Analysis of the Motion and Stability of the Holonomic Mechanical System in the Arbitrary Force Field

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In order to give an insight into the work of the machine before the production and assembly and to obtain good analysis, this paper presents detailed solutions to the specific problem occured in the field of analytical mechanics. In addition to numerical procedures in the paper, a review of the theoretical foundations was made.Various types of analysis are very common in mechanical engineering, due to the possibility of an approximation of complex machines. For the proposed system, Lagrange's equations of the first kind, covariant and contravariant equations, Hamiltons equations and the generalized coordinates, as well as insight in Coulumb friction force are provided.Also, the conditions of static equilibrium are solved numerically and using intersection of the two curves. Finally, stability of motion for the disturbed and undisturbed system was investigated.

Keywords: *Applied mechanics, Lagrange's equation, Coulomb friction Hamiltonian function, Stability of motion*

1. INTRODUCTION

Extent analysis of the mechanical systemhas been one of the most fundamental and challenging tasks, that has been largely studied for decades. Analytical mechanic proved particularly significant and useful to engineers, although it took another century after Lagrange for this to be fully realized [1]. Many studies have been done to model and examine real objects and their behaviour.A detailed review of literature related to the problems of analytical mechanic can be found in [2] and [3]. The problems considered in the present paper involve a review of references on the specific types of systems - holonomic systems. The initial motions of holonomic and nonholonomic system are investigated in [4]. Our paper suggests a different approach for modelling a specific multi-body system, including the special investigation of modelling Coulomb friction force, the problem that so far has hardly been considered. In addition to the ordinary Lagrange method, used in a traditional way, Lagrange's equations of second kind in the covariant and contravariant form are introduced. In [5-7], Lagrange's equations of second kind of rigid bodies system in a covariant form were developed. Moreover, stability of the specific mechanical system was discussed using different approaches. Namely, unlike the papers [8] and [9], where the relative advantages and disadvantages of various analytical methods of nonholonomic systems are briefly presented, the problem of the instability of the equilibrium state of a scleronomic mechanical system with linear homogeneous constraints are considered in [10], and the problem of the stability of the equilibrium state in the case with holonomic mechanical systems in [11].

From a dynamical point of view any material system can be regarded as a collection of particles [12]. The mechanical system shown in Figure 1 consists of slider-crank mechanisms M_1 and M_2, as well as the point M_3; they are tied with light rigid rods articulated to each other. Fixed plane O_{xy} coincides with the vertical plane of motion of the mechanical system, where the axis O_y is directed vertically down. The M_1 and M_2 sliders move along O_y and O_x axes, respectively. Slider-crank M_2 is connected by a damper, while the other end of the damper is attached to a fixed wall. All necessary numerical data are given in the Appendix.

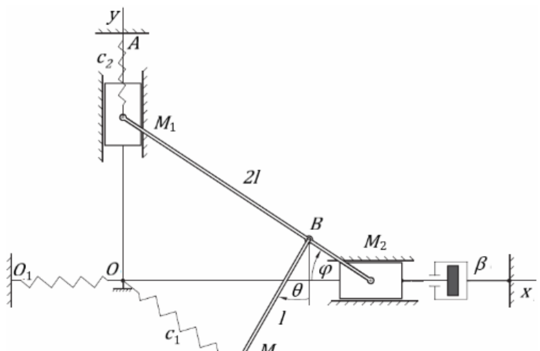

Figure 1. Mechanical system

2. CONSTRAINTS AND LAGRANGE EQUATIONS OF THE FIRST KIND

The state of a mechanical system of N points, M_v (v = 1,2,...,N), is determined in each moment t by the position and velocities of all its points in the inertial reference system (IRS). If a fixed Cartesian system is introduced into an IRS, the state of the system is determined by variable scalar quantities: coordinates x_v, y_v, z_v and velocity projections $\dot{x}_v, \dot{y}_v, \dot{z}$, which must satisfy the relations:

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$$f_u\left(x_1, y_1, z_1, ..., x_N, y_N, z_N, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., \dot{x}_N, \dot{y}_N, \dot{z}_N; t\right) \quad (1)$$
$$\mu = 1, 2, ..., m < 3N$$

The motion of the considered system is limited by the following stationary holonomic constraints (1)-(5):

$$f^1 = x_1 = 0 \quad (2)$$

$$f^2 = y_2 = 0 \quad (3)$$

$$f^3 = x_2^2 + y_1^2 - (2l)^2 = 0, \quad (4)$$

$$f^4 = (x_3 - \frac{3}{4}x_2)^2 + (y_3 - \frac{1}{4}y_1)^2 - l^2 = 0, \quad (5)$$

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial x_2} \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial x_2} \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial x_2} \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial y_3} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial y_1} & \frac{\partial f_4}{\partial x_2} \frac{\partial f_4}{\partial y_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial y_3} \end{bmatrix} \quad (6)$$

The coordinates of point B, which must be determined when forming the (5) are: $x_B = \frac{3}{4}x_2$ and $y_B = \frac{1}{4}y_1$. Since the motion of the observed system is limited only by (2)-(5), there are fourgeometric constrains (p = 4), with zero differential equations (q = 0). The values of the Jacobian matrix (6) are given as:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial y_2} = 1$$
$$\frac{\partial f_1}{\partial y_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial y_2} = \frac{\partial f_1}{\partial x_3} = \frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial y_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial x_3} =$$
$$= \frac{\partial f_2}{\partial y_3} = \frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial y_2} = \frac{\partial f_3}{\partial x_3} = \frac{\partial f_3}{\partial y_3} = \frac{\partial f_4}{\partial x_1} = \frac{\partial f_4}{\partial y_2} = 0;$$
$$\frac{\partial f_3}{\partial y_1} = 2y_1; \frac{\partial f_3}{\partial x_2} = 2x_2; \frac{\partial f_4}{\partial y_1} = -\frac{1}{2}\left(y_3 - \frac{y_1}{4}\right);$$
$$\frac{\partial f_4}{\partial x_2} = -\frac{3}{2}\left(x_3 - -\frac{3x_2}{4}\right); \frac{\partial f_4}{\partial x_3} = 2\left(x_3 - \frac{3x_2}{4}\right);$$
$$\frac{\partial f_4}{\partial y_3} = 2\left(y_3 - -\frac{y_1}{4}\right);$$

also *rank* $\boldsymbol{J} = p + q = 4$. Due to the fact that all trajectories of the points are parallel to the vertical fixed O_{xy} plane matrix $\boldsymbol{J}$ is full rank, so all the active constraints are independent. Second derivative of the (2)-(5) gives four equations with six unknown variables:

$$\begin{gathered} \ddot{x}_1 = 0, \\ \ddot{y}_2 = 0, \\ x_2\ddot{x}_2 + \dot{x}_2^2 + y_1\ddot{y}_1 + \dot{y}_1^2 = 0, \\ \left(x_3 - \frac{3x_2}{4}\right)\left(\ddot{x}_3 - \frac{3\ddot{x}_2}{4}\right) + \left(\dot{x}_3 - \frac{3\dot{x}_2}{4}\right)^2 + \\ + \left(y_3 - \frac{y_1}{4}\right)\left(\ddot{y}_3 - \frac{\ddot{y}_1}{4}\right) + \left(\dot{y}_3 - \frac{\dot{y}_1}{4}\right)^2 = 0. \end{gathered} \quad (7)$$

The system of differential equations, which represents Lagrange's equations of the first kind and which takes into account that$q$ is equal to zero, is given as (8):

$$\begin{aligned} m_\nu \ddot{x}_\nu &= X_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu}, \\ m_\nu \ddot{y}_\nu &= Y_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial y_\nu}, \\ m_\nu \ddot{z}_\nu &= Z_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial z_\nu}, \end{aligned} \quad (8)$$

where λ_i areLagrange multipliers and ν = *1, ..., N*. Applying (7) on the considered system, using D'Alambert principle to calculate N_2, by including projection of sliding friction force using the dynamic coefficient μ_d as:

$$\begin{aligned} &(F_{trd})_x = -\mu_d \left|N_2\right| \operatorname{sign}(\dot{x}_2) = \\ &-\mu_d \left|m_1 \ddot{y}_1 + m_3\ddot{y}_3 + m_1 g - c_2(2l - y_1) + \right. \\ &\left. + m_2 g + c_1 y_3 + m_3 g + F\right| \operatorname{sign}(\dot{x}_2) \end{aligned}$$

Lagrange's equations of the first kindcan be written as:

$$\begin{gathered} m_1\ddot{x}_1 = \lambda_1, \\ m_1\ddot{y}_1 = c_2(2l - y_1) - m_1 g + 2\lambda_3 y_1 - \frac{1}{2}\lambda_4\left(y_3 - \frac{y_1}{4}\right) \\ m_2\ddot{x}_2 = -\beta\dot{x}_2 + 2\lambda_3 x_2 - \frac{3}{2}\lambda_4\left(x_3 - \frac{3x_2}{4}\right) - \\ -\mu_d\left|m_1\ddot{y}_1 + m_3\ddot{y}_3 + m_1 g - c_2(2l - y_1) + m_2 g + \right. \\ \left. + c_1 y_3 + m_3 g + F\right| \operatorname{sign}(\dot{x}_2), \\ m_2\ddot{y}_2 = \lambda_2 - m_2 g, \\ m_3\ddot{x}_3 = 2\lambda_4\left(x_3 - \frac{3x_2}{4}\right) - c_1 x_3, \\ m_3\ddot{y}_3 = 2\lambda_4\left(y_3 - \frac{y_1}{4}\right) - c_1 y_3 - F - m_3 g. \end{gathered} \quad (9)$$

Incorporating: λ_1, λ_2, λ_3, and λ_4 as four new unknowns, (7) and (9) give ten equations with ten variables. Solving these equations, the expressions for Lagrange multipliers and equations of motion, i.e. explicit expressions for: x_1, x_2, x_3, y_1, y_2, y_3, λ_1, λ_2, and λ_3, are obtained.

The entire code, for the given initial conditions defined in the Appendix, was done in Wolfram Mathematica. The system of equations describes the motion in two cases. First, where the projection of the normal reaction N_2, is not a negative value, and the second when it is.Bearing in mind that normal force,as part of the friction force, figures in the explicit expressions for the $\ddot{x}_\nu$ and $\ddot{y}_\nu$, ν = 1,2,3, whereby it is the function of $\ddot{x}_\nu$ and $\ddot{y}_\nu$, ν = 1,2,3, by itself, it is necessary to determine its sign at the initial time.By solving differential equations of motion on a small time interval,for two separate cases, for the N_2 mutually close values of the same sign have been gotten. The sign that tells which system of equations to use at the initial moment.The motion of the slider M_2 will stop in the case when the absolute value of the resultant of the active forces acting on the slider M_2 is less than the force of the Coulomb friction force of the slider and also at rest.Using an example of a simple system models, papers [13] and [14] provide calculation of the minimum value of the coefficient of friction using the Coulomb laws of friction sliding. In [15] and [16] a deeper look into the necessary dynamic conditions, for the realization of motion in accordance with the system constraints, can be found. In case when $\dot{x}_2 = 0$ the friction force of the slider is equal to the limit value of the friction force at rest, whose intensity is determined by: $\mu_0 \left| N_2^* \right|$, where N_2^* is the value of the normal reaction at the moment of stopping, and μ_0 is the static coefficient of sliding friction, $\mu_0 > \mu_d$. After the condition: $\left| X_2^* \right| > \mu_0 \left| N_2^* \right|$ has been examined, the graphs of the system points over time can be seen in Figures 2-7.

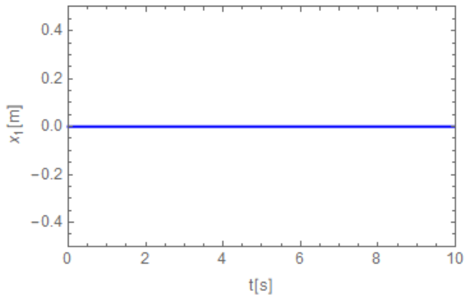

Figure 2. Graph of coordinate x_1 over time

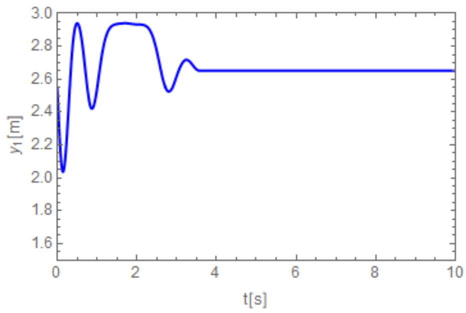

Figure 3. Graph of coordinate y_1 over time

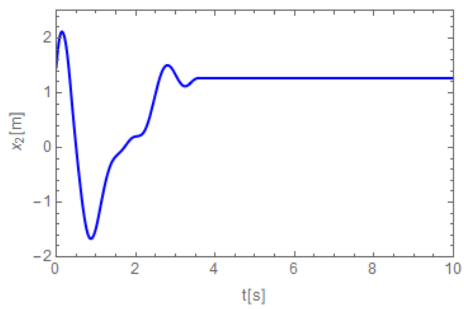

Figure 4. Graph of coordinate x_2 over time

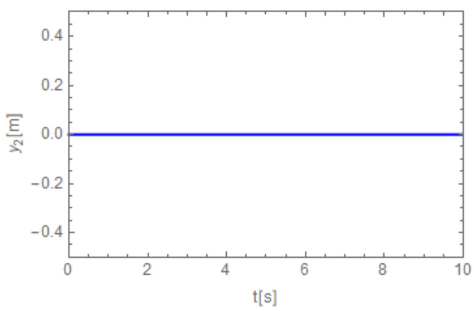

Figure 5. Graph of coordinate y_2 over time

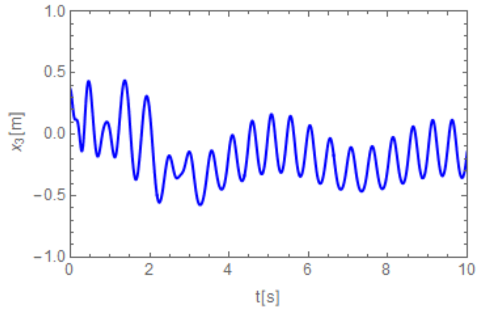

Figure 6. Graph of coordinate x_3 over time

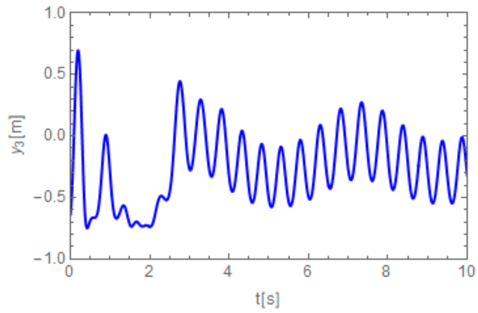

Figure 7. Graph of coordinate y_3 over time

3. VIRTUAL DISPLACEMENTS AND STATIC EQUILIBRIUM IN GENERALIZED COORDINATES

Instead of independent Cartesian coordinates, ξ, independent generalized coordinates are introduced, which also determine the position of the mechanical system. Independent generalized coordinates represent a minimum number of independent geometric parametersthat can unambiguously describe the motion of the considered mechanical system in space. Selected geometric parameters will be markedas $q(t)$. Position of the mechanical system from Figure 1 is defined by the set of Lagrange coordinates (q^1, q^2) where $q^1 = \varphi$ and $q^1 = \theta$ are the absolute angles. By introducing generalized coordinates all independent Cartesian coordinates can be expressed as:

$$\xi_{p+q+j} = \xi_{p+q+j}\left(q^1, q^2, \ldots, q^n; t\right), \quad j = 1, 2, \ldots n, \tag{10}$$

where n is the number of degrees of freedom, $n = 3N - (p + q)$ and q^j, $j = 1,2,...,n$ generalized coordinates. If independent Cartesian coordinates are y_1 and x_3, it can be written:

$$y_1 = 2l \sin\varphi \qquad x_3 = \frac{3}{2} l \cos\varphi - l \sin\theta. \tag{11}$$

The coordinates of all points can be expressed via generalized coordinates:

$$\xi_i = \xi_i\left(q^1, q^2, \ldots, q^n; t\right), \quad i = 1, 2, \ldots 3N, \tag{12}$$

but only under the condition that the determinant of the Jacobian matrix $\boldsymbol{J}_1$, of transformation of independent coordinatesthatare expressed over generalized coordinates is not equal to 0:

$$|\boldsymbol{J}_1| = \begin{vmatrix} \frac{\partial \xi_{p+q+1}}{\partial q^1} & \frac{\partial \xi_{p+q+1}}{\partial q^2} & \cdots & \frac{\partial \xi_{p+q+1}}{\partial q^n} \\ \frac{\partial \xi_{p+q+2}}{\partial q^1} & \frac{\partial \xi_{p+q+2}}{\partial q^2} & \cdots & \frac{\partial \xi_{p+q+2}}{\partial q^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_{3N}}{\partial q^1} & \frac{\partial \xi_{3N}}{\partial q^2} & \cdots & \frac{\partial \xi_{3N}}{\partial q^n} \end{vmatrix} \neq 0. \quad (13)$$

The Jacobian matrix of transformation (13) becomes:

$$\boldsymbol{J}_1 = \begin{bmatrix} \frac{\partial y_1}{\partial \varphi} & \frac{\partial y_1}{\partial \theta} \\ \frac{\partial x_3}{\partial \varphi} & \frac{\partial x_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2l\cos\varphi & 0 \\ -\frac{3}{2}l\sin\varphi & -l\cos\varphi \end{bmatrix}, \quad (14)$$

and the determinant has the value:

$$|\boldsymbol{J}_1| = -2l^2\cos\theta\cos\varphi. \quad (15)$$

Dependent Cartesian coordinates are expressed through generalized (16) and (2)–(5) become (17)-(20).

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 2l\cos\varphi, \\ y_2 &= 0, \\ y_3 &= \frac{1}{2}l\sin\varphi - l\cos\varphi. \end{aligned} \quad (16)$$

$$f^1 = 0\cos\varphi = 0 \quad (17)$$

$$f^2 = 0\sin\varphi = 0 \quad (18)$$

$$\begin{aligned} f^1 &= 0\cos\varphi = 0 \\ f^3 &= (2l\cos\varphi)^2 + (2l\sin\varphi)^2 - 4l^2 = 0 \end{aligned} \quad (19)$$

$$\begin{aligned} f^4 &= \left(\frac{3l}{2}\cos\varphi - l\sin\theta - \frac{3l}{4}2l\cos\varphi\right)^2 + \\ &+\left(l\frac{\sin\varphi}{2} - l\cos\theta - 2l\frac{\sin\varphi}{4}\right)^2 - l^2 = 0 \end{aligned} \quad (20)$$

Vector of the virtual displacement δr_ν of the point M_ν is the difference of the position vector, that determine the position of the considered point M_ν of the system, at two infinitely close time moments. Vectors of the virtual displacements are:

$$\begin{aligned} \delta \boldsymbol{r}_1 &= 2l\cos\varphi\delta\varphi\boldsymbol{j}, \\ \delta \boldsymbol{r}_2 &= -2l\sin\varphi\delta\varphi\boldsymbol{i}, \\ \delta \boldsymbol{r}_3 &= (-\frac{3l}{2}\sin\varphi\delta\varphi - l\cos\theta\delta\theta)\boldsymbol{i} + (\frac{l}{2}\cos\varphi\delta\varphi + \mathrm{l}\sin\theta\delta\theta)\boldsymbol{j}. \end{aligned} \quad (21)$$

The velocities of the points are determined by:

$$\begin{aligned} \boldsymbol{V}_1 &= 2l\cos\varphi\dot{\varphi}\boldsymbol{j}, \\ \boldsymbol{V}_2 &= -2l\sin\varphi\dot{\varphi}\boldsymbol{i} \\ \boldsymbol{V}_3 &= \left(-\frac{3}{2}l\sin\varphi\dot{\varphi} - l\cos\theta\dot{\theta}\right)\boldsymbol{i} + \\ &+\left(\frac{l}{2}\cos\varphi\dot{\varphi} + l\sin\theta\dot{\theta}\right)\boldsymbol{j}, \end{aligned} \quad (22)$$

andsystem accelerations are:

$$\begin{aligned} \boldsymbol{a}_1 &= 2l\left(\cos\varphi\ddot{\varphi} - \sin\varphi\dot{\varphi}^2\right)\boldsymbol{j}, \\ \boldsymbol{a}_2 &= 2l\left(-\cos\varphi\dot{\varphi}^2 - \sin\varphi\ddot{\varphi}\right)\boldsymbol{i}, \\ \boldsymbol{a}_3 &= \left(\left(-\sin\theta\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right)(-l) + \right. \\ &\left. +\frac{3}{2}l(-\cos\varphi\dot{\varphi}^2 - \sin\varphi\ddot{\varphi})\right)\boldsymbol{i} + \\ &+\left(\left(-\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta}\right)(-l) + \right. \\ &\left. +\frac{l}{2}(-\sin\varphi\dot{\varphi}^2 + \cos\varphi\ddot{\varphi})\right)\boldsymbol{j}. \end{aligned} \quad (23)$$

Some other equations that describe the system are: kinetic energy$T$ and generalized Lagrange – d'Alambert'sprinciple: $Q_\alpha - \sum_{\nu=1}^{N} m_\nu \boldsymbol{a}_\nu \cdot \boldsymbol{g}_{(\nu)\alpha} = 0$, α = 1,2,...,n, where $Q_\alpha = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \boldsymbol{g}_{(\nu)\alpha} = 0$ are called the generalized forces associated with the virtual displacement δr. Kinetic energy is:

$$T = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta, \quad \alpha,\beta = 1,2\ldots n \quad (24)$$

where $a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu g_{(\nu)\alpha\beta}$, $g_{(\nu)\alpha\beta} = \boldsymbol{g}_{(\nu)\alpha} \cdot \boldsymbol{g}_{(\nu)\beta}$ and $\boldsymbol{g}_{(\nu)s} = \frac{\partial \boldsymbol{r}_\nu}{\partial q^s}$, $s = 1,2,\ldots n$ is the basis vector of the curvilinear coordinate system. For the given system:

$$\begin{aligned} T = l^2 &\left(0.5m_3\dot{\theta}^2 + m_3(0.5\cos\varphi\sin\theta + 1.5\cos\theta\sin\varphi)\dot{\theta}\dot{\varphi} + \right. \\ &\left. \left((2m_1 + 0.125m_3)\cos^2\varphi + (2m_2 + 1.125m_3)\sin^2\varphi\right)\dot{\varphi}^2\right) \end{aligned} \quad (25)$$

From the Lagrange – d'Alambert's principle, it is possible to derive the general equation of statics. The total work of all ideal constraint reactions for any virtual displacements is equal to zero, so the sufficient and necessary condition is given with:

$$Q_\alpha \delta q^\alpha = 0 \quad (26)$$

Since the variations of the generalized coordinates are mutually independent and different from zero, for the last equality to be fulfilled, it must be valid: $Q_\alpha = 0$. The considered mechanical system from the task has two degrees of freedom,so the conditions of static equilibrium are given by the following two equations:

$$Q_\varphi = 0;\ Q_\theta = 0 \quad (27)$$

Obtained solutions form (27) represent static equilibrium positions. When determining static equilibrium positions three cases should be considered. In the first case, when the slider-crank M_2 tends to start moving in a direction that coincides with the positive direction of the Q_x axis, the projection of the static friction force to the axis is a negative value. In the second opposite case, the projection of the static friction force is positive. Third case considers ideally smooth surfaces. Further extension to the special cases of mechanical systems with non-ideal constraintsare presented in [17]. In the first case, after calculating the generalized forces corresponding to the elected general coordinates, a system of equations is obtained:

$$\begin{aligned}&(15264.2+820.06\cos\theta)\cos\varphi+\\&+(0.38|4725.1872+11-2460.18\sin\theta)\sin\varphi|-\\&-6355.21\sin 2\varphi=0,\end{aligned} \tag{28}$$

$$\begin{aligned}&2460.18\cos\theta\cos\varphi-\\&-\sin\theta\left(645.663+820.062\sin\varphi\right)=0\end{aligned} \tag{29}$$

$$\begin{aligned}&(15264.2+820.06\cos\theta)\cos\varphi-\\&-(0.38|4725.1872+11-2460.18\sin\theta)\sin\varphi|-\\&-6355.21\sin 2\varphi=0\end{aligned} \tag{30}$$

$$\begin{aligned}&2460.18\cos\theta\cos\varphi-\\&-\sin\theta\left(645.663+820.062\sin\varphi\right)=0\end{aligned} \tag{31}$$

The number of solutions, as well as the approximate values of the solutions - which can be used as an initial iteration to find the exact values, can be determined graphically. The solutions of the system are at the intersection of the curves whose implicit equations are the system equations. As an alternative to numerical calculation this graphical method was also proposed in [18] and [19]. The solution of the considered system of algebraic equations could be geometrically represented in the form of intersection of the corresponding surfaces [19]. The implementation of the method of crossing of the curves is achieved, same as in [20], by using the built-in ContourPlot *Mathematica* function.

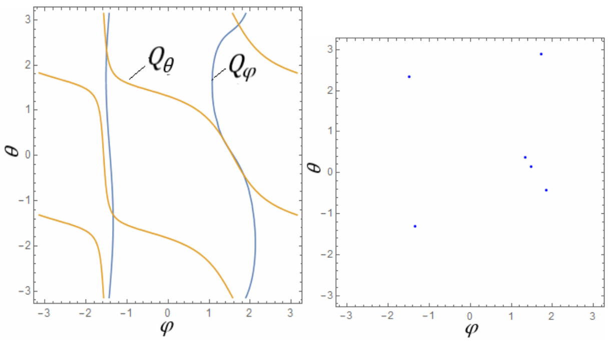

Figure 8. Graphical representation of solutions as cross sections and the points of intersection of the curves Q_φ and Q_θ in the first case

Figure 8 shows the solutions of the system, which determine static equilibrium positions, Table 1.

In the second case, after calculating the generalized forces, a system of equations is obtained in the form of (30) and (31), so the static equilibrium positions are shown in Table 2.

Finally, when it is assumed that the contact surfaces are ideally smooth, i.e. when there is no static friction force, so a system of equations is obtained as:

$$\begin{aligned}&(15264.2+820.06\cos\theta)\cos\varphi-\\&-2460.18\sin\theta\sin\varphi-6355.21\sin 2\varphi=0,\end{aligned} \tag{32}$$

$$\begin{aligned}&2460.18\cos\theta\cos\varphi+\\&+\sin\theta\left(-645.663--820.062\sin\varphi\right)=0\end{aligned} \tag{33}$$

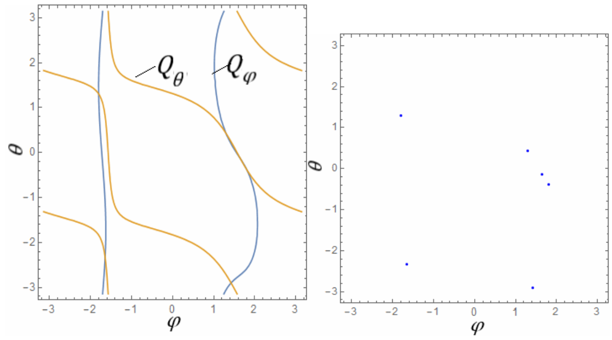

Figure 9. Intersection of the curves Q_φ and Q_θ in the second case

Table 1. Static equilibrium positions in case of negative projection of the static friction force

φ_{st1} = -1.49727	θ_{st1} = 2.33199
φ_{st2} = -1.34075	θ_{st2} = -1.30487
φ_{st3} = 1.33171	θ_{st3} = 0.383874
φ_{st4} = 1.48822	θ_{st4} = 0.0.137823
φ_{st5} = 1.71841	θ_{st5} = 2.89814
φ_{st6} = 1.83931	θ_{st6} = 0.426515

Table 2. Static equilibrium positions in case of positive projection of the static friction force

φ_{st1} = -1.80084	θ_{st1} = 1.30487
φ_{st2} = -1.64432	θ_{st2} = -2.33199
φ_{st3} = 1.30228	θ_{st3} = 0.426515
φ_{st4} = 1.42319	θ_{st4} = -2.89814
φ_{st5} = 1.65337	θ_{st5} = -0.137823
φ_{st6} = 1.80988	θ_{st6} = -0.383874

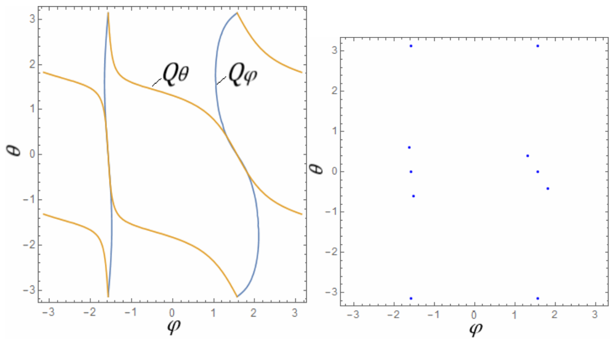

Figure 10. Graphical representation of solutions as cross sections and the points of intersection of the curves of the curves Q_φ and Q_θ in the third case

There are ten solutions that determine the positions of static equilibrium. Considering the fact that the result is in the domain: $-\pi \le \varphi \le \pi$ and $-\pi \le \theta \le \pi$, a conclusion is reached: two solutions physically represent the same position, so they are excluded from consideration. Eight different static equilibrium positions are determined by Table 3.

Table 3. Static equilibrium positions in case without static friction force

$\varphi_{st1} = \frac{\pi}{2}$	$\theta_{st1} = \pi$
$\varphi_{st2} = 1.31236$	$\theta_{st2} = 0.412071$
$\varphi_{st3} = \frac{\pi}{2}$	$\theta_{st3} = 0$
$\varphi_{st4} = 1.82924$	$\theta_{st4} = -0.412071$
$\varphi_{st5} = \frac{\pi}{2}$	$\theta_{st5} = -\pi$
$\varphi_{st6} = -1.61977$	$\theta_{st6} = 0.606987$
$\varphi_{st7} = -\frac{\pi}{2}$	$\theta_{st7} = 0$
$\varphi_{st8} = -\frac{\pi}{2}$	$\theta_{st6} = -0.606987$

4. FORMATION OF LAGRANGE'S EQUATION OF THE SECOND KIND

4.1 Covariant formulation

Using (24) the coefficients of metric tensors $a_{\alpha\beta}$ are obtained:

$$a_{\alpha\beta} = a_{\beta\alpha} = \frac{\partial^2 T}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}, \quad \alpha, \beta = 1, \ldots, n \tag{34}$$

$$\begin{aligned} a_{11} &= 2l^2\Big(\left(2m_1 + 0.125m_3\right)\cos^2\varphi + \\ &\quad + \left(2m_2 + 1.125m_3\right)\sin^2\varphi\Big), \\ a_{12} &= a_{21} = \Big(m_3 l^2 0.5\cos\varphi\sin\theta + \\ &\quad + 1.5\cos\theta\sin\varphi\Big), \\ a_{22} &= m_3 l^2. \end{aligned} \tag{35}$$

On the other hand, Christoffel symbols of the first kind are acquired as:

$$\Gamma_{\beta\gamma,\alpha} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{g}_{(\nu)\gamma}}{\partial q^\beta} \boldsymbol{g}_{(\nu)\alpha} \tag{36}$$

andcovariant form of equations is given as:

$$\begin{aligned} & a_{\alpha\beta}\ddot{q}^\beta + \Gamma_{\beta\gamma,\alpha}\dot{q}^\beta\dot{q}^\gamma = Q_\alpha, \\ & \alpha, \beta, \gamma = 1, \ldots, n \end{aligned} \tag{37}$$

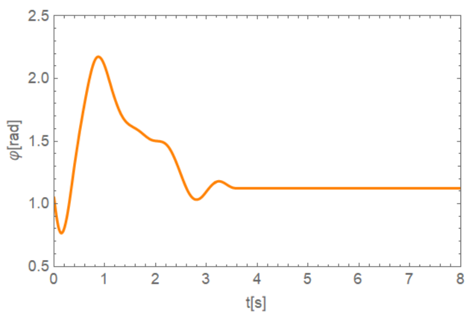

Figure 11. Graph ofgeneralized coordinate $q^1(t) = \varphi(t)$ over time

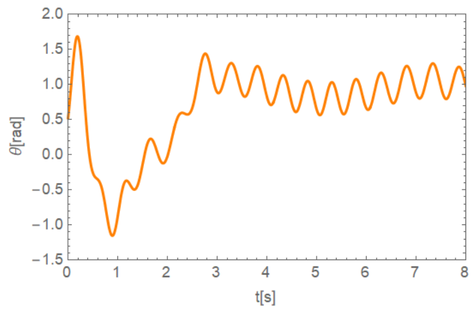

Figure 12. Graph of generalized coordinate $q^1(t) = \varphi(t)$ over time

The obtained results agree completely with the results obtained by Lagrange equations of the first kind - Figures 2-7. Also, two additional Figures 11 and 12, representing $q^i = q_i(t)$, are shown.

4.2 Contravariant formulation

When equation (37) is multiplied by the contravariant matrix tensor $a^{\delta\alpha}$, Lagrangian equations of the second kind can be derived in the contravariant form (38). If $a^{\delta\alpha}a_{\alpha\beta}\ddot{q}^\beta = \ddot{q}^\delta$, $a^{\delta\alpha}\Gamma_{\beta\gamma,\alpha} = \Gamma^\delta_{\beta\gamma}$ and $a^{\delta\alpha}Q_\alpha = Q^\delta$ for α, β, γ, δ= 1, ... ,n ,the contravariant equations of motion of the system are obtained:

$$\begin{aligned} & a^{\delta\alpha}a_{\alpha\beta}\ddot{q}^\beta + a^{\delta\alpha}\Gamma_{\beta\gamma,\alpha}\dot{q}^\beta\dot{q}^\gamma = a^{\delta\alpha}Q_\alpha \\ & \alpha, \beta, \gamma = 1, \ldots, n \end{aligned} \tag{38}$$

$$\ddot{q}^\delta + \Gamma^\delta_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma = Q^\delta, \quad \beta, \gamma, \delta = 1, \ldots, n \tag{39}$$

where the expression denoted by $\Gamma^\delta_{\beta\gamma}$ represents Christoffel symbols of the second kind and Q^δ represents the contravariant coordinates of the generalized forces.

The positions of the points obtained on the basis of the contravariant Lagrange's equations of the second kind are confirmed graphically and are the same as in Figures 2-7 and 11-12.

5. LAGRANGE FUNCTION AND HAMILTON EQUATION

Differential equations of motion can be represented in the form of Lagrange equations of the second kind (expressed only as a function of independent generalized coordinates, q^1, q^2, ..., q^n, and their first and second derivatives in time), i.e. in the form of:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = Q_\alpha, \quad \alpha = 1, \ldots, n \tag{40}$$

where Q_α is the generalized force of the system of active forces acting on the considered system, which corresponds to the independent generalized coordinate q^α. Generalized forces can be givenas the sum of potential $\boldsymbol{F}_\nu^\Pi$ and nonpotential $\tilde{\boldsymbol{F}}_\nu$ forces:

$$\begin{aligned} & Q_\alpha = \sum_{\nu=1}^{N}\left(\boldsymbol{F}_\nu^\Pi + \tilde{\boldsymbol{F}}_\nu\right)\frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} = -\frac{\partial \Pi}{\partial q} + \tilde{Q}_\alpha \\ & \alpha = 1, 2, \ldots, n \end{aligned} \tag{41}$$

In the considered system, the following potential forces act on the observed mechanical system: the weight forces of the sliders M_1, M_2 and point M_3, the forces exerted by a springs, stiffness c_1 and c_2. Nonpotential forces acting on the considered mechanical system are: viscous damping force, force of friction and externalforce $\boldsymbol{F}$. Potential energy takes the shape:

$$\begin{aligned} &= m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + \\ &+ \frac{1}{2} c_1 \left(x_3^2 + y_3^2 \right) + \frac{1}{2} c_2 (2l - y_1)^2 + C_0 \end{aligned} \tag{42}$$

or:

$$\begin{aligned} &= 0.5l \Big(4c_2 l(-1+\sin\varphi)^2 + \\ &+ 4m_1 g \sin\varphi + m_3 g(\sin\varphi - 2\cos\theta) + \\ &\big(+0.25 c_1 l (9 + 4\cos 2\varphi - 12\cos\varphi \sin\theta - -4\cos\theta \sin\varphi) \big) + C_0 \end{aligned} \tag{43}$$

Components of generalized forces due to the action of non-potential forces, in the notation, $\tilde{Q}_\alpha$, in (41) are obtained by determining the work of non-potential forces on virtual displacements. Virtual work of force $\boldsymbol{F}$on virtual displacement is:

$$\delta' A(\boldsymbol{F}) = \boldsymbol{F} \delta \boldsymbol{r}_3 = -F\left(\frac{1}{2} l \cos\varphi \, \delta\varphi + l \sin\theta \, \delta\theta\right) \tag{44}$$

so it is true.

For the dynamics friction force $\boldsymbol{F}_{trd}$ generalized forces are obtained as:

$$\begin{aligned} &\tilde{Q}_\varphi(\boldsymbol{F}) = -\frac{1}{2} F l \cos\varphi, \\ &\tilde{Q}_\theta(\boldsymbol{F}) = -F l \sin\theta \end{aligned} \tag{45}$$

The Lagrange function is introduced as the difference between kinetic and potential energy: $L = T - \Pi$. By transforming the expression starting from Lagrange equations of the second kind and including potential and non-potential forces, it was obtained that:

$$\begin{aligned} &\tilde{Q}_\varphi(\boldsymbol{F}_{trd}) \\ &= -2l \mu_d \big| g m_1 + g m_2 + g m_3 + F \\ &- c_2 (2l - 2l \sin\varphi) + c_1(-l\cos\varphi + 0.5 l \sin\varphi) \\ &+ 2l m_1 \left(-\sin\varphi \dot{\varphi}^2 + cos\varphi \ddot{\varphi} \right) \big| + m_3(-l(-\cos\dot{\theta}\dot{\theta}^2 - \sin\theta \ddot{\theta}) \\ &+ 0.5l(-\sin\varphi \ddot{\varphi}^2 + cos\varphi \ddot{\varphi})) \big| \operatorname{sign}(l \sin\varphi \dot{\varphi}) \sin\varphi \\ &\tilde{Q}_\theta(\boldsymbol{F}_{trd}) = 0 \end{aligned} \tag{46}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = \tilde{Q}_\alpha, \quad \alpha = 1, \ldots, n \tag{47}$$

Observing (47), it is noticed that the motion of the system in the configuration space of generalized coordinates is described through scalar functions, kinetic and potential energy. Kinetic and potential energies of the system are already given (25) and (43). Using Rayleigh function, where the relative velocitybetween the piston and the cylinder is $\dot{x}_2$, equations and laws of motion of the system can be easily obtained.It has been proved that the figures are the same as Figures 2-7 and 11-12.Unlike Lagrange's way of describing the system defined as a function of time, position and velocity of material points, the Hamiltonian system use time, position and generalized impulses: $p_\alpha = \dfrac{\partial T}{\partial \dot{q}^\alpha}$, α = 1, ... ,$n$. Canonical (Hamilton's) differential equations of motion are given as follows:

$$\dot{q}^\alpha = \frac{\partial H}{\partial p^\alpha}, \dot{p}_\alpha = -\frac{\partial H}{\partial p^\alpha} + \tilde{Q}_\alpha \tag{48}$$

In order to determine the motion of a given system, it is first necessary to define the Hamiltonian function H, which, for the scleronomic system whose kinetic energy does not explicitly depend on time, has the form:

$$H = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta + \Pi \tag{49}$$

where $p_\varphi = \dfrac{\partial T}{\partial \dot{\varphi}}$ and $p_\theta = \dfrac{\partial T}{\partial \dot{\theta}}$ are generalized momenta.

For the proposed system Hamiltonian function is given by (50). When a mechanical system is described by Hamiltonian equations (variables), it is necessary to solve $2n$ differential equations of the first order, which can determine $2n$ functions, which describe the system: $q^\alpha = q^\alpha(\mathrm{t}); p_\alpha = p_\alpha(\mathrm{t})$.

$$\begin{aligned} &H = 0.5l(4c_2 l(-1+\sin\varphi)^2 + 4g m_1 \sin\varphi + \\ &+ g m_3(-2\cos\theta + \sin\varphi) + \\ &+ 0.25 c_1 l (9 + 4\cos 2\varphi - 12\cos\varphi \sin\theta - -4\cos\theta \sin\varphi) + \\ &+ (-2 m_3 p_\varphi^2 + + m_3 p_\varphi p_\theta (2\cos\varphi \sin\theta + 6\cos\theta \sin\varphi) + \\ &+ p_\theta^2 \big((-8m_1 - 0.5m_3)\cos^2\varphi + (-8m_2 - 4.5m_3)\sin^2\varphi \big) \big) \\ &/(l^2 m_3 (\cos^2\varphi \left(-16m_1 - -m_3 + m_3 \sin^2\theta \right) \\ &- \left(16m_2 + 9m_3 + 9m_3\cos^2\theta \right) \sin^2\varphi + 1.5 m_3 \sin 2\varphi \sin 2\theta)). \end{aligned} \tag{50}$$

Graphs in generalized coordinates, as well as the positions of material points were obtained. It can be concluded that the Hamiltonian equations also obtained the same solutions as by applying Lagrange equations of the second kind in the covariant and contravariant form, in Figures 2-7 and 11-12.

6. STABILITY OF MOTION

One of the important requirements that are named in terms of the functioning of technical system is the stability of their work [21]. If it is necessary to determine the position of static equilibrium, it is possible to start considering it based on Lagrange’s equations of the second kind.As the system is static, generalized velocities and accelerations must be equal to zero. Based on the previous statement, Lagrange's equations of the second kind (40) are equal to zero. The conservative mechanical system is scleronomic and exposed exclusively to the action of conservative forces (potential forces whose potential energy does not depend explicitly on time). In this case, the required equilibrium conditions (54), due to the absence of nonconservative forces, take the form:

$$\frac{\partial \Pi}{\partial q^{\alpha}} = 0, \quad \alpha = 1,\ldots,n \tag{51}$$

so the equilibrium stability test comes down to considering the potential energy of the mechanical system. The analysis can also be approached through a Lagrange-Dirichlet theorem:

$$\Pi \approx \frac{1}{2} c_{\alpha\beta} q^{\alpha} q^{\beta}, \text{with}: c_{\alpha\beta} = \left(\frac{\partial^2 \Pi}{\partial q^{\alpha} \partial q^{\beta}} \right)_0 \quad \alpha, \beta = 1,\ldots,n \tag{52}$$

where $()_0$ indicates that the value of the expression in parentheses is calculated in the equilibrium position. The behaviour of the potential energy in the environment of the equilibrium position corresponds to the behaviour of a homogeneous quadratic form with constant coefficients $c_{\alpha\beta}$. If the equilibrium position is stable, the quadratic form (52) is positive definite.

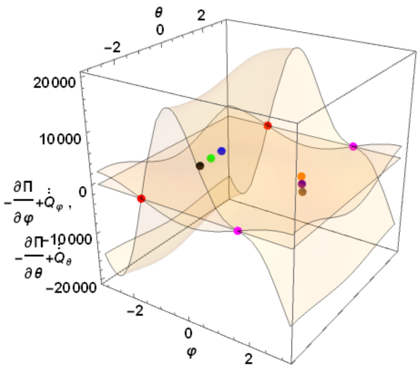

Figure 13. Spatial arrangement of equilibrium positions

This is examined with the help of Sylvestercriteria. Generalized coordinates, prescribed by the task, are included in the (41), so the obtained equations are the same as (32)-(33). In both ways, the previously obtained result given in Table 3 is obtained. Thus, there are eight static equilibrium positions (bearing in mind that there are two physically identical positions), whose plane arrangement *φ-θ* is shown in Figure 13, and the spatial arrangement is given in the following Figure 14.

Using Lagrange-Dirichlet theorem potential energy is obtained as:

$$\begin{aligned} \Pi = {} & 12840.64 - 2996.06\cos^2\varphi - \\ & -2365.31\cos\varphi\sin\theta + \\ & +\cos\theta(-651.10 - 788.44\sin\varphi) - \\ & -14385.86\sin\varphi + 2996.06\sin^2\varphi. \end{aligned} \tag{54}$$

The following Figure 14 shows the law of change of potential energy $\Pi = \Pi(\varphi,\theta)$ according to (53), where $-\pi \le \varphi \le \pi$, $-\pi \le \theta \le \pi$, while the points represent numerically determined equilibrium positions.

According to Lagrage-Dirichlet's theory, in order that the equilibrium position of a mechanical system is stable, a necessary and sufficient condition is that the potential energy around the equilibrium position has a positive value. This condition is met if the potential energy is defined as a positive definite quadrature form. According to Sylvester's criterion, for the Hermit matrix to be positively definite, it is necessary and sufficient for all its major minors to be positive.

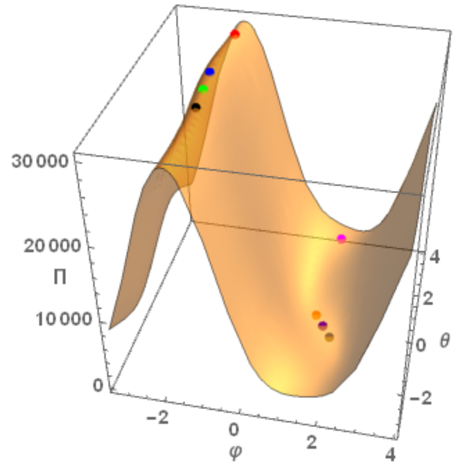

Figure 14. Potential energy

Table 4. Equilibrium positions of the system

$C = c_{\alpha\beta}$	Equlibrium position of the system
$\begin{pmatrix} 1733.71 & -2460.18 \\ -2460.18 & -1465.72 \end{pmatrix}$	unstable
$\begin{pmatrix} 4685.56 & 2263.33 \\ 2263.33 & 1569.90 \end{pmatrix}$	stable
$\begin{pmatrix} 3373.83 & 2460.18 \\ 2460.18 & 1465.72 \end{pmatrix}$	unstable
$\begin{pmatrix} 4685.56 & 2263.33 \\ 2263.33 & 1569.90 \end{pmatrix}$	stable
$\begin{pmatrix} -27154.54 & 2460.18 \\ 2460.18 & 174.40 \end{pmatrix}$	unstable
$\begin{pmatrix} -28636.85 & -2041.20 \\ -2041.20 & -211.13 \end{pmatrix}$	unstable
$\begin{pmatrix} -28794.66 & -2460.18 \\ -2460.18 & -174.40 \end{pmatrix}$	unstable
$\begin{pmatrix} -28636.85 & -2041.20 \\ -2041.20 & -211.13 \end{pmatrix}$	unstable

Based on the above, in order to determine the stability of the equilibrium position, it is necessary to calculate the main minors of the matrix C. Another way of examining the stability of the undisturbed motion of the considered system is by using the Hurwitz criterion, i.e. via the basic principal minors of the Hurwitz matrix and polynomial. The result of the program code, for the second case in Table 5, is in Figures 15 and 16, which confirms that the position of static equilibrium is correctly determined.

Table 5. Stability of static equilibrium position based on the roots of characteristic polynomials

Characteristic polynomial	Undisturbed motion of the system is:
$8216+254.36\lambda+243.49\lambda^2-1.5\lambda^3-\lambda^4$	unstable
$2102.77+250.79\lambda+193.25\lambda^2+1.38\lambda^3+\lambda^4$	stable
$-1058.73+254.36\lambda+163.72\lambda^2+1.5\lambda^3+\lambda^4$	unstable
$2102.77+250.79\lambda+193.25\lambda^2+1.38\lambda^3+\lambda^4$	stable
$-10314.12+30.26\lambda-261.97\lambda^2+1.5\lambda^3+\lambda^4$	unstable
$1711.84-34.82\lambda-292.43\lambda^2+1.43\lambda^3+\lambda^4$	unstable
$-985.47-30.26\lambda-322.36\lambda^2+1.5\lambda^3+\lambda^4$	unstable
$1711.85-34.82\lambda-292.43\lambda^2+1.43\lambda^3+\lambda^4$	unstable

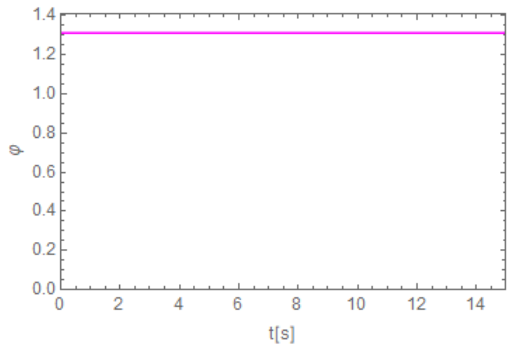

Figure 15. Graph of generalized coordinate *φ*(*t*) over time

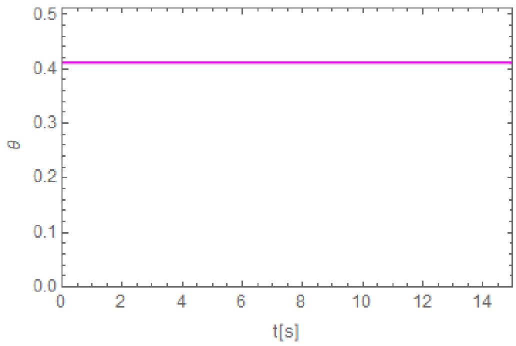

Figure 16. Graph of generalized coordinate *θ*(*t*) over time

It is possible to establish the disturbed equations of motion as:

$$\bar{q}^{\alpha}=\bar{q}^{\alpha}\left(t;t_0,\bar{\boldsymbol{q}}_0,\dot{\bar{\boldsymbol{q}}}_0\right),$$
$$\dot{\bar{q}}^{\alpha}=\dot{\bar{q}}^{\alpha}\left(t;t_0,\bar{\boldsymbol{q}}_0,\dot{\bar{\boldsymbol{q}}}_0\right),\quad \alpha=1,\ldots,n \tag{54}$$

where: $\bar{\boldsymbol{q}}_0=(\bar{q}_{01},\bar{q}_{02},\ldots,\bar{q}_{0n})$, $\left(\dot{\bar{q}}_{01},\dot{\bar{q}}_{02},\ldots,\dot{\bar{q}}_{0n}\right)$ and $\bar{q}^{\alpha}(t)-q^{\alpha}(t)=\xi^{\alpha}(t)$, $\dot{\bar{q}}^{\alpha}(t)-\dot{q}^{\alpha}=\eta^{\alpha}(t)$ are disturbances. For the initial disturbances values: $\xi_1=0.1$, $\xi_2=0$, $\eta_1=0.1$, $\xi_1=0$, it is expected that the norm of disturbed motion tends to zero during time (since earlier results have shown that the second position of static equilibrium is stable).

The earlier conclusion on the stability of the position of static equilibrium, based on the Lagrange - Dirichlet theorem and based on the root of the characteristic polynomial, second case shown in Tables 4 and 5, does not change. The check was performed for all other equilibrium points and earlier conclusions were confirmed.

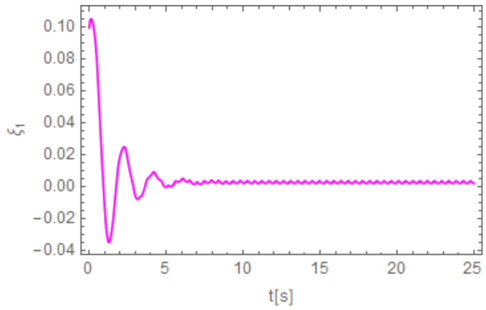

Figure 17. Graph of $\xi_1(t)$ over time

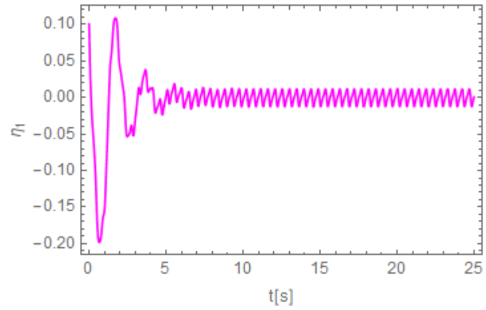

Figure 18. Graph of $\eta_1(t)$ over time

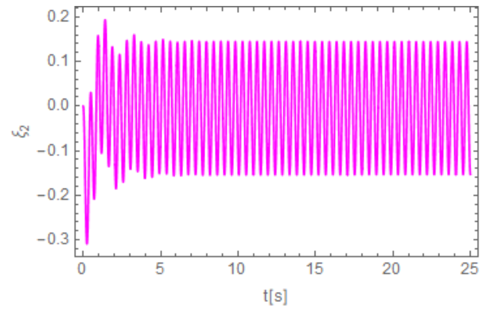

Figure 19. Graph of $\xi_2(t)$ over time

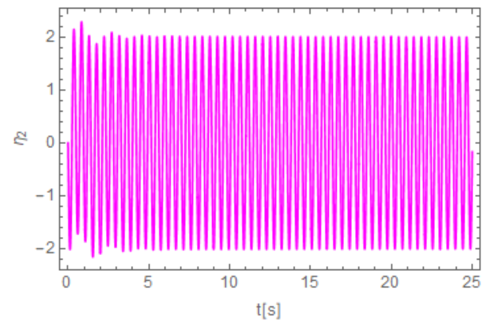

Figure 20. Graph of $\eta_1(t)$over time

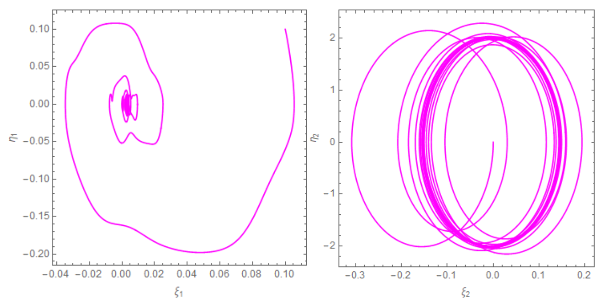

Figure 21. $\xi_i(t)$ - $\eta(t)$, i = 1,2 dependency

7. CONCLUSION

In the beginning of this paper, using the example of a simple mechanical construction model with constrained motion, we have proved that it is possible to perform an analysis of the motion of a mechanical systemby applying Lagrange's equations of the first and second kind, as well as Hamilton's equations. By applying Lagrange's equations of the first kind, the system of six differential equations was obtained. The system had two degrees of freedom, so there are only two independent coordinates, and the other four can be expressed through those two. Based on the above, it is concluded that the analysis of the systemby applying Lagrange equations of the first kind is complex and redundant. Much more elegant and better way of analyzing the system is by applying the Lagrange equations of the second kind. Based on determination of the total kinetic energy of the system and potential generalized forces, as well as their partial derivatives,two differential equations can be obtained. These differential equations depend on only twogeneralized coordinates andthe final equations of motion are obtained.By applying Hamilton's equations instead of secondorder differential equations, solving the problemis simplified. Moreover, a model has been obtained with consideration for Coulomb friction force. The stability of the system for disturbed and undisturbed motion was checked. Different methods were confirmed by drawing the same conclusions.

APPENDIX

Absolut angles are: $q^1 = \varphi, q^2 = \theta$

Distances between points are: $M_1M_2 = 2l$, $BM_3 = l$, $BM_2 = l/2$

Spring whose stiffness is c_1 has a length: $l_{01} = O_1O$

Spring whose stiffness is c_2 has a length: $l_{02} = l/2$

Coordinate: A(0, 5l/2)

Other data: $\boldsymbol{q}_0 = \left[q_0^1 q_0^2 \right]^T = \left[\frac{\pi}{3} \frac{\pi}{6} \right]^T$;

$\dot{\boldsymbol{q}}_0 = \left[\dot{q}_0^1 \dot{q}_0^2 \right]^T = \left[-3.95 \ 5.85 \right]^T$; v

$m_1 = 14\,\text{kg}; m_2 = 14\,\text{kg}; m_3 = 4\,\text{kg};$

$l = 1.47\text{m}; \ c_1 = 759\frac{\text{N}}{\text{m}}; c_2 = 1850\frac{\text{N}}{\text{m}};$

$\beta = \frac{21\text{Ns}}{\text{m}}; \boldsymbol{F}_{st} = 400\,\text{N}; \ g = 9.80665\frac{\text{m}}{\text{s}^2}$

$\mu_d = 0.11; \mu_0 = 0.13$

$\boldsymbol{F} = -\boldsymbol{F}_0 e^{-\alpha t} \sin \Omega t; \boldsymbol{F}_0 = 255\,\text{N};$

$\Omega = 1.5\text{s}^{-1}, \alpha = 0.09.$

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АНАЛИЗА КРЕТАЊА И СТАБИЛНОСТИ ХОЛОНОМНОГ МЕХАНИЧКОГ СИСТЕМА У ПРОИЗВОЉНОМ ПОЉУ СИЛА

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У циљу добијања увида у рад машине пре њене монтаже и производње, као и добијања добре анализе, овај рад представља детаљна решења специфичног проблема из области аналитичке механике. Поред нумеричких поступака у раду, извршен је и преглед теоријских основа. Разне врсте анализа су врло честе у машинском инжењерству, због могућности апроксимације сложених машина. За предложени систем дате су Лагранжове једначине прве врсте, коваријантне и контраваријантне једначине, Хамилтонове једначине, генералисане координате, као и увид у Кулонову силу трења. Такође, решени су и услови статичке равнотеже уз помоћ нумеричких и графичких поступака - пресеком две криве. Коначно, разматрана је и стабилност кретања поремећеног и непоремећеног кретања.