GRADIMIR MILOVANOVIĆ – A MASTER IN APPROXIMATION AND COMPUTATION PART II

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This Special Issue of the journal *Applicable Analysis and Discrete Mathematics* is dedicated to the 70th birth anniversary of the eminent Serbian mathematician and Academician professor Gradimir V. Milovanović. This Issue is split into two parts, with the first being published in 2019, and the second one in 2020. Most of the papers featured within this Special Issue have been presented at the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas in Antalya-Turkey, October 26–29, 2018 (http://micopam2018.akdeniz.edu.tr/home/).

The Preface in Part II of this Special Issue of the journal *Applicable Analysis and Discrete Mathematics* is devoted to an analysis of Milovanović's scientific results obtained during the last decade.

A detailed account of Prof. G.V. Milovanović's mathematical work, on the occasion of his 60th birthday, is featured in the papers:

[1] Aleksandar Ivić: *The scientific work of Gradimir V. Milovanović*,

[2] Walter Gautschi: *My collaboration with Gradimir V. Milovanović*,

[3] Themistocles M. Rassias: *On Some Major Trends in Mathematics*,

which are published in the book: "Approximation and Computation: In Honor of Gradimir V. Milovanović, Springer Optimization and Its Applications 42, W. Gautschi et al. (eds.), DOI 10.1007/978-1-4419-6594-3 16, Springer Science+Business Media, New York, LLC 2011." Here, we review again, in short, some important scientific results of Milovanović from that period, in order to show their influence on and connection with the results obtained in the last decade.

Milovanović has obtained important results in several areas of Numerical Analysis, Approximation Theory, and Special Functions, with many applications in various fields of Mathematics, Physics, and Engineering.

Figure 1: Mediterranean International Conference of Pure & Applied Mathematics and Related Areas in Antalya-Turkey, October 26–29, 2018

The topics to which he made his most important contributions include: Polynomials (extremal problems, inequalities and zeros); Orthogonal polynomials and systems (constructive theory, new applications, software implementation); Approximations by polynomials and splines; Interpolation processes; Quadrature processes (constructive theory of quadratures of Gaussian type, error estimate, quadrature with multiple nodes, integration of highly oscillating functions); Integral equations; Summation of slowly convergent series; Special functions, polynomials and special numbers; etc.

Henceforth, instead of Prof. Gradimir V. Milovanović we mostly use GVM for shortness.

Prof. Gradimir V. Milovanović is one of the best mathematicians that Serbia has ever had, as stressed by Serbian academician Ivić in [1]. Ivić quotes also that: "A topic which had absorbed GVM for a long time is the "introduction of new concepts of orthogonality". For example, one can mention orthogonality on the semicircle, on a circular arc, on the radial rays, orthogonality of Müntz polynomials, multiple orthogonality, s- and σ-orthogonality, etc." The renowned

mathematician Walter Gautschi, one of the founders of modern numerical analysis, wrote in [**2**]: "My collaboration with Gradimir V. Milovanović over a time interval of about 15 years, from 1983 to 1997, is described concerning work on a variety of topics in the area of orthogonal polynomials, Gauss-type quadrature, and some of their applications." and "In looking back on my collaboration with Gradimir, I can only marvel at the spontaneity and originality of his input, which often reduced my own role to one of implementor and organizer. It has been truly a pleasure to work together with Gradimir, and I am sure I am sharing this feeling with the many other individuals who have had the privilege of collaborating with Gradimir."

Their cooperation continued in the following years. As already mentioned, this collaboration began in the mid-1980s, just as Walter was developing his *constructive theory of orthogonal polynomials*, which significantly influenced the development of Gradimir's scientific career, and he became one of Walter's closest collaborators. The constructive theory of orthogonal polynomials opened the door for extensive computational work on orthogonal polynomials and their various applications.

Figure 2: ACTA 2017 (November 30 – December 2, 2017), Belgrade: L. Reichel, GVM, M.M. Spalević (left); Y. Simsek, W. Gautschi, GVM and G. Mastroianni (right)

At the invitation of Walter Gautschi, and on the occasion of Elsevier's project "Numerical analysis 2000", Milovanović was invited to write a chapter "Quadrature with multiple nodes, power orthogonality, and moment-preserving spline approximation" as an expert in the field of quadrature with multiple nodes for Vol. 5 "Quadrature and orthogonal polynomials" (W. Gautschi, F. Marcellan, and L. Reichel, eds.), which was published also in the journal J. Comput. Appl. Math. **127** (2001), 267–86. As part of the extensive three-volume project "Walter Gautschi: Selected Works with Commentaries" (C. Brezinski, A. Sameh, eds.), published by Birkhäuser, Basel, 2014, Milovanović wrote two chapters: Chapter 11: Orthogonal polynomials on the real line, and Chapter 23: Computer algorithms and software packages, showing Walter's scientific work in these areas. At the end of 2018, Milovanović and his associate Spalević organized an international symposium ACTA

2017: Approximation and Computation – Theory and Applications at the Serbian Academy of Sciences and Arts in Belgrade on the occasion of the 90th anniversary of Walter Gautschi with a large number of prominent scientists from around the world. On the same occasion, in the spring of the following year, Milovanović was one of the three plenary lecturers at Purdue Conference on Scientific Computing and Approximation (March 30–31, 2018), organized at the Department of Computer Science at Purdue University, IN, USA (`https://www.cs.purdue.edu/sca/`).

Figure 3: Purdue Conference (2018): Nick Higham, Alex Pothen, W. Gautschi, GVM (left); Ron Devore and GVM (right)

In 2018 Gautschi and Milovanović published an interesting paper in the journal Electron. Trans. Numer. Anal. (ETNA) on Binet-type orthogonal polynomials and their zeros, and very recently they completed a work on orthogonal polynomials relative to a Marchenko-Pastur probability measure, which was introduced in 1967 by the Ukrainian mathematicians Vladimir Alexandrovich Marchenko and Leonid Andreevich Pastur, working on the asymptotic theory of large random matrices. Gautschi and Milovanović have introduced two-parameter generalization and provided an efficient algorithm for constructing the corresponding orthogonal polynomials, as well as the coefficients in their three-term recurrence relations. In special cases, these orthogonal polynomials are identified in terms of Chebyshev polynomials of all four kinds and explicit expressions for all coefficients are derived.

In the next sections, we mention a few fields where most important contributions by GVM lie.

1. GVM AND POLYNOMIALS

The crowning achievement of GVM in this field is his extensive monograph (written jointly with D.S. Mitrinović and Th. M. Rassias) "*Topics in Polynomials: Extremal Problems, Inequalities, Zeros*", World Scientific, Singapore, 1994, XIV+822 pp. This is a famous work, called by many the "Bible of Polynomials".

Th.M. Rassias, D.S. Mitrinović and G.V. Milovanović (Belgrade, 1988)

1.1. Classical orthogonal polynomials

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree not exceeding n. Define the norms

$$||f||_\infty := \max_{-1 \leqslant t \leqslant 1} |f(t)| \quad \text{and} \quad ||f||_r := \left(\int_{-\infty}^{\infty} |f(t)|^r \mathrm{d}\lambda(t) \right)^{1/r}, \quad r \geqslant 1,$$

where $\mathrm{d}\lambda(t)$ is a given non-negative measure on the real line (with compact support or otherwise), for which all the moments

$$\mu_k := \int_{-\infty}^{\infty} t^k \, \mathrm{d}\lambda(t), \qquad k = 0, 1, \ldots,$$

exist and $\mu_0 > 0$. When $r = 2$ we have the norm

$$||f||_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}\lambda(t) \right)^{1/2},$$

and then the inner product is defined by $(f, g) := \int_{-\infty}^{\infty} f(t)\overline{g(t)} \, \mathrm{d}\lambda(t)$. A standard case of orthogonal polynomials is when $\mathrm{d}\lambda(t) = w(t)\mathrm{d}t$, where $w(t)$ is non-negative, all its moments exist and $\mu_0 > 0$. An important case are the *classical orthogonal polynomials* on an interval of orthogonality $(a, b) \in \mathbb{R}$. One can take the following intervals:

$$(a, b) = (-1, 1), (0, +\infty) \ \text{or} \ (-\infty, +\infty)$$

with the inner product

$$(f, g) = \int_a^b f(t)\overline{g(t)}w(t) \, \mathrm{d}t.$$

Orthogonal polynomials $\{Q_n(t)\}$ on (a, b) with this inner product are called *classical orthogonal polynomials* if $w(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(A(t)w(t)\Big) = B(t),$$

where B is a polynomial of the first degree and

$$A(t) = \begin{cases} 1 - t^2, & \text{if } (a, b) = (-1, 1), \\ t, & \text{if } (a, b) = (0, \infty), \\ 1, & \text{if } (a, b) = (-\infty, \infty). \end{cases}$$

The classical orthogonal polynomial $Q_n(t)$ is a solution of $A(t)y'' + B(t)y' + \lambda_n y = 0$ with an explicit constant λ_n.

1.2. Extremal problems of Markov-Bernstein type for polynomials

There are many results on extremal problems and inequalities for the class $\mathcal{P}_n$ of all algebraic polynomials of degree at most n. A. A. Markov (1889) solved the extremal problem of determining

$$A_n = \sup_{P \in \mathcal{P}_n} \frac{||P'||_\infty}{||P||_\infty}.$$

The best constant is $A_n = n^2$ and the extremal polynomial is $P^*(t) = cT_n(t)$, where $T_n(t) = \cos(n \arccos t)$ is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant, alternatively $A_n = T_n'(1)$,

$$||P'||_\infty \leqslant n^2 ||P||_\infty \qquad (P \in \mathcal{P}_n).$$

V. Markov (half brother of A. Markov) in 1892 proved that

$$||P^{(k)}||_\infty \leqslant T_n^{(k)}(1)||P||_\infty \qquad (P \in \mathcal{P}_n),$$

and Bernstein in 1912 proved that

$$||P'||_\infty \leqslant n||P|| \qquad \Big(||f|| = \max_{|z| \leqslant 1} |f(z)|, \ P \in \mathcal{P}_n\Big).$$

When combined these results yield

$$(1) \qquad |P'(t)| \leqslant \min\Big\{n^2, \frac{n}{\sqrt{1 - t^2}}\Big\} \qquad (-1 \leqslant t \leqslant 1).$$

Excluding only certain particular cases, the best constant $A_n(k, p)$ in a general Markov type inequality

$$||P^{(k)}||_p \leqslant A_n(k, p)||P||_p \quad (1 \leq k \leq n)$$

in L^p-norm $(p \geqslant 1)$ is still not known, even for $p = 2$ and $w(t) = 1$ on $[-1, 1]$. The case $p = 2$ was independently investigated by Dörfler (1987) and GVM (1987). GVM obtained the best constant in terms of eigenvalues of a five diagonal matrix and reduced it to two sequences of some orthogonal polynomials.

Guessab and Milovanović (1994) considered a weighted L^2-analogue of Bernstein's inequality which can be stated as

$$||\sqrt{1 - t^2}\, P'(t)||_\infty \;\leqslant\; n||P||_\infty.$$

Using the norm $||f||^2 = (f, f)$ with a classical weight $w(t)$ they determined the best constant $C_{n,m}(w)$ $(1 \leqslant m \leqslant n)$ in the inequality

(2) $$||A^{m/2}P^{(m)}|| \;\leqslant\; C_{n,m}(w)||P||,$$

where A is defined before. Namely for all polynomials $P \in \mathcal{P}$ the inequality (2) holds true, with the best constant

$$C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}},$$

where $\lambda_{n,k} = -(n - k)\,(1/2 \cdot (m + k - 1)A''(0) + B'(0))$. Equality holds in (2) if and only if P is a multiple of the classical polynomial $Q_n(t)$ orthogonal with respect to the weight function $w(t)$.

Agarwal and Milovanović (1991) proved that, for all polynomials $P \in \mathcal{P}$,

$$\left(2\lambda_n + B'(0)\right)||\sqrt{A}\, P'||^2 \;\leqslant\; \lambda_n^2||P||^2 + ||AP''||^2,$$

with equality again iff P is a multiple of the classical polynomial $Q_n(t)$ orthogonal with respect to the weight function $w(t)$. Here, $\lambda_n = \lambda_{n,0}$.

Extremal problems of Markov-Bernstein type and corresponding inequalities have recently attracted the interest of GVM, who jointly with his colleagues Narendra Govil and Robert Gardner is currently working on the completion of a monograph on this subject which is to be published by Elsevier.

1.2.1. L^2–inequalities with Laguerre measure for nonnegative polynomials

Varma (1981) investigated the problem of determining the best constant $C_n(\alpha)$ in the L^2–inequality

$$||P'||^2 \;\leqslant\; C_n(\alpha)||P||^2$$

for polynomials with nonnegative coefficients, with respect to the generalized Laguerre weight function $w(t) = t^\alpha e^{-t}$ $(\alpha > -1)$ on $[0, \infty)$. If P_n is an algebraic polynomial of degree n with nonnegative coefficients, then for $\alpha \geqslant \frac{1}{2}(\sqrt{5}-1)$ Varma proved that

$$\int_0^\infty (P_n'(t))^2 w(t)\, \mathrm{d}t \leqslant \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} \int_0^\infty (P_n(t))^2 w(t)\, \mathrm{d}t.$$

The equality holds for $P_n(t) = t^n$. For $0 \leqslant \alpha \leqslant 1/2$

(3)
$$\int_0^\infty (P_n'(t))^2 w(t)\, \mathrm{d}t \leqslant \frac{1}{(2+\alpha)(1+\alpha)} \int_0^\infty (P_n(t))^2 w(t)\, \mathrm{d}t.$$

Moreover (3) is the best possible in the sense that for $P_n(t) = t^n + \lambda t$ the expression on the left-hand side of (3) can be made arbitrarily close to the one on the right-hand side if λ is sufficienlty large. The ranges $\alpha < 0$ and $1/2 < \alpha < 1/2 \cdot (\sqrt{5}-1)$ are not covered by Varma's results. This gap was filled by GVM (1985). He determined

(4)
$$C_n(\alpha) \;=\; \sup_{P \in W_n} \frac{||P'||^2}{||P||^2}$$

for all $\alpha \in (-1,\ \infty)$, where

$$W_n := \left\{ P \ \middle| \ P(t) = \sum_{\nu=0}^n a_\nu t^\nu,\ a_0 \geqslant 0,\ a_1 \geqslant 0,\ \ldots,\ a_{n-1} \geqslant 0,\ a_n > 0 \right\}.$$

There are several other results obtained by GVM, including extremal problems for higher derivatives, other weight functions, as well as dirrerent metrics.

1.2.2. Extremal problems for the Lorentz class of polynomials

Extremal problems for the Lorentz class of polynomials with respect to the Jacobi weight $w(t) = (1-t)^\alpha (1+t)^\beta$, $\alpha, \beta > -1$, were investigated by Milovanović and Petković (1988). Let L_n be the Lorentz class of polynomials

$$P(t) \;:=\; \sum_{\nu=0}^n b_\nu (1-t)^\nu (1+t)^{n-\nu} \quad (b_\nu \geqslant 0 \ \text{ for } \ \nu = 0, 1, \ldots, n).$$

In their work they determined the best constant $C_n^{(k)}(\alpha, \beta) \;:=\; \sup ||P'||^2/||P||^2$, where the supremum is over the polynomials from L_n for which $P^{(i-1)}(\pm 1) = 0$ for $i = 1, \ldots, k$. A particular result was already obtained by Erdős and Varma (1986).

1.3. Orthogonal polynomials on radial rays

Milovanović (1997, 2002) studied orthogonal polynomials on radial rays in the complex plane and presented several applications of his results. Let $M \in \mathbb{N}$ and

$$a_s > 0, \quad s = 1, 2, \ldots, M, \quad 0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_M < 2\pi.$$

Consider points $z_s = a_s \varepsilon_s, \varepsilon_s = \mathrm{e}^{\mathrm{i}\theta_s}$ $(s = 1, 2, \ldots, M)$ and define the inner product

$$(f, g) := \sum_{s=1}^M \mathrm{e}^{-\mathrm{i}\theta_s} \int_{\ell_s} f(z)\overline{g(z)}\, |w_s(z)|\, \mathrm{d}z,$$
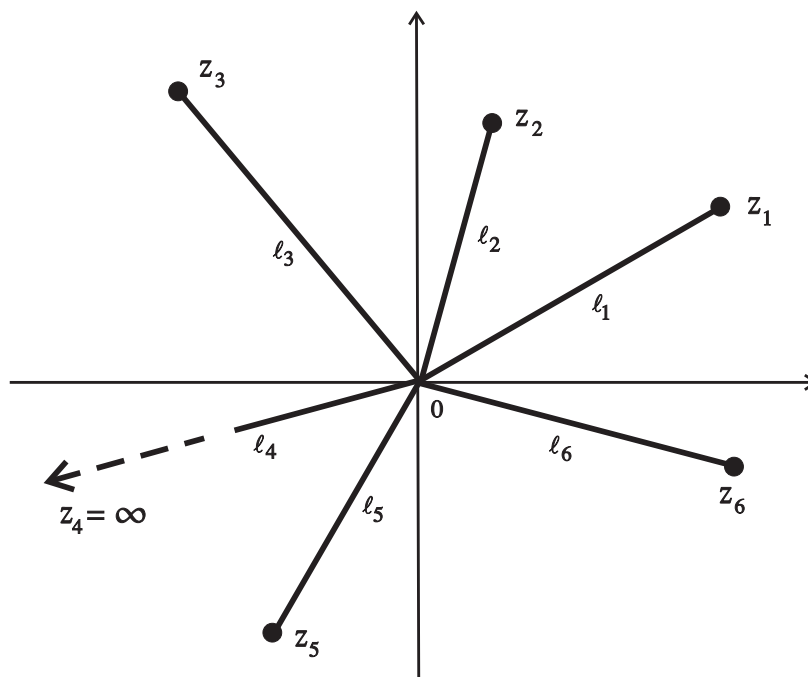
Figure 4: The rays in the complex plane ($M = 6$)

where ℓ_s are the radial rays in the complex plane which connect the origin and the points z_s, while $w_s(z)$ are suitable complex weights. The case $M = 6$ is shown in Fig. 4. Precisely, $\omega_s(x) = |w_s(z)| = |w_s(x\varepsilon_s)|$ are weight functions on $(0, a_s)$. One can write this as

$$(f, g) := \sum_{s=1}^{M} \int_0^{a_s} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\omega_s(x)\,\mathrm{d}x,$$

and for $M = 2, \theta_1 = 0, \theta_2 = \pi$,

$$(f, g) = \int_0^{a_1} f(x)\overline{g(x)}\omega_1(x)\,\mathrm{d}x + \int_0^{a_2} f(-x)\overline{g(-x)}\omega_2(x)\,\mathrm{d}x,$$

that is,

$$(f, g) = \int_a^b f(x)\overline{g(x)}\omega(x)\,\mathrm{d}x$$

with $a = -a_2,\ b = a_1$ and

$$\omega(x) = \begin{cases} \omega_1(x), & \text{if } 0 < x < b, \\ \omega_2(-x), & \text{if } a < x < 0. \end{cases}$$

GVM proved the existence and uniqueness of orthogonal polynomials on radial rays $\pi_N(z)$ ($N = 0, 1, 2, \ldots$). He considered the numerical construction of these

polynomials, recurrence relations, connections with standard polynomials orthogonal on $\mathbb{R}$, as well as many other properties, including some interesting classes of orthogonal polynomials with rays of equal lengths, distributed equidistantly in the complex plane, and with same weights on the rays.

In the symmetric case with even numbers of rays ($M = 2m$), GVM obtained analytic results for the recurrence coefficients for all classical weight functions (Jacobi, generalized Laguerre, Hermite). In the simple symmetric (Legendre) case with four rays ($M = 4$) and

$$(f,g) = \int_0^1 \left[f(x)\overline{g(x)} + f(\mathrm{i}x)\overline{g(\mathrm{i}x)} + f(-x)\overline{g(-x)} + f(-\mathrm{i}x)\overline{g(-\mathrm{i}x)} \right] \mathrm{d}x,$$

he proved the recurrence relation

$$\pi_{N+2}(z) = z^2 \pi_N(z) - b_N \pi_{N-2}(z), \quad N \geq 2; \quad \pi_N(z) = z^N, \ \ N \leq 3,$$

where the coefficient b_N ($N = 4n + \nu$; $n = [N/4]$) is given by

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n + 2\nu - 3)(8n + 2\nu + 1)} & \text{if } \nu = 0, 1, \\[4mm] \dfrac{(4n + 2\nu - 3)^2}{(8n + 2\nu - 3)(8n + 2\nu + 1)} & \text{if } \nu = 2, 3. \end{cases}$$

In the general case, using some kind of the discretized Stieltjes-Gautschi procedure, GVM numerically constructed the coefficients β_{kj} in the relation

$$\pi_k(z) = z\pi_{k-1}(z) - \sum_{j=1}^k \beta_{kj} \pi_{j-1}(z), \quad \beta_{kj} = \frac{(z\pi_{k-1}, \pi_{j-1})}{(\pi_{j-1}, \pi_{j-1})} \quad (1 \leq j \leq k).$$

Regarding the zero distribution of $\pi_N(z)$, Milovanović proved that *all the zeros of the orthogonal polynomial $\pi_N(z)$ lie in the convex hull of the rays $L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_M$.*

Among several examples we mention here one of his interesting examples for asymmetric case, with five rays ($M = 5$), defined by points in the complex plane (2018): $z_1 = 6$, $z_2 = 5\mathrm{e}^{9\pi\mathrm{i}/14}$, $z_3 = 2\mathrm{e}^{4\pi\mathrm{i}/5}$, $z_4 = 5\mathrm{e}^{6\pi\mathrm{i}/5}$, $z_5 = 5\mathrm{e}^{7\pi\mathrm{i}/4}$, with weight functions transformed to $(0, 1)$: $\omega_1(x) = 1$ (Legendre weight), $\omega_2(x) = 1/\sqrt{x(1 - x)}$ (Chebyshev weight of the first kind), $\omega_3(x) = \sqrt{x(1 - x)}$ Chebyshev weight of the second kind), $\omega_4(x) = \sqrt{x/(1 - x)}$ (Chebyshev weight of the fourth kind), $\omega_5(x) = \sqrt{(1 - x)/x}$ (Chebyshev weight of the third kind), respectively.

Zeros of $\pi_N(z)$ for $N = 20$ and $N = 100$ are presented in Figure 5.

Following Ivić [1] we mention here two of Milovanović's applications of these polynomials in Physics and Electrostatic. The first one is a physical problem connected with a non-linear diffusion equation, where the equations for the dispersion of a buoyant contaminant can be approximated by the Erdogan-Chatwin equation

$$\partial_t c = \partial_y \left\{ \left[D_0 + (\partial_y c)^2 D_2 \right] \partial_y c \right\},$$
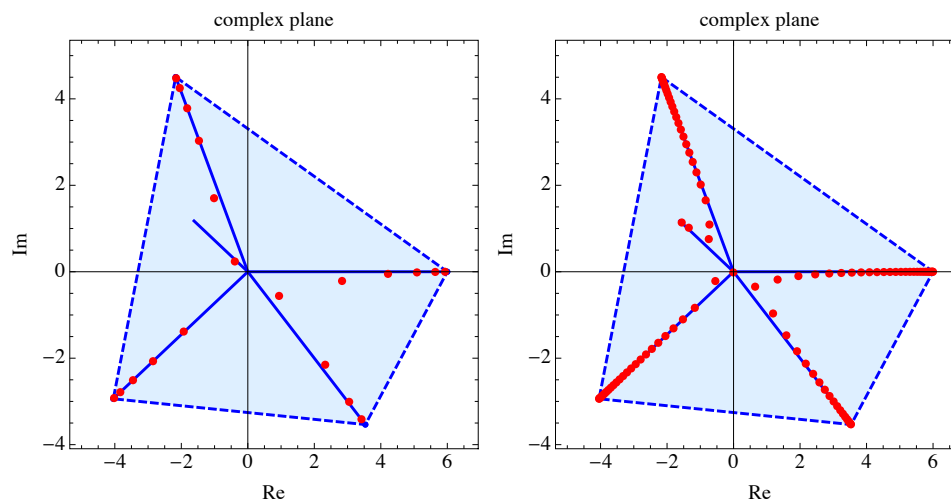
Figure 5: Zeros of $\pi_N(z)$ for $N = 20$ (left) and $N = 100$ (right)

where D_0 is the dispersion coefficient appropriate for neutrally-buoyant contaminants, and D_2 represents the increased rate of dispersion associated with the buoyancy-driven currents. Smith (1982) obtained analytic expressions for the similarity solutions of this equation in the limit of strong non-linearity ($D_0 = 0$), i.e.,

$$\partial_t c = D_2 \partial_y \Big[\big(\partial_y c \big)^3 \Big],$$

both for a concentration jump and for a finite discharge. He also investigated the asymptotic stability of these solutions. It is interesting that the stability analysis for the finite discharge involves a family of orthogonal polynomials $Y_N(z)$, such that

$$(1 - z^4) Y_N'' - 6z^3 Y_N' + N(N+5) z^2 Y_N = 0.$$

The degree N is restricted to the values $0, 1, 4, 5, 8, 9, \ldots$, so that the first few (monic) polynomials are:

$$1, \; z, \; z^4 - \frac{1}{3}, \; z^5 - \frac{5}{11} z, \; z^8 - \frac{14}{17} z^4 + \frac{21}{221}, \; z^9 - \frac{18}{19} z^5 + \frac{3}{19} z, \; \ldots \; .$$

These polynomials are a special case of Milovanović's polynomials orthogonal on the radial rays in the complex plane for $M = 4$ and $\omega(x) = (1 - x^4)^{1/2} x^2$.

The second application is an electrostatic interpretation of the zeros of polynomials $\pi_N(z)$. It is a nontrivial generalization of the first electrostatic interpretation of the zeros of Jacobi polynomials given by Stieltjes in 1885. Namely, *an electrostatic system of M positive point charges all of strength q, which are placed at fixed points ξ_s given by*

$$\xi_s = \exp\Big(\frac{2(s-1)\pi \mathrm{i}}{M} \Big) \quad (s = 1, 2, \ldots, M)$$

and a charge of strength p ($> -(M-1)/2$) at the origin $z = 0$, as well as N positive free unit charges, positioned at z_1, z_2, ..., z_N, is in electrostatic equilibrium if these points z_k are zeros of the polynomial $\pi_N(z)$ orthogonal with respect to the inner product

$$(f, g) = \int_0^1 \left(\sum_{s=1}^{M} f(x e_s) \overline{g(x e_s)} \right) \omega(x)\, dx,$$

with the weight function $\omega(x) = (1 - x^M)^{2q-1} x^{M+2(p-1)}$. This polynomial can be expressed in terms of the monic Jacobi polynomials

$$\pi_N(z) = 2^{-n} z^\nu \hat{P}_n^{(2q-1,(2p+2\nu-1)/M)}(2z^M - 1),$$

where $N = Mn + \nu$, $n = [N/M]$.

This area has recently been in the focus of Milovanović's interest again.

2. GVM AND INTERPOLATION PROCESSES AND INTEGRAL EQUATIONS

Interpolation of functions is one of the basic parts of Approximation Theory. There are many books on approximation theory, yet only a few of them are exclusively devoted to interpolation processes. The classical books on interpolation address numerous negative results, i.e., results on divergent interpolation processes, usually constructed over some equidistant system of nodes. In this section, following [1], we give an account of the recent comprehensive monograph "*Interpolation Processes: Basic Theory and Applications*" (Springer, 2008), written jointly by GVM and Giuseppe Mastroianni. This work is a crowning achievement of GVM's work in this expanding field. This new book of GVM and Mastroianni deals mainly with new results on convergent interpolation processes in uniform norm, for algebraic and trigonometric polynomials, not yet published in other textbooks and monographs on Approximation Theory and Numerical Mathematics. Basic tools in this field (orthogonal polynomials, moduli of smoothness, K-functionals, etc.), as well as some selected applications in numerical integration, integral equations, moment-preserving approximation and summation of slowly convergent series are also given.

The first chapter provides an account of basic facts on approximation by algebraic and trigonometric polynomials introducing the most important concepts regarding the approximation of functions.

The second chapter of this nice book is devoted to orthogonal polynomials on the real line and the weighted polynomial approximation. For polynomials orthogonal on the real line the authors give the basic properties and introduce and discuss the associated polynomials, functions of the second kind, Stieltjes polynomials, as well as the Christoffel functions and numbers. The *classical orthogonal polynomials* as the most important class of orthogonal polynomials on the real line are treated in detail, as well as new results on the so-called *nonclassical orthogonal polynomials*, including methods for their numerical construction. Introducing the weighted

functional spaces, moduli of smoothness and K-functionals, the weighted best polynomial approximations on $(-1, 1)$, $(0, +\infty)$ and $(-\infty, +\infty)$ are also treated, as well as the weighted polynomial approximation of functions having interior isolated singularities.

Trigonometric approximation is considered in Chapter 3. Approximations by sums of Fourier and Fejér and de la Vallée Poussin means are presented. Their discrete versions and the Lagrange trigonometric operator are also investigated. As a basic tool for studying approximating properties of the Lagrange and de la Vallée Poussin operators the authors consider the so-called Marcinkiewicz inequalities. Besides the uniform approximation they also investigate the Lagrange interpolation error in the L^p-norm $(1 < p < +\infty)$ and give some estimates in the L^1-Sobolev norm, including some weighted versions.

Chapter 4 treats algebraic interpolation processes $\{L_n(\mathcal{X})\}_{n \in \mathbb{N}}$ in the uniform norm, starting with the so-called optimal system of nodes $\mathcal{X}$, which provides Lebesgue constants of order $\log n$ and the convergence of the corresponding interpolation processes. Moreover, the error of these approximations is near to the error of the best uniform approximation. Beside two classical examples of the well-known optimal systems of nodes (zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(-1 < \alpha, \beta \leqslant -1/2)$ and the so-called Clenshaw's abscissas), they introduce more general results for constructing interpolation processes at nodes with an *arc sine distribution* having Lebesgue constants of order $\log n$.

The final chapter provides some selected applications in numerical analysis. In the first section on quadrature formulae they present some special Newton-Cotes rules, the Gauss-Christoffel, Gauss-Radau and Gauss-Lobatto quadratures, the so-called product integration rules, as well as a method for the numerical integration of periodic functions on the real line with respect to a rational weight function. Also, they include the error estimates of Gaussian rules for some classes of functions. The second section is devoted to methods for solving the Fredholm integral equations of the second kind. The methods are based on the so-called *Approximation and Polynomial Interpolation Theory* and lead to the construction of polynomial sequences converging to the exact solutions in some weighted uniform norms. Also, the authors consider some kinds of moment-preserving approximations by polynomials and splines, as well as two recent methods of summation of slowly convergent series based on integral representations of series and an application of Gaussian quadratures. We will write about these methods in more detail in the next section.

2.1. Fredholm integral equations of the second kind

Some basic facts on integral equations, in particular for Fredholm integral equations of the second kind (FK2), have been presented in the last chapter of the previous mentioned monograph "*Interpolation Processes: Basic Theory and Applications*" by Mastroianni and Milovanović (2008). In some classifications one can find the

so-called *Fredholm integral equations of the third kind* (FK3),

$$h(y)f(y) + \mu \int_A k(x,y)f(x)w(x)\,\mathrm{d}x = g(y), \quad y \in A,$$

where $k(x,y)$ is the kernel, w is a given weight function, g and h are known functions, $\mu \in \mathbb{R}$ is a parameter, and f is an unknown function.

If $h(y) \neq 0$ on A, then after dividing FK3 by h and absorbing it into k and g, FK3 reduces to FK2

(5) $$f(y) + \mu \int_A k(x,y)f(x)w(x)\,\mathrm{d}x = g(y), \quad y \in A.$$

However, if $h(y) = 0$ on A, FK3 reduces to FK1

$$\mu \int_A k(x,y)f(x)w(x)\,\mathrm{d}x = g(y), \quad y \in A.$$

In the literature many numerical methods have been proposed for solving integral equations. Sometimes, they are developed for specific type of kernels. In his research GVM, jointly with Mastroianni and sometimes with his associates, mainly investigated numerical methods for computing approximate solutions of some classes of Fredholm integral equations of the second kind. Such methods are based on *Approximation and Polynomial Interpolation Theory* and lead to the construction of a polynomial sequence converging to the exact solution in some weighted uniform norm. However, the construction of such a sequence requires the solution of systems of linear equations that might be ill-conditioned. In their approach, special attention is paid to providing well-conditioned systems of linear equations (except for the usual log factor) for Fredholm integral equations of the second kind (FK2) given by (5), as well as for its two-dimensional analogous (2013)

(6) $$f(\mathbf{y}) + \mu \int_{\mathcal{D}} k(\mathbf{x},\mathbf{y})f(\mathbf{x})w(\mathbf{x})\mathrm{d}\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} \in \mathcal{D}.$$

For example, in the cases when $A = [-1,1]$ and $A = [0,+\infty)$, with $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, and $w(x) = x^\alpha \mathrm{e}^{-x^\beta}$, $\alpha > -1$, $\beta > 1/2$, respectively, Mastroianni and GVM (2009) considered the corresponding Fredholm integral equations (5) in the spaces of continuous functions equipped with certain uniform weighted norms. Assuming the continuity of the kernel $k(x,y)$ they used Nyström methods and proved the stability, the convergence, as well as the well-conditioning of the corresponding matrices. The last property is derived only from the continuity of the kernel and not from its special form. Error estimates and numerical tests are also included.

An interesting class of Fredholm integral equations of the second kind (5), with respect to the exponential weight function $w(x) = \exp(-(x^{-\alpha} + x^\beta))$, $\alpha > 0$, $\beta > 1$, on $A = (0,+\infty)$, was considered by Mastroianni, GVM and Notarangelo (2017). The kernel $k(x,y)$ and the function $g(x)$ in such kind of equations can

grow exponentially with respect to their arguments, when they approach to 0^+ and/or $+\infty$. The authors proposed a simple and suitable Nyström-type method for solving these equations. The study of the stability and the convergence of this numerical method is based on the results on weighted polynomial approximation and "truncated" Gaussian rules presented in earlier papers of these authors. A priori error estimates are derived and some numerical examples are presented, as well as a comparison with other Nyström methods.

3. GVM AND QUADRATURE PROCESSES

Numerical integration begins with Newton's idea (1676) for finding the weight coefficients A_1, A_2, ..., A_n in the so-called *n-point quadrature formula*

$$(7) \qquad I(f) = \int_a^b f(t)\,\mathrm{d}t \approx Q_n(f) = A_1 f(\tau_1) + A_2 f(\tau_2) + \cdots + A_n f(\tau_n),$$

for given (usually equidistant) n points (nodes) τ_1, τ_2, ..., τ_n, such that (7) is exact for all algebraic polynomials of degree of precision d at most $n-1$, i.e., for each $f \in \mathcal{P}_{n-1}$. We write $d = d(Q_n) \geq n-1$.

Starting from the work of Newton and Cotes and combining it with his earlier work on the hypergeometric series, Gauss (1814) develops his famous method of numerical integration which dramatically improved the earlier method of Newton and Cotes. Today these formulae with maximal degree of precision d are known as the *Gauss-Christoffel quadrature formulae*. It has been proven in the meanwhile that the nodes τ_k, $k = 1, 2, \ldots, n$, are zeros of the polynomial of degree n, which is orthogonal to $\mathcal{P}_{n-1}$ with respect to a given measure $\mathrm{d}\mu(x)$.

3.1. Construction of Gaussian quadratures

Passing to modern theory, we mention some non-classical measures: $\mathrm{d}\mu(t) = w(t)\,\mathrm{d}t$ for which the recursion coefficients $\alpha_k(\mathrm{d}\mu)$, $\beta_k(\mathrm{d}\mu)$, $k = 0, 1, \ldots, n-1$, in the fundamental three-term recurrence relation for the corresponding (monic) orthogonal polynomials,

$$(8) \qquad \pi_{k+1}(t) = (t - \alpha_k(\mathrm{d}\mu))\pi_k(t) - \beta_k(\mathrm{d}\mu)\pi_{k-1}(t), \quad k = 0, 1, \ldots,$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$, have been provided in the literature and used in the construction of the Gaussian quadratures,

$$\int_{\mathbb{R}} f(t)w(t)\mathrm{d}t = \sum_{k=1}^n A_k f(\tau_k) + R_n(f),$$

using the Golub-Welsch algorithm (1969). For example, such well-known weight functions are:

1. *One-sided Hermite weight* $w(t) = \exp(-t^2)$ on $[0, c]$, $0 < c \leqslant +\infty$.

2. *Logarithmic weight* $w(t) = t^\alpha \log(1/t)$, $\alpha > -1$ on $(0,1)$.

3. *Airy weight* $w(t) = \exp(-t^3/3)$ on $(0,+\infty)$.

4. *Reciprocal gamma function* $w(t) = 1/\Gamma(t)$ on $(0,+\infty)$.

5. *Bose–Einstein's* and *Fermi's weight functions* on $(0,+\infty)$,

$$w_1(t) = \varepsilon(t) = \frac{t}{e^t - 1} \quad \text{and} \quad w_2(t) = \varphi(t) = \frac{1}{e^t + 1}\,.$$

For $w_1(t)$, $w_2(t)$, $w_3(t) = \varepsilon(t)^2$ and $w_4(t) = \varphi(t)^2$, Gautschi and Milovanović (1985) performed the first systematic investigation on the derivation of quadrature rules with high-precision, determined the recursion coefficients α_k and β_k, and presented an application of the corresponding Gauss-Christoffel quadratures to the summation of slowly convergent series, whose general term is expressible in terms of a Laplace transform or its derivative (the *method of Laplace transform*). Moreover, integrals with these weights frequently occur in connection with the evaluation, in the independent particle approximation, of thermodynamic variables for solid state physics problems both for boson systems (which associate measures $d\mu(t) = \varepsilon(t)dt$) and for fermion systems (which associate measures $d\mu(t) = \varphi(t)dt$).

6. *The hyperbolic weights* on $(0,+\infty)$,

$$(9) \qquad\qquad w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t}\,.$$

The recursion coefficients α_k, β_k, for $k < 40$ were obtained by Milovanović (1995). The main application of these quadratures is the summation of slowly convergent series, with the general term $a_k = f(k)$. Such a method known as the *method of contour integration* was given by GVM (1995).

Recent progress in *symbolic computation* and *variable–precision arithmetic* now makes it possible to generate the coefficients α_k and β_k in the three–term recurrence relation (8) directly by using the original Chebyshev method of moments in sufficiently high precision or even in symbolic form. Respectively symbolic/variable–precision software for orthogonal polynomials is available: Gautschi's package SOPQ in MATLAB and the MATHEMATICA package OrthogonalPolynomials, developed by Cvetković and Milovanović (2004). Thus, all that is required is a procedure for symbolic or numerical calculation of the moments in variable–precision arithmetic. Such an approach enables us to overcome the numerical instability.

For example, GVM and Cvetković (2012) determined the moments for the previous weight functions.

The moments of the Bose-Einstein's weight function can be calculated exactly

$$\mu_k(\varepsilon) = \int_0^{+\infty} t^k \varepsilon(t) dt = (k+1)! \zeta(k+2), \quad k \in \mathbb{N}_0,$$

where the zeta function can be evaluated to arbitrary numerical precision, while the moments of the Fermi-Dirac function as

$$\mu_k(\varphi) = \int_0^{+\infty} \frac{t^k}{e^t + 1}\,\mathrm{d}t = \begin{cases} \log 2, & k = 0, \\ (1 - 2^{-k})k!\zeta(k+1), & k > 0. \end{cases}$$

A general problem with the weight function $w(t) = [\varepsilon(t)]^r$, where $r \in \mathbb{N}$, can be also considered in a similar way. In that case, the corresponding moments $\mu_k^{(r)}(\varepsilon)$, $r > 1$, can be obtained recursively by

$$\mu_k^{(r)}(\varepsilon) = \frac{k+r}{r-1}\,\mu_k^{(r-1)}(\varepsilon) - \mu_{k+1}^{(r-1)}(\varepsilon).$$

For example, $\mu_k^{(2)}(\varepsilon) = (k+2)![\zeta(k+2) - \zeta(k+3)]$, $k \in \mathbb{N}_0$

The moments for hyperbolic functions (9) are

$$(10) \qquad \mu_k^{(1)} = \int_0^{+\infty} t^k w_1(t)\mathrm{d}t = \begin{cases} 1, & k = 0, \\ \log 2, & k = 1, \\ C_k \zeta(k), & k \geq 2, \end{cases}$$

where $C_k = (2^{k-1} - 1)k!/4^{k-1}$, and

$$\mu_k^{(2)} = \int_0^{+\infty} t^k w_2(t)\mathrm{d}t = \begin{cases} 1, & k = 0, \\ k\left(\dfrac{\pi}{2}\right)^k |E_{k-1}|, & k\,(\text{odd}) \geq 1, \\ \dfrac{2k}{4^k}\left(\psi^{(k-1)}(1/4) - \psi^{(k-1)}(3/4)\right), & k\,(\text{even}) \geq 2, \end{cases}$$

where E_k are Euler's numbers, defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{k=0}^{+\infty} E_k \frac{t^k}{k!},$$

and $\psi(z)$ is the so–called digamma function, i.e., the logarithmic derivative of the gamma function, given by $\psi(z) = \Gamma'(z)/\Gamma(z)$.

7. *The weight $w^{(\alpha,\beta)}(t) = \exp(-t^{-\alpha} - t^\beta)$, $\alpha, \beta > 0$, on $(0, +\infty)$.* In the case $\alpha = \beta$, the moments are

$$\mu_k^{(\beta,\beta)} = \int_0^{+\infty} t^k w^{(\beta,\beta)}(t)\,\mathrm{d}t = \frac{2}{\beta}K_{(k+1)/\beta}(2), \quad k \in \mathbb{N}_0,$$

where $K_r(z)$ is the modified Bessel function of the second kind. In the MATHE-MATICA package this function is implemented as `BesselK[r,z]`, and its value can be evaluated with an arbitrary precision. GVM (2015) considered also the more complicated cases when $\alpha \neq \beta$, and obtained, that in some cases for integer (or

rational) values of parameters, the moments can be expressed in terms of the Meijer G–function, e.g.,

$$\mu_k^{(1,2)} = \frac{1}{2^{k+2}\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4} \,\middle|\, \begin{matrix} -\,;\,- \\ -\frac{k+1}{2}, -\frac{k}{2}, 0;\, - \end{matrix}\right), \quad k \geq 0;$$

$$\mu_k^{(2,1)} = \frac{2^k}{\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4} \,\middle|\, \begin{matrix} -\,;\,- \\ 0, \frac{k+1}{2}, \frac{k+2}{2};\, - \end{matrix}\right), \quad k \geq 0;$$

$$\mu_k^{(3,1)} = \frac{3^{k+1/2}}{2\pi} G_{2,5}^{4,1}\left(\frac{1}{27} \,\middle|\, \begin{matrix} -\,;\,- \\ 0, \frac{k+1}{3}, \frac{k+2}{3}, \frac{k+3}{3};\, - \end{matrix}\right), \quad k \geq 0;$$

$$\mu_k^{(1/2,3/2)} = \frac{1}{3^{2k+5/2}\pi} G_{2,5}^{4,1}\left(\frac{1}{27} \,\middle|\, \begin{matrix} -\,;\,- \\ -\frac{2k+2}{3}, -\frac{2k+1}{3}, -\frac{2k}{3}, 0;\, - \end{matrix}\right), \quad k \geq 0.$$

In some cases GVM obtained the recursive coefficients α_k and β_k in symbolic form, very often for some exotic weights.

8. *The Stieltjes-Wigert weight function*

$$w(x) := \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\dfrac{\log^2(x)}{2\sigma^2}\right], & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases}$$

has the moments given by

$$\mu_k = \int_0^{+\infty} x^k w(x)\mathrm{d}x = q^{k^2/2}, \quad k \in \mathbb{N}_0 \quad (q = \mathrm{e}^{\sigma^2}).$$

GVM (2015) obtained the coefficients in the three-term recurrence relation in an analytic form,

$$\alpha_k = q^{k-1/2}\left(q^{k+1} + q^k - 1\right); \quad \beta_0 = 1, \; \beta_k = q^{3k-2}\left(q^k - 1\right), \quad k = 0, 1, \dots .$$

8. For *the weight function* on $\mathbb{R}$ given by

$$w(x) = \frac{x^2 \mathrm{e}^{-\pi x}}{(1 - \mathrm{e}^{-\pi x})^2} = \left(\frac{x}{2\sinh(\pi x/2)}\right)^2 = \frac{1}{4}[w^A(x/2)]^2,$$

where $w^A(x)$ is the Abel weight on $\mathbb{R}$, GVM (2015) determined the moments in terms of Bernoulli numbers

$$\mu_k = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2} 2^{k+2} \dfrac{B_{k+2}}{\pi}, & k \text{ is even}, \end{cases}$$

and then the recurrence coefficients

$$\beta_0 = \mu_0 = \frac{2}{3\pi}, \quad \beta_k = \frac{k(k+1)^2(k+2)}{(2k+1)(2k+3)}, \quad k \in \mathbb{N}.$$

9. Similarly, for *the weight function* on $\mathbb{R}$, given by

$$w(x) = x^2 \frac{\mathrm{e}^{\pi x/2} + \mathrm{e}^{-\pi x/2}}{(\mathrm{e}^{\pi x/2} - \mathrm{e}^{-\pi x/2})^2} = 2 \cosh \frac{\pi x}{2} \left(\frac{x}{2 \sinh(\pi x/2)} \right)^2,$$

GVM (2015) obtained the moments

$$\mu_k = \begin{cases} 0, & k \text{ is odd,} \\ \dfrac{2^{k+3}}{\pi}(2^{k+2} - 1)|B_{k+2}|, & k \text{ is even,} \end{cases}$$

and then the recurrence coefficients

$$\beta_0 = \mu_0 = \frac{4}{\pi}, \quad \beta_k = \begin{cases} (k+1)^2, & k \text{ is odd,} \\ k(k+2), & k \text{ is even.} \end{cases}$$

10. Recently GVM (Numer. Algorithms, 2017) has considered symbolic-numeric computation of orthogonal polynomials and Gaussian quadratures with respect to the cardinal B-spline.

3.2. Moment-preserving spline approximation and quadratures

An interesting application of Gaussian–type formulas concerns the so-called *moment-preserving spline approximation* of a given function f on $[0, +\infty)$ (or on a finite interval, e.g. $[0, 1]$). Such kind of problems appeared in Physics, for example in the approximation of the Maxwell velocity distribution by a linear combination of Dirac δ-functions or in the corresponding approximation by a linear combination of Heaviside step functions. In order to get a stable method for this kind of approximation, Gautschi and Milovanović (1986) found new applications of Gaussian type of quadratures.

Let f be a given function defined on the positive real line $\mathbb{R}_+ = [0, +\infty)$ and $s_{n,m}$ be a spline of the form

$$s_{n,m}(t) = \sum_{\nu=1}^{n} a_\nu (t_\nu - t)_+^m, \quad 0 \leqslant t < +\infty,$$

where the plus sign on the right is the cutoff symbol, meaning that $u_+ = u$ if $u > 0$ and $u_+ = 0$ if $u \leqslant 0$, $0 < t_1 < \cdots < t_n$, $a_\nu \in \mathbb{R}$. They considered the moment-preserving spline approximation $f(t) \approx s_{n,m}(t)$ such that

$$\int_0^{+\infty} s_{n,m}(t) t^j \mathrm{d}V = \int_0^{+\infty} f(t) t^j \mathrm{d}V, \quad j = 0, 1, \ldots, 2n - 1,$$

where $\mathrm{d}V$ is the volume element depending on the geometry of the problem. In some concrete applications in Physics, up to unimportant numerical factors, $\mathrm{d}V = t^{d-1}\mathrm{d}t$, where $d = 1, 2, 3$ for rectilinear, cylindric, and spherical geometry, respectively.

For fixed $n, m \in \mathbb{N}$, $d \in \{1, 2, 3\}$ and certain conditions on f they proved (1986) that the spline function $s_{n,m}$ exists uniquely if and only if the measure

$$d\lambda_m(t) = \frac{(-1)^{m+1}}{m!} t^{m+d} f^{(m+1)}(t) dt \quad \text{on} \quad [0, +\infty)$$

admits an n-point Gauss-Christoffel quadrature formula

$$\int_0^{+\infty} g(x) \, d\lambda_m(x) = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} g(\tau_\nu^{(n)}) + R_n(g; d\lambda_m)$$

with distinct positive nodes $\tau_\nu^{(n)}$, where $R_n(g; d\lambda_m) = 0$ for all $g \in \mathcal{P}_{2n-1}$.

Approximation on a compact interval was considered by Frontini, Gautschi and Milovanović (1987).

3.3. Summation of slowly convergent series

There are many methods for fast summation of slowly convergent series. GVM mainly worked on the so-called *summation/integration* procedures. The basic idea in such procedures is to transform the sum to an integral with respect to some weight function on $\mathbb{R}$ (or $\mathbb{R}_+$), and then to approximate this integral by a finite quadrature sum,

$$\sum_{k=1}^{+\infty} (\pm 1)^k f(k) = \int_{\mathbb{R}} g(x) w(x) dx \approx \sum_{\nu=1}^{N} A_\nu g(x_\nu),$$

where the function g is connected with f in some way. Thus, these procedures need two steps:

(a) Methods of transformation $\sum \Rightarrow \int$;

(b) Construction of Gaussian quadratures

$$\int_{\mathbb{R}} g(x) w(x) dx = \sum_{\nu=1}^{N} A_\nu g(x_\nu) + R_n(f),$$

where w is a *non-classical* weight.

3.3.1. Method of Laplace transform

In this part we mention only the basic idea of the *method of Laplace transform*.

Suppose that the general term of series is expressible in terms of the Laplace transform, or its derivative, of a known function.

Let $f(s) = \int_0^{+\infty} e^{-st} g(t) dt$, $\operatorname{Re} s \geq 1$. Then

$$T = \sum_{k=1}^{+\infty} f(k) = \sum_{k=1}^{+\infty} \int_0^{+\infty} e^{-kt} g(t) dt = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} e^{-kt} \right) g(t) dt,$$

i.e.,

$$T = \int_0^{+\infty} \frac{e^{-t}}{1 - e^{-t}} g(t)\mathrm{d}t = \int_0^{+\infty} \frac{t}{e^t - 1} \frac{g(t)}{t}\mathrm{d}t.$$

Thus, the *summation of series* is now transformed to an *integration problem* with respect to the Bose-Einstein weight function $\varepsilon(t) = t/(e^t - 1)$ on $\mathbb{R}^+$, which is considered by Gautschi and GVM (1985) (se also Subsection 3.1).

Similarly, for "alternating" series, we have

(11)
$$S = \sum_{k=1}^{+\infty}(-1)^k f(k) = \int_0^{+\infty} \frac{1}{e^t + 1}(-g(t))\mathrm{d}t,$$

where the Fermi-Dirac weight function on $\mathbb{R}^+$, $\varphi(t) = 1/(e^t + 1)$, appears on the right-hand side in (11).

3.3.2. Hyperbolic weight functions and $\sum \Rightarrow \int$ transformation

In this part we present an idea of GVM from 1994 on an alternative summation/integration procedure for the series

(12)
$$T_{m,n} = \sum_{k=m}^{n} f(k) \quad \text{and} \quad S_{m,n} = \sum_{k=m}^{n}(-1)^k f(k),$$

where $m, n \in \mathbb{Z}$ ($m < n \leq +\infty$) and the function f is holomorphic in the region

(13)
$$\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha, \; m - 1 < \alpha < m\}.$$

The method of transformation "sum" to "integral" requires the indefinite integral F of f chosen so as to satisfy the following decay properties,

(C1) F *is a holomorphic function in the region* (13);

(C2) $\lim\limits_{|t| \to +\infty} e^{-c|t|} F\left(x + \mathrm{i}t/\pi\right) = 0$, *uniformly for $x \geq \alpha$;*

(C3) $\lim\limits_{x \to +\infty} \int_{\mathbb{R}} e^{-c|t|}\left|F\left(x + \mathrm{i}t/\pi\right)\right|\mathrm{d}t = 0$,

where $c = 2$ or $c = 1$, when we consider $T_{m,n}$ or $S_{n,m}$, respectively.

Let $m - 1 < \alpha < m$, $n < \beta < n + 1$, $\delta > 0$, and

$$G = \left\{z \in \mathbb{C} \; : \; \alpha \leq \operatorname{Re} z \leq \beta, \; |\operatorname{Im} z| \leq \frac{\delta}{\pi}\right\}.$$

Using contour integration of a product of functions $z \mapsto f(z)g(z)$ over the rectangle $\Gamma = \partial G$ in the complex plane, where $g(z) = \pi/\tan \pi z$ and $g(z) = \pi/\sin \pi z$, by Cauchy's residue theorem, GVM obtained

$$T_{m,n} = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma f(z)\frac{\pi}{\tan \pi z}\mathrm{d}z \quad \text{and} \quad S_{m,n} = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma f(z)\frac{\pi}{\sin \pi z}\mathrm{d}z.$$

After integration by parts, these formulas reduce to

$$T_{m,n} = \frac{1}{2\pi i} \oint_\Gamma \left(\frac{\pi}{\sin \pi z}\right)^2 F(z) \mathrm{d}z, \quad S_{m,n} = \frac{1}{2\pi i} \oint_\Gamma \left(\frac{\pi}{\sin \pi z}\right)^2 \cos \pi z \, F(z) \mathrm{d}z,$$

where F is an integral of f.

Finally, setting $\alpha = m - 1/2$, $\beta = n + 1/2$, and letting $\delta \to +\infty$, under conditions (C1), (C2), and (C3), the previous integrals over Γ reduce to the weighted integrals over $(0, +\infty)$, which yield the transformations

$$(14) \qquad \sum_{k=m}^{+\infty} f(k) = \int_0^{+\infty} w_1(t) \Phi\left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t$$

and

$$(15) \qquad \sum_{k=m}^{+\infty} (-1)^k f(k) = (-1)^m \int_0^{+\infty} w_2(t) \Psi\left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t,$$

where the weight functions are given by

$$(16) \qquad w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t},$$

respectively. Here F is an integral of f, as well as

$$\Phi(x, y) = -\frac{1}{2} \left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y)\right] = -\operatorname{Re} F(x + \mathrm{i}y)$$

and

$$\Psi(x, y) = \frac{1}{2\mathrm{i}} \left[F(x + \mathrm{i}y) - F(x - \mathrm{i}y)\right] = \operatorname{Im} F(x + \mathrm{i}y).$$

The second task is a numerical construction of Gaussian quadratures with respect to the hyperbolic weights w_1 and w_2, defined in (16),

$$(17) \qquad \int_0^{+\infty} g(t) w_s(t) \mathrm{d}t = \sum_{\nu=1}^N A_{\nu,s}^N g(\tau_{\nu,s}^N) + R_{N,s}(g) \quad (s = 1, 2),$$

with weights $A_{\nu,s}^N$ and nodes $\tau_{\nu,s}^N$, $\nu = 1, \ldots, N$ ($s = 1, 2$), which are exact for all $g \in \mathcal{P}_{2N-1}$, was solved by GVM (1994).

The moments of the hyperbolic weights w_1 and w_2 have been presented earlier in Subsection 3.1. The convergence of the previous quadrature formulas (17) is very fast for sufficiently regular integrands.

3.4. Quadratures with multiple nodes and error estimate

One may consider *Quadratures with multiple nodes*, where $\eta_1, \ldots, \eta_m$ ($\eta_1 < \cdots < \eta_m$) are given *fixed* (or *prescribed*) nodes, with multiplicities $m_1, \ldots, m_m$, and

$\tau_1, \ldots, \tau_n$, with $\tau_1 < \cdots < \tau_n$, are *free* nodes, with given multiplicities $n_1, \ldots, n_n$, respectively. The quadrature formula is

$$(18) \qquad I(f) = \int_{\mathbb{R}} f(t) \, \mathrm{d}\lambda(t) \cong Q(f),$$

where

$$(19) \qquad Q(f) = \sum_{\nu=1}^{n} \sum_{i=0}^{n_\nu - 1} A_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\nu=1}^{m} \sum_{i=0}^{m_\nu - 1} B_{i,\nu} f^{(i)}(\eta_\nu),$$

with an algebraic degree of exactness at least $M + N - 1$ where $M = \sum_{\nu=1}^{m} m_\nu$ and $N = \sum_{\nu=1}^{n} n_\nu$. If $m = 0$ and the last quadrature formula is of Gaussian type, i. e., it has the maximal degree of exactness $N + n - 1$ (n_ν are positive integers), then it is called Chakalov-Popoviciu quadrature formula, and in the special case (n_ν are equal positive integers) it is called Turán quadrature formula. It has been proved (Stancu) that the Chakalov-Popoviciu quadrature formula is based on the zeros of a σ-orthogonal, and the Turán quadrature formula is based on the zeros of an s-orthogonal polynomial, respectively.

At the Third Conference on Numerical Methods and Approximation Theory (Niš, 1987) GVM presented a stable method with quadratic convergence for numerically constructing s-orogonal polynomials, whose zeros are nodes of Turán quadratures. The basic idea of the method of numerically constructing s-orthogonal polynomials with respect to the measure $\mathrm{d}\mu(t)$ on the real line $\mathbb{R}$ is a reinterpretation of the *s-orthogonality* in terms of implicitly defined standard orthogonality. Further progress in this direction was made by Gautschi and Milovanović (1997). After Milovanović's survey (2001), where he established a connection between quadratures, s and σ-orthogonality and moment-preserving approximation with defective splines, the interest for this subject rapidly increased. A very efficient method for constructing quadratures with multiple nodes was given recently by Milovanović, Spalević and Cvetković (2004).

For Turán and Chakalov-Popoviciu quadrature formulas (with $m = 0$ in (19)), including their Kronrod, Radau and Lobatto extensions, of integrands analytic on the interior of a simple closed curve Γ in the complex plane encompassing the interval $[-1, 1]$, Milovanović and Spalević (jointly with their younger collaborators Pranić and Pejčev) performed a detailed and rigorous analysis for effective error estimates.

Therefore, let Γ be a simple closed curve in the complex plane encompassing the interval $[-1, 1]$ and let $\mathcal{D}$ be its interior. Here in (18), $\mathrm{d}\lambda(t) = w(t) \, \mathrm{d}t$ and the closed support is the interval $[-1, 1]$. Suppose f is a function that is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$. Taking any system of n distinct points $\{\tau_1, \ldots, \tau_n\}$ in $\mathcal{D}$ (in particular in $(-1, 1)$) and n positive integers $n_1, \ldots, n_n$, the error in the Hermite interpolating polynomial of f at a point $t \in \mathcal{D}$ (in particular in $(-1, 1)$) can be expressed in the form (Gončarov (1954), Mysovskih (1969))

$$(20) \qquad r_n(f; t) = f(t) - \sum_{\nu=1}^{n} \sum_{i=0}^{n_\nu - 1} l_{\nu,i} f^{(i)}(\tau_\nu) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{f(z) \Omega_n(t)}{(z - t) \Omega_n(z)} \, \mathrm{d}z,$$

where $l_{\nu,i}(t)$ are the fundamental functions of Hermite interpolation and

$$\Omega_n(z) = \prod_{\nu=1}^{n}(z - \tau_\nu)^{n_\nu}.$$

By multiplying (20) by the weight function $w(t)$ and integrating in t over $(-1, 1)$ we get a contour integral representation for the remainder term $R_n(f)$ in a quadrature formula with multiple nodes:

$$(21) \qquad R_n(f) = I(f) - \sum_{\nu=1}^{n}\sum_{i=0}^{n_\nu-1} A_{i,\nu} f^{(i)}(\tau_\nu) = \frac{1}{2\pi\,\mathrm{i}} \oint_\Gamma K_n(z; w) f(z)\, \mathrm{d}z,$$

where now, the integral (18) has the form $I(f) = \int_{-1}^{1} f(t)w(t)\, \mathrm{d}t$,

$$A_{i,\nu} = \int_{-1}^{1} l_{\nu,i}(t)w(t)\, \mathrm{d}t$$

and the *kernel* $K_n(z) = K_n(z; w)$ is given by

$$K_n(z; w) = \frac{\varrho_n(z; w)}{\Omega_n(z)}, \quad \varrho_n(z; w) = \int_{-1}^{1} \frac{\Omega_n(t)}{z - t}\, w(t)\, \mathrm{d}t, \quad z \in \mathbb{C} \setminus [-1, 1].$$

The integral representation (21) leads directly to the error estimate

$$(22) \qquad |R_n(f)| \le \frac{l(\Gamma)}{2\pi} \left(\max_{z \in \Gamma} |K_n(z)| \right) \left(\max_{z \in \Gamma} |f(z)| \right),$$

where $l(\Gamma)$ is the length of the contour Γ.

On the other hand, one can consider also the error bound:

$$(23) \qquad |R_n(f)| \le \frac{1}{2\pi} \left(\oint_\Gamma |K_n(z)|\,|\mathrm{d}z| \right) \left(\max_{z \in \Gamma} |f(z)| \right),$$

which is evidently stronger than (22) since

$$\oint_\Gamma |K_n(z)|\,|\mathrm{d}z| \le l(\Gamma) \left(\max_{z \in \Gamma} |K_n(z)| \right).$$

In order to obtain the estimates (22) and (23) one needs to study the magnitude of $|K_n(z)|$ on Γ or the quantity

$$L_n(\Gamma) := \frac{1}{2\pi} \oint_\Gamma |K_n(z)|\,|\mathrm{d}z|,$$

respectively.

A common choice for the contour Γ is one of the confocal ellipses with foci at the points ∓ 1 and the sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} :\ z = \frac{1}{2} \left(\rho\mathrm{e}^{\mathrm{i}\theta} + \rho^{-1}\mathrm{e}^{-\mathrm{i}\theta} \right),\ 0 \le \theta < 2\pi \right\}.$$

For such Γ Milovanović and Spalević studied the estimates (22) and (23) (2003, 2005), including some that are based on the expansion of the error term $R_n(f)$ in series, for the Gauss-Turán quadrature formulas ($n_\nu = 2s+1$, $\nu = 1, 2, \ldots, n$; $s \in \mathbb{N}_0$) when w is one of four generalized Chebyshev weight functions. These results were generalisations of the analogous ones obtained for the standard Gauss quadratures and their error estimates by Gautschi and Varga (1983, 1990) and Hunter (1995). Milovanović and Spalević published on the subject a sequence of more than 10 papers in the period 2003-2010. In two of their papers (2005) and (2007) on the subject, they use circles instead of ellipses. In 2013 they, jointly with Pejčev, solved a conjecture posted in the paper from 2003, proving in that way a general error bound based on expansion of the error term $R_n(f)$ in the series, for the Gauss-Turán quadrature formula with the Chebyshev weight function of the first kind.

In respect to the quadratures with multiple nodes, we mention here the recent results by Milovanović and Spalević (2014, 2019). They continued with analyzing quadrature formulas of high degree of precision for computing the Fourier coefficients in expansions of functions with respect to a system of orthogonal polynomials, initiaded recently by Bojanov and Petrova (2009). Milovanović and Spalević extend the results by Bojanov and Petrova. Construction of new Gaussian quadrature formulas for the Fourier coefficients of a function, based on the values of the function and its derivatives, is considered. Milovanović and Spalević proved the existence and uniqueness of Kronrod extensions with multiple nodes of standard Gaussian quadrature formulas with multiple nodes for several weight functions, in order to construct some new generalizations of quadrature formulas for the Fourier coefficients. For the quadrature formulas for the Fourier coefficients based on the zeros of the corresponding orthogonal polynomials they construct Kronrod extensions with multiple nodes and highest algebraic degree of precision. For this very desirable kind of extensions there do not exist any results in the theory of standard quadrature formulas. A numerically stable procedure for the construction of some quadrature formulas with multiple nodes for Fourier coefficients has been proposed recently by them jointly with R. Orive (2019). Finally, we mention a recent survey paper by Milovanović, Pranić and Spalević (2019) dealing with new contributions to the theory of Gaussian quadrature formulas with multiple nodes published after 2001, including numerical construction, error analysis, and applications. The first part was published in "Numerical analysis 2000" by GVM.

3.5. Orthogonality with respect to a moment functional and corresponding quadratures

In the previous sections the inner product was always positive definite provided the existence of the corresponding orthogonal polynomials with real zeros in the support of the measure. Such zeros appeared as the nodes of the Gaussian formulas. However, there are more general concepts of orthogonality with respect to a given linear moment functional L on the linear space $\mathcal{P}$ of all algebraic polynomials. Due to linearity, the value of the linear functional L at every polynomial is known if the

values of L are known at the set of all monomials, i.e., if we know $L(x^k) = \mu_k$, for each $k \in \mathbb{N}_0$. In that case we can introduce a system of orthogonal polynomials $\{\pi_k\}_{k \in \mathbb{N}_0}$ with respect to the functional L if for all nonnegative integers k and n we have that $\pi_k(x)$ is a polynomial of degree k, $L(\pi_k(x)\pi_n(x)) = 0$, if $k \neq n$, $L(\pi_n^2(x)) \neq 0$.

3.5.1. Orthogonality on the semicircle and quadratures

Let w be a weight function which is positive and integrable on the open interval $(-1, 1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \ \mathrm{Im}\, z > 0\}$. Consider the following "inner product":

$$(f, g) = \int_\Gamma f(z) g(z) w(z) (\mathrm{i}z)^{-1} \mathrm{d}z = \int_0^\pi f(\mathrm{e}^{\mathrm{i}\theta}) g(\mathrm{e}^{\mathrm{i}\theta}) w(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta,$$

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist (possibly) as appropriately defined improper integrals. The existence of the corresponding orthogonal polynomials $\{\pi_n\}_{n \in \mathbb{N}_0}$ is not guaranteed.

The case $w = 1$ was considered by Gautschi and Milovanović (1986). The existence and uniqueness of polynomials orthogonal on the unit semicircle were proved.

A more general case of the complex weight was considered by Gautschi, Landau and Milovanović (1987). Under the condition $\mathrm{Re}\, \mu_0 = \mathrm{Re} \int_0^\pi w(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta \neq 0$, they proved that the orthogonal polynomials $\{\pi_n\}_{n \in \mathbb{N}_0}$ exist uniquely and that they can be represented in the form

$$\pi_n(z) = p_n(z) - \mathrm{i}\theta_{n-1} p_{n-1}(z),$$

where

$$\theta_{n-1} = \theta_{n-1}(w) = \frac{\mu_0 p_n(0) + \mathrm{i}q_n(0)}{\mathrm{i}\mu_0 p_{n-1}(0) - q_{n-1}(0)}.$$

Here the p_k's are standard (real) polynomials orthogonal with respect to the inner product

$$[f, g] = \int_{-1}^1 f(x)\overline{g(x)} w(x) \mathrm{d}x,$$

and the q_k's are the corresponding associated polynomials of the second kind:

$$q_k(z) = \int_{-1}^1 \frac{p_k(z) - p_k(x)}{z - x} w(x) \mathrm{d}x.$$

3.6.2. Orthogonal polynomials for oscillatory weights

Let w be a given weight function on $[-1, 1]$ and $\mathrm{d}\mu(x) = xw(x)\mathrm{e}^{\mathrm{i}\zeta x}\mathrm{d}x$, where $\zeta \in \mathbb{R}$. One could consider the existence of the orthogonal polynomials π_n with respect to

the functional

$$L(p) = \int_{-1}^{1} p(x)\mathrm{d}\mu(x), \quad \mu_k = L(x^k), \ k \in \mathbb{N}_0.$$

Two cases are intensively studied by Milovanović–Cvetković (2005):

1. *Case $w(x) = 1$, $\zeta = m\pi$, $m \in \mathbb{Z} \setminus \{0\}$,*

2. *Case $w(x) = 1/\sqrt{1 - x^2}$, $\zeta \in \mathbb{R}$.*

A possible application of these quadratures is in the numerical calculation of integrals involving highly oscillatory integrands, in particular for the calculation of Fourier coefficients.

3.6. Nonstandard quadratures of Gaussian type

All previous quadrature rules use the information on the integrand only at some selected points x_k, $k = 1, \ldots, n$ (the values of the function f and its derivatives in the cases of rules with multiple nodes). Such quadratures will be called the *standard quadrature formulae*. However, in many cases in Physics and technical sciences it is not possible to measure the exact value of the function f at points x_k, so that a standard quadrature cannot be applied. On the other side, some other information on f can be available, like the average

$$\frac{1}{2h_k} \int_{I_k} f(x)\mathrm{d}\mu(x)$$

of this function over some nonoverlapping subintervals I_k, with length of I_k equal to $2h_k$, and their union which is a proper subset of $\mathrm{supp}(d\mu)$. One may also know a fixed linear-combination of the values of this function, e.g.

$$af(x - h) + bf(x) + cf(x + h)$$

at some points x_k, where a, b, c are constants and h is a sufficiently small positive number, etc.

If the information data $\{f(x_k)\}_{k=1}^{n}$ in the standard quadrature is replaced by $\{(\mathcal{A}^{h_k}f)(x_k)\}_{k=1}^{n}$, where $\mathcal{A}^h$ is an extension of some linear operator $\mathcal{A}^h : \mathcal{P} \to \mathcal{P}$, $h \geqslant 0$, we get a non-standard quadrature formula

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) = \sum_{k=1}^{n} w_k(\mathcal{A}^{h_k}f)(x_k) + R_n(f).$$

This kind of quadratures is based on operator values for a general family of linear operators $\mathcal{A}^h$, acting of the space on algebraic polynomials, such that the degrees of polynomials are preserved.

A typical example for such operators is the *average (Steklov) operator* mentioned above, i.e.,

$$(\mathcal{A}^h p)(x) = \frac{1}{2h} \int_{x-h}^{x+h} p(x)\mathrm{d}x, \qquad h > 0, \ p \in \mathcal{P}.$$

The first idea involves the so-called *interval quadratures*. Let $h_1, \ldots, h_n$ be nonnegative numbers such that

(24) $\quad a < x_1 - h_1 \leqslant x_1 + h_1 < x_2 - h_2 \leqslant x_2 + h_2 < \cdots < x_n - h_n \leqslant x_n + h_n < b,$

and let $w(x)$ be a given weight function on $[a, b]$. Using the previous inequalities it is obvious that we have $2(h_1 + \cdots + h_n) < b - a$.

Bojanov and Petrov (2001) proved that the Gaussian interval quadrature rule of the maximal algebraic degree of exactness $2n - 1$ exists, i.e.,

(25) $$\int_a^b f(x)w(x)\mathrm{d}x = \sum_{k=1}^n \frac{w_k}{2h_k} \int_{x_k-h_k}^{x_k+h_k} f(x)w(x)\mathrm{d}x + R_n(f),$$

where $R_n(f) = 0$ for each $f \in \mathcal{P}_{2n-1}$. If $h_k = h$, $1 \leqslant k \leqslant n$, they also proved the uniqueness of (25). In 2003 they proved the uniqueness of (25) for the Legendre weight $(w(x) = 1)$ for any set of lengths $h_k \geqslant 0$, $k = 1, \ldots, n$, satisfying the condition (24).

Milovanović–Cvetković (2004), by using properties of the topological degree of non-linear mappings, proved that the Gaussian interval quadrature formula is unique for the Jacobi weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, on $[-1, 1]$ and they proposed an algorithm for numerical construction. For the special case of the Chebyshev weight of the first kind and the special set of lengths, an analytic solution can be given.

Bojanov and Petrov (2005) proved the existence and uniqueness of the weighted Gaussian interval quadrature formula for a given system of continuously differentiable functions, which constitute an ET system of order two on $[a, b]$.

The cases with interval quadratures on unbounded intervals with the classical generalized Laguerre and Hermite weights have been investigated by Milovanović and Cvetković (2005, 2007).

They also considered the nonstandard quadratures with some special operators of the form

$$(\mathcal{A}^h p)(x) = \frac{1}{2h} \int_{x-h}^{x+h} p(t)\mathrm{d}t, \qquad (\mathcal{A}^h p)(x) = \sum_{k=-m}^m a_k p(x + kh)$$

or

$$(\mathcal{A}^h p)(x) = \sum_{k=-m}^{m-1} a_k p\left(x + (k + 1/2)h\right) \quad \text{and} \quad (\mathcal{A}^h p)(x) = \sum_{k=0}^m \frac{b_k h^k}{k!} \mathcal{D}^k p(x),$$

where m is a fixed natural number and $\mathcal{D}^k = \mathrm{d}^k/\mathrm{d}x^k$, $k \in \mathbb{N}_0$.

3.7. Generalized Birkhoff-Young quadrature formulas

In 1950 G. Birkhoff and D. M. Young derived a quadrature formula for numerical integration of analytic and harmonic functions in the complex domain over the line segment $[a-h, a+h]$ in five points a, $a\pm h$, $a\pm ih$ ($i = \sqrt{-1}$), with algebraic degree $d = 5$. After several results by Lyness and Delves (1967), Lyness and Moler (1967), Lyness (1969), Lether (1976), in 1978 D. Đ. Tošić derived a one-parametric family of five-point quadrature rules of this type with degree of exactness $d = 7$. Later in 1982 GVM and R. Ž. Đorđević obtained such a formula with nine nodes and degree of exactness $d = 13$.

In 2011, jointly With his collaborators Cvetković and Stanić, GVM derived a generalized N-point Birkhoff-Young quadrature of interpolatory type, with the Chebyshev weight, whose nodes are characterized by an orthogonality relation.

Several types of quadratures of Birkhoff-Young type, as well as a sequence of the weighted generalized quadrature rules and their connection with multiple orthogonal polynomials, have been recently considered by Milovanović (2017). Besides a result on the generalized $(4n+1)$-point Birkhoff-Young quadrature, general weighted quadrature formulas of Birkhoff-Young type with the maximal degree of exactness have been established. It includes a characterization and uniqueness of such rules, as well as numerical construction of nodes and weight coefficients. An explicit form of the node polynomial of such kind of quadratures with respect to the generalized Gegenbauer weight function is obtained. Finally, a sequence of generalized quadrature formulas is studied and their node polynomials are interpreted in terms of multiple orthogonal polynomials. The construction of multiple orthogonal polynomials, defined using orthogonality conditions spread out over r different measures, as well as weighted quadratures of Gaussian type were considered earlier by GVM and Stanić in 2003 (see also Chapter 26 in the monograph NONLINEAR ANALYSIS: STABILITY, APPROXIMATION, AND INEQUALITIES (P. Georgiev, P.M. Pardalos, H.M. Srivastava, eds.).

3.8. Gaussian quadrature for Müntz systems

Gaussian integration can be extended in a natural way to non-polynomial functions, considering a system of linearly independent functions

$$(26) \qquad \{P_0(x), P_1(x), P_2(x), \ldots\} \qquad (x \in [a,b]),$$

usually chosen to be complete in some suitable space of functions. If $d\sigma(x)$ is a given nonnegative measure on $[a,b]$ and the quadrature rule

$$(27) \qquad \int_a^b f(x)\, d\sigma(x) = \sum_{k=1}^n A_k f(x_k) + R_n(f)$$

is such that it integrates exactly the first $2n$ functions in (26), we call the rule (27) *Gaussian with respect to the system* (26). The existence and uniqueness of

557

a Gaussian quadrature rule (27) with respect to the system (26), or for short a *generalized Gaussian formula*, is always guaranteed if the first $2n$ functions of this system constitute a Chebyshev system on $[a, b]$. Then, all the weights $A_1, \ldots, A_n$ in (27) are positive.

The generalized Gaussian quadratures for Müntz systems go back to Stieltjes in 1884. Taking $P_k(x) = x^{\lambda_k}$ on $[a, b] = [0, 1]$, where $0 \leqslant \lambda_0 < \lambda_1 < \cdots$, Stieltjes showed the existence of Gaussian formulae. A numerical algorithm for constructing the generalized Gaussian quadratures was recently investigated by Ma, Rokhlin and Wandzura (1996), but it is ill conditioned.

Milovanović and Cvetković (2005) presented an alternative numerical method for constructing the generalized Gaussian quadrature (5) for Müntz polynomials, which is exact for each

$$f \in M_{2n-1}(\Lambda) = \text{span}\big\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_{2n-1}}\big\}.$$

Besides the properties of orthogonal Müntz polynomials on $(0, 1)$ and their connection with orthogonal rational functions, GVM presented also a method for the numerical evaluation of such generalized polynomials (1999). His method is rather stable and simpler than the previous one, since it is based on orthogonal Müntz systems.

Figure 6: At a relaxing moment with the family: Vladica, Irena, Gradimir and Dobrila (Restaurant "Kalenić", Belgrade, 2017)

4. GVM AND OTHER FIELDS

Professor Milovanović has further notable works in other fields as well, such as the integration of fast oscillatory functions, special functions, especially of hypergeometric type and theorems on their summation, special numbers, as well as in the field of iterative processes, optimization theory and the domain of inequalities, with which his scientific career began. The applications of his theoretical results in the fields of Electrical Engineering, Physics and Telecommunications are also significant.

5. THE BIBLIOGRAPHY OF GVM

The full bibliography, covering all periods of the rich scientific activity of GVM, can be found at the web site:

`http://www.mi.sanu.ac.rs/~gvm/`

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In this volume the following contributions of Gradimir's friends and collaborators are featured:

The paper by Kashuri and Rassias is concerned with the fractional trapezium-type inequalities for strongly exponentially generalized preinvex functions; In their paper Srivastava, Jena and Paikray introduce and study the notion of statistical probability convergence for sequences of random variables as well as the concept of statistical convergence for sequences of real numbers, which are defined over a Banach space via deferred weighted summability mean; In their paper Malešević, Lutovac, Rašajski and Banjac consider error-functions in double-sided Taylor's approximations; In the paper by Guessab, Driouch and Nouisser a new modified moving asymptotes method is presented; The aim of the paper by Simsek is to define new families of combinatorial numbers and polynomials associated with Peters polynomials; The authors Park, Kim and So consider a result for binomial convolution sums of restricted divisor functions; In his paper Özarslan introduces the Jain-Appell operators by applying the Gamma transform to the Jakimovski-Leviatan operators, and investigates their weighted approximation properties and computes the error of approximation by using certain Lipschitz class functions; The paper by Vukelić is concerned with Levinson's inequality involving averages of 3-convex functions; The paper by Delen, Togan, Yurttas, Ana and Cangul is concerned with the effect of edge and vertex deletion on omega invariant; A note on polylogarithms and incomplete gamma function is presented by Aygüneş; The goal of the paper by Savas is to study some new sequence spaces of order α that are defined using modulus function and infinite matrix; The paper by Kim, Bayad and Ahn is concerned with a study of Möbius-Bernoulli numbers; In their paper Behl, Gutiérrez, Argyros and Alshomrani consider efficient optimal families of

higher-order iterative methods with local convergence; In their paper Branquinho, García-Ardila and Marcellán study matrix biorthogonal polynomials sequences that satisfy a nonsymmetric three term recurrence relation with unbounded matrix coefficients; The paper by Jovanović and Voß describes a matheuristic approach for solving the 2-connected dominating set problem; In their paper Landon, Carley and Mohapatra consider two operators based on the K^λ means of the Fourier series of and conjugate series of functions of class L^p, $p > 1$.

We wish to express once again our warmest thanks to all the mathematicians who have contributed their papers, and to all the referees for their assistance on judging the merits of the submissions as well as for their useful comments and propositions, which led to the composition of this Special Issue dedicated to the 70th birth anniversary of the academician Professor Gradimir V. Milovanović.

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