in other words the case without the slip effect. The second approximation for all variables includes the effect of gas rarefaction. Further, as the outlet cross section is the same for all examples of geometry, the solutions at the outlet cross section are also the same for all of the three geometry cases.

3.1 Pressure distribution

According to the pressure distribution (Fig. 2.) it is noticeable that the pressure decreases with the increase of the longitudinal coordinate *z*. The first approximation of pressure (Kn_e=0) shows the same behavior.

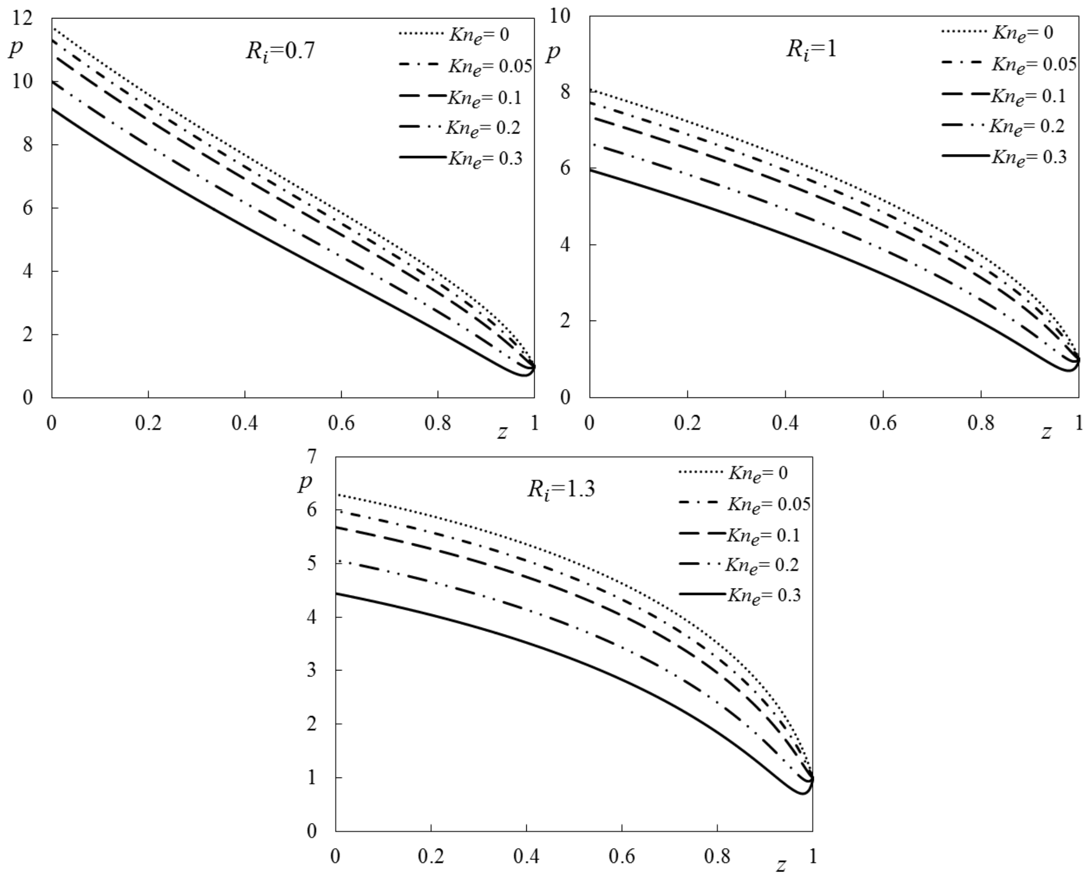

Fig. 2. The pressure distribution in a microtube with different Knudsen numbers and different geometries: divergent microtube (R_i=0.7); microtube with constant radius (R_i=1) and convergent microtube (R_i=1.3).

As the first approximation has the greatest impact on the pressure value obtained as summation of the two approximations, the behavior of overall pressure is similar to the behavior of the first approximation.

The results presented for different Knudsen number values show that its influence, or an increase in Knudsen number, reduces the pressure value independently of the microtube geometry (Fig. 2.). The comparison of the pressure field results for the same Knudsen number, and for different microtube geometries, gives that the dimensionless pressure at the inlet cross-section is highest in the divergent microtube, and the smallest in the convergent one.

In order to obtain pressure values for all geometries, the same conditions at the outlet cross-section are given. For obtaining the same value of dimensionless pressure at the outlet cross-section, the greatest dimensionless inlet pressure is required in the divergent tube, then in the tube with constant cross section, and the smallest ratio of the dimensionless inlet and outlet pressure is required in the convergent tube.

3.2 Mass flow distribution

With the known pressure field it is possible to investigate the behavior of the mass flow field for different geometries. Results are presented as dependency between $\dot{m}/\dot{m}_0 - 1$, where $\dot{m}$ is dimensionless mass flow rate, $\dot{m}_0$ is dimensionless mass flow rate for the continuum, and the ratio of the inlet and the outlet pressure values. It is possible to show that the ratio $\dot{m}/\dot{m}_0$ can be presented as β/β_0 .

From Fig. 3 it is obvious that the gas rarefaction increase leads to the smaller increasement of the mass flow than in the case when slip is neglected.

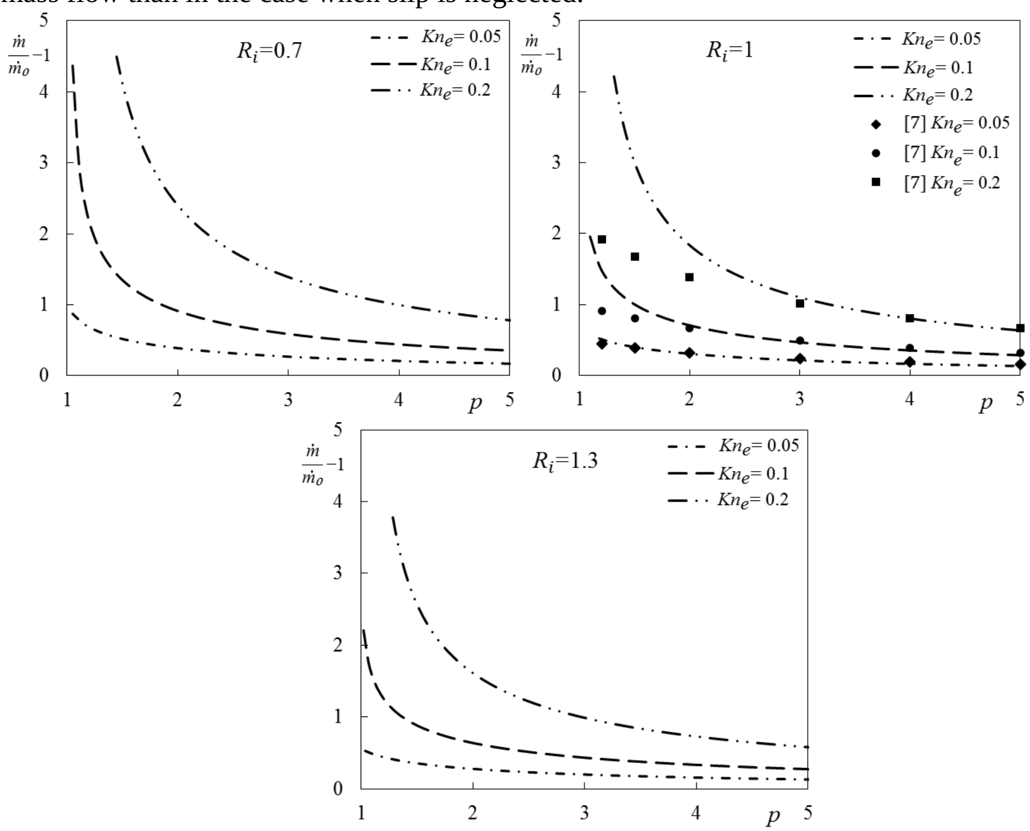

Fig. 3. The mass flow distribution in a microtube, at different cross-sections and with different Knudsen numbers and different geometries: divergent microtube (R_i=0.7); microtube with constant radius (R_i=1) and convergent microtube (R_i=1.3).

The solution for the mass flow rate of a rarefied gas through microtube with constant radius (R_i=1) exist in the literature [7,8], where results are reached numerically from the linearized Boltzmann equation. The comparison of results from [7,8] with the results obtained in this paper is also presented and good agreement is achieved (Fig.3). It is concluded that for the smaller pressure ratio and for the higher values of Knudsen number it is necessary to go with more approximations in perturbation series.

3. Conclusions

In this paper the case of compressible isothermal slip gas flow at low Reynolds numbers is analyzed. Gas flow occurred at low Reynolds numbers due to the pressure difference at the inlet and outlet cross section and thorough microtube with variable cross section. Three cases of geometries are analyzed: convergent microtube, divergent microtube and microtube with constant radius. According to all conditions it is possible to define the small parameter ε and appropriate relations between ε and Mach, Reynolds and Knudsen numbers. Each term contribution in governing equations is estimated in this way. The obtained solutions for the pressure and mass flow field are assumed by perturbation series, where two approximations are achieved. The validity of used method is confirmed comparing with available numerical results from the literature.

The further research plan is related to the microtube gas flow at higher Reynolds numbers and to the solution with the higher number of approximations.

Acknowledgment

This work was partially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grants 35046 and 174014).

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