Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 2840482, 7 pages
https://doi.org/10.1155/2020/2840482

Research Article

A Common Fixed Point Theorem for Nonlinear Quasi-Contractions on b-Metric Spaces with Application in Integral Equations

Zoran D. Mitrović,[1] **Ivan D. Aranđelović**,[2] **Vesna Mišić**,[3] **Abdollah Dinmohammadi**,[4] **and Vahid Parvaneh**[5]

[1]Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina
[2]Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11000 Beograd, Serbia
[3]Faculty of Transport and Traffic Engineering, University of East Sarajevo, Vojvode Mišića 52, 53000 Doboj, Bosnia and Herzegovina
[4]Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
[5]Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran

Correspondence should be addressed to Abdollah Dinmohammadi; mathdin@yahoo.com and Vahid Parvaneh; zam.dalahoo@gmail.com

Received 12 June 2020; Revised 4 September 2020; Accepted 14 September 2020; Published 29 September 2020

Academic Editor: Sergey Shmarev

In this paper, we present a common fixed point result for a pair of mappings defined on a b-metric space, which satisfies quasi-contractive inequality with nonlinear comparison functions. An application in solving a class of integral equations will support our results.

1. Introduction

In 1974, Ćirić presented the first fixed point result for quasi-contractive mappings. This Ćirić's theorem is one of the most general results with linear comparison function in classical metrical fixed point theory (see [1, 2]). The existence and uniqueness of fixed point for mappings defined on metric spaces, which satisfies a quasi-contractive inequality with a nonlinear comparison function, were considered by Danes [3], Ivanov [4], Aranđelović et al. [5], and Bessenyei [6]. Alshehri et al. [7] proved a fixed point theorem for quasi-contractive mappings, defined by linear quasi-contractive conditions on b-metric spaces. Common fixed point generalizations of Ćirić result was obtained by Das and Naik [8], with linear comparison functions and by Di Bari and Vetro [9], with a nonlinear comparison function.

The notion of symmetric spaces, which is the oldest and one of the most important generalizations of metric spaces, was introduced by Fréchet [10]. He used the name E-space for a symmetric space. In the last 50 years, many authors (see [11–16]) called them semimetric (in German *halb-metrisher*) spaces. Now, the term symmetric space is usual. After 1955, the term semimetric space is widely used to denote a symmetric space in which the closure operator is idempotent, which started the papers of Heath, Brown, Mc Auley, Jones, and Burke (see [17, 18]). Fixed point investigation was started by Cicchese [19] and Jachymski et al. [18] on semimetric spaces and by Hicks and Rhoades [20] on symmetric spaces.

In [10], Fréchet also considered the class of E-spaces with regular écart which include the class of b-metric spaces. Important examples of b-metric spaces are quasi-normed spaces introduced by Bourgin [21] and Hyers [22] and spaces of homogeneous type which have many applications in the theory of analytic functions (see Coifman and Weiss [23]). First, fixed point results on b-metric spaces were presented by Bakhtin [24] and Czerwik [25].

In this paper, we present a common fixed point result for a pair of mappings defined on a b-metric space, which satisfies a quasi-contractive inequality with a nonlinear comparison function.

2. Symmetric Spaces and b-Metric Spaces

The ordered pair (Δ, μ), where Δ is a nonempty set and $\mu : \Delta^2 \to [0,\infty)$, is a symmetric space, if and only if it satisfies:

(W1) $\mu(\iota, \kappa) = 0$ if and only if $\iota = \kappa$

(W2) $\mu(\iota, \kappa) = \mu(\kappa, \iota)$ for any $\iota, \kappa \in \Delta$

The difference between symmetric spaces and more convenient metric spaces is in the absence of triangle inequality, but many notions in symmetric spaces are defined similar to those in metric spaces. For instance, in symmetric space (Δ, μ), the limit point of a sequence (ι_n) is defined by

$$\lim \mu(\iota_n, \iota) = 0 \Leftrightarrow \lim \iota_n = \iota. \tag{1}$$

Also, we say that a sequence $\{\iota_n\} \subseteq \Delta$ is a Cauchy sequence, if for any given $\varepsilon > 0$, there exists a positive integer n_0 such that $\mu(\iota_m, \iota_n) < \varepsilon$ for every $m, n \geq n_0$. If each Cauchy sequence in symmetric space (Δ, μ) is convergent, then we say that (Δ, μ) is a complete symmetric space.

By

$$\operatorname{diam}(A) = \sup_{\iota,\kappa\in A} \mu(\iota, \kappa), \tag{2}$$

we indicate the diameter of the set A.

Let (Δ, μ) be a symmetric space. We can introduce the topology τ_d by defining the family of all closed sets as follows: a set $A \subseteq \Delta$ is closed if and only if for each $\iota \in \Delta$, $\mu(\iota, A) = 0$ implies $\iota \in A$, where

$$\mu(\iota, A) = \inf \{\mu(\iota, a) : a \in A\}. \tag{3}$$

The convergence of a sequence (ι_n) in the topology τ_d need not imply $\mu(\iota_n, \iota) \to 0$, but the converse is true.

Let (Δ, μ) be a symmetric space, $(\iota_n), (\kappa_n), (z_n) \subseteq \Delta$ and $\iota, \kappa \in \Delta$. We considered the following seven properties as partial replacements for the triangle inequality:

(W3) $\lim \mu(\iota_n, \iota) = 0 \wedge \lim \mu(\iota_n, \kappa) = 0 \Rightarrow \iota = \kappa$

(W4) $\lim \mu(\iota_n, \iota) = 0 \wedge \lim \mu(\iota_n, \kappa_n) = 0 \Rightarrow \lim \mu(\kappa_n, \iota) = 0$

(HE) $\lim \mu(\iota_n, \iota) = 0 \wedge \lim \mu(\kappa_n, \iota) = 0 \Rightarrow \lim \mu(\iota_n, \kappa_n) = 0$

(CC) $\lim \mu(\iota_n, \iota) = 0 \Rightarrow \lim \mu(\iota_n, \kappa) = \mu(\iota, \kappa)$

(W) $\lim \mu(\iota_n, \kappa_n) = 0 \wedge \lim \mu(\kappa_n, z_n) = 0 \Rightarrow \lim \mu(\iota_n, z_n) = 0$

(JMS) $\lim \mu(\iota_n, \kappa_n) = 0 \wedge \lim \mu(\kappa_n, z_n) = 0 \Rightarrow \overline{\lim} \mu(\iota_n, z_n) \neq \infty$;

(SC) $\lim_{n\to\infty} \mu(\iota_n, \iota) = 0$ implies $\overline{\lim}_{n\to\infty} \mu(\iota_n, \kappa) = \mu(\iota, \kappa)$

The property (W3) has been introduced by Fréchet [10]; (W4), (HE), and (W) by Pitcher and Chittenden [14]; (CC) by Sims [15]; (JMS) by Jachymski et al. [18]; and (SC) by Aranđelović and Kečkić [17]. Note that $(W) \Rightarrow (W4) \Rightarrow (W3)$, $(W) \Rightarrow (JMS)$, $(W) \Rightarrow (HE)$, $(CC) \Rightarrow (W3)$, and $(CC) \Rightarrow (SC)$ (see [17, 26]).

In [26], the authors give examples for the following relationships: $(W3) \Rightarrow (W4)$, $(W4) \Rightarrow (HE)$, $(W4) \Rightarrow (CC)$, $(W3) \Rightarrow (HE)$, $(W3) \Rightarrow (CC)$, $(CC) \Rightarrow (W4)$, $(HE) \Rightarrow (CC)$, $(HE) \Rightarrow (W3)$, $(HE) \Rightarrow (W4)$, and $(CC) \Rightarrow (HE)$. The fact that $(W) \Rightarrow (CC)$ has been proved in [27].

Definition 1. Let Δ be a nonempty set, $\mu : \Delta \times \Delta \to [0,\infty)$. (Δ, μ) is said to be a b -metric space if there exists $s \in [0,\infty)$ such that

(1) $\mu(\iota, \kappa) = 0$ if and only if $\iota = \kappa$

(2) $\mu(\iota, \kappa) = \mu(\kappa, \iota)$ for any $\iota, \kappa \in \Delta$

(3) $\mu(\iota, z) \leq s[\mu(\iota, \kappa) + \mu(\kappa, z)]$for all $\iota, \kappa, z \in \Delta$.

Any $s \in [0,\infty)$ which satisfies inequality (3) of Definition 1 for all $\iota, \kappa, z \in \Delta$, where (Δ, μ) is a b-metric space, is said to be the b constant of space (Δ, μ). It is clear that if $s = 1$, then (Δ, μ) is a metric space.

Lemma 2. *Let (Δ, μ) be a b-metric space with b constant s. Then, $s \geq 1$.*

Proof. Let $\iota, \kappa \in \Delta$. Then, $\mu(\iota, \kappa) \leq s[\mu(\iota, \kappa) + \mu(\kappa, \kappa)] = s\mu(\iota, \kappa)$, which implies that $s \geq 1$.

In [17], the following result was proved.

Lemma 3 (see [17]). *Let (Δ, μ) be a b-metric space. Then, (Δ, μ) is a symmetric space which satisfies the properties (W3), (W4), (HE), (W), and (JMS).*

3. Comparison Functions

Let $\chi : [0,\infty) \to [0,\infty)$ be a function such that $\chi(\iota) = 0$ if and only if $\iota = 0$. Define:

(1) $\chi \in \Xi_0$ if and only if $\chi(r) < r$ for each $r > 0$

(2) $\chi \in \Xi_1$ if and only if $\overline{\lim}_{tr+} \chi(t) = \chi(r)$ for each $r > 0$

(3) $\chi \in \Xi_2$ if and only if $\overline{\lim}_{tr} \chi(t) \leq \chi(r)$ for any $r > 0$

(4) $\chi \in \Xi_3$ if and only if $\overline{\lim}_{tr+} \chi(t) < r$ for any $r > 0$

(5) $\chi \in \Xi_4$ if and only if $\overline{\lim}_{tr} \chi(t) < r$ for all $r > 0$

(6) $\chi \in \Xi_5$ if and only if $\lim_{t\to\infty} (t - \chi(t)) = \infty$

(7) $\chi \in \Xi_6$ if and only if $I - \chi : [0,\infty) \to [0,\infty)$ is a strictly increasing surjection

(8) $\chi \in \Xi_7$ if and only if $\{\iota : (I - \chi)(\iota) < r\}$ is bounded for every $r > 0$

(9) $\chi \in \Xi_8$ if and only if χ is monotone nondecreasing

If $\chi \in \cap_{i=0}^{7} \Xi_i$, then we say that χ is a comparison function.

If $\chi \in \Xi_1$, then χ is continuous from the right on $(0, \infty)$.

If $\chi \in \Xi_2$, then χ is upper semicontinuous on $(0, \infty)$.

The class of $\Xi_0 \cap \Xi_1 \cap \Xi_6 \cap \Xi_8$ has been applied in the theory of nonlinear quasi-contractions by Danes [18],$\Xi_0 \cap \Xi_1 \cap \Xi_5 \cap \Xi_8$ by Ivanov [4], $\Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$ by Aranđelović et al. [5] and Di Bari and Vetro [9], $\Xi_0 \cap \Xi_4 \cap \Xi_5$ by Aranđelović et al. [5], and the class of $\Xi_0 \cap \Xi_2 \cap \Xi_7 \cap \Xi_8$ by Bessenyei [6].

Note that $\Xi_4 \subseteq \Xi_3$, $(\Xi_1 \cap \Xi_0) \subseteq (\Xi_3 \cap \Xi_0)$, and $(\Xi_2 \cap \Xi_0) \subseteq (\Xi_4 \cap \Xi_0)$. Some further inclusion between different classes of comparison functions will be presented in the next statements.

Proposition 4. *If $\chi \in \Xi_3 \cap \Xi_8$, then $\chi \in \Xi_4$.*

Proof. For any $r > 0$, from $\overline{\lim}_{tr+} \chi(t) < r$, we get that $\overline{\lim}_{tr} \chi(t) < r$, because χ is monotone nondecreasing. So, we obtain that $\overline{\lim}_{tr} \chi(t) < r$, for every $r > 0$. ⋄

Proposition 5. $\Xi_5 = \Xi_7$.

Proof. Let $\chi \in \Xi_5$. If there exists $r > 0$ such that $\{\iota : \iota - \chi(\iota) < r\}$ is unbounded; then, for every $M > 0$, there exists $t > 0$ such that $t - \chi(t) < r$. So,

$$\underline{\lim}_{t\to\infty} \chi(t) \le r < \infty, \tag{4}$$

which is a contradiction with $\chi \in \Xi_5$.

Let $\chi \in \Xi_7$. Suppose that there exists an increasing sequence $(t_n) \subseteq (0,\infty)$ such that $\lim t_n = \infty$ and $R > 0$ such that $(t_n - \chi(t_n)) < R$, for each n. Hence, $(t_n) \subseteq \{\iota : \iota - \chi(\iota) < R\}$. So, $\{\iota : \iota - \chi(\iota) < R\}$ is unbounded which implies that $\chi \in \Xi_7$.

Proposition 6. *If $\chi \in \Xi_6$, then $\chi \in \Xi_5$.*

Proof. Let $\chi \in \Xi_6$. Suppose that there exists a strictly increasing sequence $(t_n) \subseteq (0,\infty)$ such that $\lim t_n = \infty$ and $R > 0$ such that $(t_n - \chi(t_n)) < R$, for each positive integer n. So, for any $\iota > 0$, we have $\iota - \chi(\iota) < R$, because there exists t_n such that $\iota < t_n$, which implies that χ is not a surjection. Hence, $\lim_{t\to\infty} (t - \chi(t)) = \infty$.⋄

Two following two lemmas have been proved in [5].

Lemma 7. *Let $\chi \in \Xi_0 \cap \Xi_4 \cap \Xi_5$. Then, there exists $\Omega \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$ such that*

$$\chi(\iota) \le \Omega(\iota) < \iota, \tag{5}$$

for each $\iota > 0$.

Lemma 8. *Let $\chi_1, \cdots, \chi_n \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$. Then, there exists $\Omega \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$ such that*

$$\chi_k(\iota) \le \Omega(\iota) < \iota, \tag{6}$$

for each $1 \le k \le n$ and $\iota > 0$.

4. Main Results

First, recall some standard terminology and notations from the fixed point theory.

Let Δ be a nonempty set, and let $Y : \Delta \to \Delta$ be an arbitrary mapping.

Let Δ and Λ be nonempty sets, $Y, \Gamma : \Delta \to \Lambda$, and $Y(\Delta) \subseteq \Gamma(\Delta)$. Choose a point $\iota_1 \in \Delta$ such that $Y(\iota_0) = \Gamma(\iota_1)$. Continuing this process, having $\iota_n \in \Delta$, we obtain $\iota_{n+1} \in \Delta$ such that $Y(\iota_n) = \Gamma(\iota_{n+1})$. $Y(\iota_n)$ is called a Jungck sequence with an initial point ι_0. Note that a Jungck sequence might not be determined by its initial point ι_0.

Let Δ be a nonempty set and $Y, \Gamma : \Delta \to \Delta$. Y and Γ are called weakly compatible if they commute at their coincidence points.

Lemma 9 (see [28]). *Let Δ be a nonempty set and let $Y, \Gamma : \Delta \to \Delta$ be weakly compatible self mappings. If Y and Γ have a unique point of coincidence $\kappa = Y(\iota) = \Gamma(\iota)$, then κ is the unique common fixed point of Y and Γ.*

Now, we present our main result. Before stating the result, we make a convention to abbreviate $Y(\iota)$ and $\Gamma(\iota)$ in order to avoid too much parenthesis.

Theorem 10. *Let (Λ, μ) be a b -metric space with b constant s and let $Y, \Gamma : \Delta \to \Lambda$ be two mappings. Suppose that the range of Γ contains the range of Y and that $\Gamma(\Delta)$ is a complete subspace of Λ. If there exist $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5 : [0,\infty) \to [0,\infty)$ such that*

$$s \cdot \chi_1, s \cdot \chi_2, s \cdot \chi_3, s \cdot \chi_4, s \cdot \chi_5 \in \Xi_0 \cap \Xi_4 \cap \Xi_5 \text{ and} \tag{7}$$

$$\mu(Y\iota, Y\kappa) \le \max\{\chi_1(\mu(\Gamma\iota, \Gamma\kappa)), \chi_2(\mu(\Gamma\iota, Y\iota)), \chi_3 \cdot (\mu(\Gamma\kappa, Y\kappa)), \chi_4(\mu(\Gamma\iota, Y\kappa)), \chi_5(\mu(Y\iota, \Gamma\kappa)\}, \tag{8}$$

for any $\iota, \kappa \in \Delta$, then there exists $z \in \Lambda$ which is the limit of every Jungck sequence defined by Y and Γ. Further, z is the unique point of coincidence of Y and Γ. Moreover, if $\Delta = \Lambda$ and Y, Γ are weakly compatible, then z is the unique common fixed point for Y and Γ.

Proof. We shall, first, reduce the statement to the case $\chi_1 = \cdots = \chi_5$ and $s \cdot \chi_i \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$. Indeed, from Lemma 7, it follows that there exist functions $\chi_k^* : [0,\infty) \to [0,\infty)$ such that $s \cdot \chi_k^* \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$ and

$$\chi_k(\iota) \le \chi_k^*(\iota) < \iota, \tag{9}$$

for each $\iota > 0$ and for all $1 \le k \le 5$, whereas from Lemma 8, it follows that there exists a real function $\chi : [0,\infty) \to [0,\infty)$ such that $s \cdot \chi \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$ and

$$\chi_k^*(\iota) \le \chi(\iota) < \frac{\iota}{s}, (1 \le k \le 5) \text{ for each } \iota > 0, \tag{10}$$

which implies

$$\mu(Y\iota, Y\kappa) \le \max\{\chi(\mu(\Gamma\iota, \Gamma\kappa)), \chi(\mu(\Gamma\iota, Y\iota)), \chi \cdot (\mu(\Gamma\kappa, Y\kappa)), \chi(\mu(\Gamma\iota, Y\kappa)), \chi(\mu(Y\iota, \Gamma\kappa))\}. \quad (11)$$

Thus, we can assume that $\chi_j = \chi$ for all $1 \le j \le 5$ and $s \cdot \chi \in \Xi_0 \cap \Xi_3 \cap \Xi_5 \cap \Xi_8$.

Let $\iota_0 \in \Delta$ be arbitrary and let (ι_n) be an arbitrary sequence such that $Y(\iota_n)$ is a Jungck sequence with an initial point ι_0.

Let $d_0 = \mu(\Gamma\iota_0, Y\iota_0)$. We will prove that there exists a real number $r_0 > 0$ such that:

$$r_0 - s \cdot \chi(r_0) \le d_0 \text{ and } r - s \cdot \chi(r) > d_0 \text{ for } r > r_0. \quad (12)$$

Consider the set $D = \{r \mid t - s \cdot \chi(t) > d_0 \forall t > r\}$ which is nonempty, since $r - \chi(r) \to \infty$ as $r \to \infty$. Also, if $q \in D$ and $p > q$ imply $p \in D$, and hence, D is an unbounded interval. Set $r_0 = \inf D$. For each positive integer n, there is $r_n \notin D$ such that $r_0 - 1/n < r_n$, and therefore, there is $r_0 \ge t_n > r_n > r_0 - 1/n$ such that $t_n - s \cdot \chi(t_n) \le d_0$. Since χ is nondecreasing, we have $s \cdot \chi(t_n) \le s \cdot \chi(r_0)$ which implies that $t_n - s \cdot \chi(r_0) \le d_0$. Taking the limit as $n \to \infty$, we get $r_0 - s \cdot \chi(r_0) \le d_0$.

For any $j \ge 0$, define $\mathcal{O}_n(\iota_j) = \{Y\iota_k \mid k = j, j+1, j+2, \cdots, j+n\}$ and $\mathcal{O}(\iota_j) = \{Y(\iota_k) \mid k = j, j+1, j+2, \cdots\}$. Also, let $\mathrm{diam}(A)$ denote the diameter of A.

Next, we prove that

$$\delta(\mathcal{O}_n(\iota_k)) \le \chi(\delta(\mathcal{O}_{n+1}(\iota_{k-1}))), \quad (13)$$

for all positive integer k, n.

Since χ is nondecreasing, it commutes with max, and for all $k \le i, j \le k+n$, we have

$$\begin{aligned} \mu(Y\iota_i, Y\iota_j) &\le \chi(\max\{\mu(\Gamma\iota_i, \Gamma\iota_j), \mu(\Gamma\iota_i, Y\iota_i), \mu(\Gamma\iota_j, Y\iota_j), \mu \cdot (\Gamma\iota_i, Y\iota_j), \mu(\Gamma\iota_j, Y\iota_i)\}) \\ &= \chi(\max\{\mu(Y\iota_{i-1}, Y\iota_{j-1}), \mu(Y\iota_{i-1}, Y\iota_i), \mu \cdot (Y\iota_{j-1}, Y\iota_j), \mu(Y\iota_{i-1}, Y\iota_j), \mu(Y\iota_{j-1}, Y\iota_i)\}) \\ &\le \chi(\mathrm{diam}(\mathcal{O}_{n+1}(\iota_{k-1}))). \end{aligned} \quad (14)$$

By induction, from (13), we obtain that

$$\delta(\mathcal{O}_n(\iota_k)) \le \chi^l(\mathrm{diam}(\mathcal{O}_{n+l}(\iota_{k-l}))). \quad (15)$$

For $1 \le i, j \le n$, we have $Y\iota_i, Y\iota_j \in \mathcal{O}_{n-1}(\iota_1)$, and hence, by (13)

$$\mu(Y\iota_i, Y\iota_j) \le \mathrm{diam}(\mathcal{O}_{n-1}(\iota_1)) \le \chi(\mathrm{diam}(\mathcal{O}_n(\iota_0))) < \mathrm{diam}(\mathcal{O}_n(\iota_0)). \quad (16)$$

Therefore, there is $1 \le k \le n$ such that

$$\begin{aligned} \mathrm{diam}(\mathcal{O}_n(\iota_0)) = \mu(Y\iota_0, Y\iota_k) &\le s \cdot [\mu(Y\iota_0, Y\iota_1) + \mu(Y\iota_1, Y\iota_k)] \\ &\le s \cdot d_0 + s \cdot \mathrm{diam}(\mathcal{O}_{n-1}(\iota_1)) \\ &\le s \cdot d_0 + s \cdot \chi(\mathrm{diam}(\mathcal{O}_n(\iota_0))). \end{aligned} \quad (17)$$

Hence, we get

$$\mathrm{diam}(\mathcal{O}_n(\iota_0)) - s \cdot \chi(\mathrm{diam}(\mathcal{O}_n(\iota_0))) \le d_0, \quad (18)$$

which implies that $\mathrm{diam}(\mathcal{O}_n(\iota_0)) \le r_0$, and hence

$$\mathrm{diam}(\mathcal{O}(\iota_0)) = \sup_n \mathrm{diam}(\mathcal{O}_n(\iota_0)) \le r_0. \quad (19)$$

Hence, all Jungck sequences defined by Y and Γ are bounded.

Now, we shall prove that our Jungck sequence is a Cauchy sequence. Let $m > n$ be positive integers. Then, $Y\iota_n, Y\iota_m \in \mathcal{O}_{m-n+1}(\iota_n)$. Using (15) (with $l = n$) and (19), we get

$$\mu(Y\iota_n, Y\iota_m) \le \mathrm{diam}(\mathcal{O}_{m-n+1}(\iota_n)) \le \chi^n(\mathrm{diam}(\mathcal{O}_{m+1}(\iota_0))) \le \chi^n(r_0) \to 0, \quad (20)$$

as $m, n \to \infty$. Since $Y(\Delta) \subseteq \Gamma(\Delta)$, and $\Gamma(\Delta)$ is complete, it follows that $Y\iota_n$ is convergent. Let $\kappa \in \Delta$ be its limit.

Clearly, $\kappa \in \Gamma(\Delta)$. So, there is $z \in \Delta$ such that $\Gamma(z) = \kappa$. Let us prove that $Y(z)$ is also equal to κ. By (8), we have

$$\begin{aligned} \mu(Y\iota_n, Yz) &\le \chi(\max\{\mu(\Gamma\iota_n, \Gamma z), \mu(\Gamma\iota_n, Y\iota_n), \mu(\Gamma z, Yz), \mu \cdot (\Gamma\iota_n, Yz), \mu(\Gamma z, Y\iota_n)\}) \\ &= \chi(\max\{\mu(Y\iota_{n-1}, \kappa), \mu(Y\iota_{n-1}, Y\iota_n), \mu(\kappa, Yz), \mu \cdot (Y\iota_{n-1}, Yz), \mu(\kappa, Y\iota_n)\}). \end{aligned} \quad (21)$$

If $n \to \infty$, then the left-hand side in the previous inequality tends to $\mu(\kappa, Yz)$, and the first, the second, and the fifth argument of max tend to $\mu(\kappa, \kappa) = 0$, whereas the third and the fourth tend to $\mu(\kappa, Yz)$. Thus, we have

$$\mu(\kappa, Yz) \le \chi(\mu(\kappa, Yz)), \quad (22)$$

which is impossible, unless $\mu(\kappa, Yz) = 0$.

Finally, we prove that the point of coincidence is unique. Suppose that there is two points of coincidence κ and κ' obtained by z and z', i.e., $Yz = \Gamma z = \kappa$ and $Yz' = \Gamma z' = \kappa'$. Then, by (8) we have

$$\begin{aligned} \mu(\kappa, \kappa') = \mu(Yz, Yz') &\le \chi(\max\{\mu(\Gamma z, \Gamma z'), \mu(\Gamma z, Yz), \mu \cdot (\Gamma z', Yz'), \mu(\Gamma z, Yz'), \mu(\Gamma z', Yz)\}) \\ &= \chi(\max\{\mu(\kappa, \kappa'), 0, 0, \mu(\kappa, \kappa'), \mu(\kappa', \kappa)\}) \\ &= \chi(\mu(\kappa, \kappa')) < \mu(\kappa, \kappa'), \end{aligned} \quad (23)$$

unless $\mu(\kappa, \kappa') = 0$. Since every Jungck sequence converges to some point of coincidence, and the point of coincidence is unique, it follows that all Jungck sequences converge to the same limit.

Let $\Delta = \Lambda$ and let Y, Γ be weakly compatible. By Lemma 3, we get that $\kappa = z$ which is the unique common fixed point of Y and Γ.

The previous theorem extended earlier results for nonlinear contractions on metric spaces obtained by Danes [3], Ivanov [4], Aranđelović et al. [5], and Bessenyei [6] and common fixed point results of Das and Naik [8] and Di Bari and Vetro [9]. It also generalizes the fixed point theorem of Aleksić et al. [7] which proved the fixed point theorems for quasi-contractive mappings on b-metric spaces, defined by linear quasi-contractive conditions.

Example 1. Let $\Delta = \Lambda = \{0, 1, 2, 3\}$ be equipped with the following b-metric $\mu : X \times X \to \mathbb{R}^+$ by $\mu(\iota, \kappa) = |\iota - \kappa|^2$.

It is easy to see that $(\Delta = \Lambda, \mu)$ is a complete b-metric space with $s = 2$.

Define the self-maps Y and Γ by

$$
\begin{aligned}
Y &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\Gamma &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix}.
\end{aligned}
\tag{24}
$$

We see that $\Gamma(\Delta) \supseteq Y(\Delta)$.

Define $\chi_i : [0,\infty) \to [0,\infty)$ by $\chi(t) = t - \sinh^{-1} t$. One can easily check that Y satisfies condition (8). Indeed, we have some cases as follows:

(1) $(\iota, \kappa) = (0, 2)$. Then,

$$
\begin{aligned}
\mu(Y\iota, Y\kappa) = |Y0 - Y2|^2 = 1 &\le 9 - \sinh^{-1}(9) \\
&\le \chi_1(\mu(\Gamma\iota, \Gamma\kappa)) \\
&\le \max\{\chi_1(\mu(\Gamma\iota, \Gamma\kappa)), \chi_2(\mu(\Gamma\iota, Y\iota)), \chi_3 \\
&\quad \cdot (\mu(\Gamma\kappa, Y\kappa)), \chi_4(\mu(\Gamma\iota, Y\kappa)), \chi_5(\mu(Y\iota, \Gamma\kappa)\}.
\end{aligned}
\tag{25}
$$

(2) $(\iota, \kappa) = (1, 2)$. Then,

$$
\begin{aligned}
\mu(Y\iota, Y\kappa) = |Y1 - Y2|^2 = 1 &\le 9 - \sin h^{-1}(9) \\
&= \chi_1(\mu(\Gamma\iota, \Gamma\kappa)) \\
&\le \max\{\chi_1(\mu(\Gamma\iota, \Gamma\kappa)), \chi_2(\mu(\Gamma\iota, Y\iota)), \chi_3 \\
&\quad \cdot (\mu(\Gamma\kappa, Y\kappa)), \chi_4(\mu(\Gamma\iota, Y\kappa)), \chi_5(\mu(Y\iota, \Gamma\kappa)\}.
\end{aligned}
\tag{26}
$$

(3) $(\iota, \kappa) = (3, 2)$. Then,

$$
\begin{aligned}
\mu(Y\iota, Y\kappa) = |Y1 - Y2|^2 = 1 &\le 4 - \sin h^{-1}(4) \\
&= \chi_1(\mu(\Gamma\iota, \Gamma\kappa)) \\
&\le \max\{\chi_1(\mu(\Gamma\iota, \Gamma\kappa)), \chi_2 \\
&\quad \cdot (\mu(\Gamma\iota, Y\iota)), \chi_3(\mu(\Gamma\kappa, Y\kappa)), \chi_4 \\
&\quad \cdot (\mu(\Gamma\iota, Y\kappa)), \chi_5(\mu(Y\iota, \Gamma\kappa)\}.
\end{aligned}
\tag{27}
$$

Thus, all the conditions of Theorem 10 are satisfied, and hence, Y and Γ have a common fixed point. Indeed, 0 is the unique common fixed point of Y and Γ.

5. Application

The existence of the solution for the following integral equation is the main purpose in this section.

$$
\sigma(\iota) = f\left(\iota, \int_0^{\rho(\iota)} g(\iota, \kappa, \sigma(\rho(\kappa)))d\kappa\right),
\tag{28}
$$

where $\iota \in [0,\infty)$.

We will ensure such an existence by applying Theorem 10.

Let $BC[0, \infty)$ be the space of all real, bounded and continuous functions on the interval $[0, \infty)$. We endow it with the b-metric

$$
d(\iota, \kappa) = \sup\{|\iota(t) - \kappa(t)|^p : t \in [0,\infty)\},
\tag{29}
$$

where $p \ge 1$.

Theorem 11. *Suppose that the following assumptions are satisfied:*

(i) $\rho, \varrho : [0,\infty) \to [0,\infty)$ *are continuous functions so that*

$$
\Lambda^p = \sup\{|\varrho(t)|: t \in [0,\infty)\} < 1,
\tag{30}
$$

(ii) The function $f : [0,\infty) \times R \to R$ *is continuous so that*

$$
|f(\iota, \sigma_1) - f(\iota, \sigma_2)| \le |\sigma_1 - \sigma_2|,
\tag{31}
$$

(iii) For all $\iota \in [0,\infty)$ *and* $\sigma_i \in \mathbb{R}$

$$
|g(\iota, \kappa, \sigma_1(\rho(\kappa))) - g(\iota, \kappa, \sigma_2(\rho(\kappa)))| \le |\sigma_1(\rho(\kappa)) - \sigma_2(\rho(\kappa))|,
\tag{32}
$$

where $g : [0,\infty)^2 \times R \to R$ *is continuous.*

(iv) $M = \max\{f(\iota, 0): \iota \in [0,\infty)\} < \infty$ *and* $G = \sup\{|g(\iota, \kappa, 0)|: \iota \in [0,\infty)\} < \infty$

Then, the integral equation (28) admits at least one solution in the space $(BC[0, \infty))$.

Proof. Let us consider the operator $Y : BC[0,\infty) \to BC[0,\infty)$ defined by

$$
Y(\sigma)(\iota) = f\left(\iota, \int_0^{\rho(\iota)} g(\iota, \kappa, \sigma(\rho(\kappa)))d\kappa\right).
\tag{33}
$$

In view of the given assumptions, we infer that the function $Y(\sigma)$ is continuous for arbitrarily $\sigma \in BC[0,\infty)$. Now, we show that $Y(\sigma)$ is bounded in $BC[0, \infty)$. As

$$\begin{aligned}|Y(\sigma)(\iota)| &= \left|f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma(\rho(\kappa)))d\kappa\right)\right| \\ &\le \left|f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma(\rho(\kappa)))d\kappa\right) - f(\iota,0)\right| + |f(\iota,0)|,\end{aligned} \tag{34}$$

we have

$$\begin{aligned}&\left|f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma(\rho(\kappa)))d\kappa\right) - f(\iota,0)\right| \\ &\quad\le \left|\int_0^{\rho(\iota)} g(\iota,\kappa,\sigma(\rho(\kappa)))d\kappa\right| \\ &\quad\le \Lambda\|\sigma\| + \Lambda G.\end{aligned} \tag{35}$$

Thus,

$$\left|f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma(\rho(\kappa)))d\kappa\right) - f(\iota,0)\right| \le \Lambda\|\sigma\| + \Lambda G. \tag{36}$$

From the above calculations, we have

$$\|Y(\sigma)(\iota)\| \le \Lambda\|\sigma\| + \Lambda G + M. \tag{37}$$

Due to the above inequality, the function Y is bounded.

Now, we show that Y satisfies all the conditions of Theorem 10. Let σ_1, σ_2 be some elements of $BC[0,\infty)$. Then, we have

$$\begin{aligned}|Y(\sigma_1)(\iota) - Y(\sigma_2)(\iota)|^p &\le \left|f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma_1(\rho(\kappa)))d\kappa\right)\right. \\ &\quad \left. - f\left(\iota, \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma_2(\rho(\kappa)))d\kappa\right)\right|^p \\ &\le \left|\int_0^{\rho(\iota)} g(\iota,\kappa,\sigma_1(\rho(\kappa)))d\kappa\right. \\ &\quad \left. - \int_0^{\rho(\iota)} g(\iota,\kappa,\sigma_2(\rho(\kappa)))d\kappa\right|^p \\ &\le \left(\left(\int_0^{\rho(\iota)} 1^q d\kappa\right)^{1/q} \int_0^{\rho(\iota)} |g(\iota,\kappa,\sigma_1(\rho(\kappa)))\right. \\ &\quad \left. - g(\iota,\kappa,\sigma_2(\rho(\kappa)))|^p d\kappa)^{1/p}\right)^p \\ &\le (\rho(\iota))^{p/q}\left(\int_0^{\rho(\iota)} \chi_1(|\sigma_1(\rho(\kappa))d\kappa\right. \\ &\quad \left. - \sigma_2(\rho(\kappa))|^p)d\kappa)^{1/p}\right)^p \\ &\le (\rho(\iota))^{p/q+1}\chi_1(d(\sigma_1,\sigma_2)) \\ &\le \Lambda^p\chi_1(d(\sigma_1,\sigma_2)) \le M(\sigma_1,\sigma_2),\end{aligned} \tag{38}$$

where $M(\sigma_1, \sigma_2)$ is defined by

$$\begin{aligned}M(\sigma_1,\sigma_2) = \max\{&\chi_1(\mu(\iota,\kappa)), \chi_2(\mu(\iota, Y\iota)), \chi_3(\mu(\kappa, Y\kappa)), \chi_4 \\ &\cdot(\mu(\iota, Y\kappa)), \chi_5(\mu(Y\iota,\kappa)\}.\end{aligned} \tag{39}$$

Thus, we obtain that

$$\mu(Y(\sigma_1), Y(\sigma_2)) \le M(\sigma_1,\sigma_2). \tag{40}$$

Using Theorem 10, we obtain that the operator Y admits a fixed point. Thus, the functional integral equation (28) admits at least one solution in $BC[0,\infty)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Authors' Contributions

All authors read and approved the final manuscript.

Acknowledgments

The second author was supported by the Ministry of Education, Science and Technological Development of Serbia, Grant no. 451-03-68 / 2020-14 / 200105.

References

[1] I. D. Aranđelović, M. Mateljević, and B. Ljubomir, "Ćirić (1935–2016)," *Filomat*, vol. 31, no. 11, pp. 3035–3040, 2017.

[2] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.

[3] J. Danes, "Two fixed point theorems in topological and metric spaces," *Bulletin of the Australian Mathematical Society*, vol. 14, no. 2, pp. 259–265, 1976.

[4] A. A. Ivanov, "Fixed points of mappings on metric spaces," *Journal of Soviet Mathematics*, vol. 66, pp. 5–102, 1976.

[5] I. D. Aranđelović, M. Rajović, and V. Kilibarda, "On nonlinear quasi-contractions," *Fixed Point Theory*, vol. 9, no. 2, pp. 387–394, 2008.

[6] M. Bessenyei, "Nonlinear quasicontractions in complete metric spaces," *Expositiones Mathematicae*, vol. 33, no. 4, pp. 517–525, 2015.

[7] S. Alshehri, I. Aranđelović, and N. Shahzad, "Symmetric spaces and fixed points of generalized contractions," *Abstract and Applied Analysis*, vol. 2014, Article ID 763547, 8 pages, 2014.

[8] K. M. Das and K. V. Naik, "Common fixed point theorems for commuting maps on a metric space," *Proceedings of the American Mathematical Society*, vol. 77, no. 3, pp. 369–373, 1979.

[9] C. Di Bari and P. Vetro, "Nonlinear contractions of Ćirić type," *Fixed Point Theory*, vol. 13, no. 2, pp. 453–460, 2012.

[10] M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22, no. 1, pp. 1–72, 1906.

[11] E. W. Chittenden, "On the equivalence of ecart and voisinage," *Transactions of the American Mathematical Society*, vol. 18, pp. 161–166, 1917.

[12] K. Menger, "Untersuchungen über allgemeine metrik," *Mathematische Annalen*, vol. 100, no. 1, pp. 75–163, 1928.

[13] V. W. Niemytzki, "On the "Third Axiom of Metric Space"," *Transactions of the American Mathematical Society*, vol. 29, no. 3, pp. 507–513, 1927.

[14] A. D. Pitcher and E. W. Chittenden, "On the foundations of the Calcul Fonctionnel of Frechet," *Transactions of the American Mathematical Society*, vol. 19, pp. 66–78, 1918.

[15] B. T. Sims, *Some Properties and Generalizations of Semi-Metric Spaces*, Ph. D. dissertation, Iowa State University, 1962.

[16] W. A. Wilson, "On semi-metric spaces," *American Journal of Mathematics*, vol. 53, no. 2, pp. 361–373, 1931.

[17] I. D. Aranđelović and D. J. Kečkić, "Symmetric spaces approach to some fixed point results," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 13, pp. 5157–5168, 2012.

[18] J. Jachymski, J. Matkowski, and T. Świątkowski, "Nonlinear contractions on semimetric spaces," *Journal of Applied Analysis*, vol. 1, no. 2, pp. 125–133, 1995.

[19] M. Cicchese, "Questioni di completezza e contrazioni in spazi metrici generalizzati," *Bollettino dell'Unione Matematica Italiana*, vol. 13-A, no. 5, pp. 175–179, 1976.

[20] T. L. Hicks and B. E. Rhoades, "Fixed point theory in symmetric spaces with applications to probabilistic spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 36, no. 3, pp. 331–344, 1999.

[21] T. H. Bourgin, "Linear topological spaces," *American Journal of Mathematics*, vol. 65, no. 4, pp. 637–659, 1943.

[22] D. H. Hyers, "Locally bounded linear topological spaces," *Revista de Ciencias Lima*, vol. 41, pp. 555–574, 1939.

[23] R. R. Coifman and G. Weiss, "Extensions of hardy spaces and their use in analysis," *Bulletin of the American Mathematical Society*, vol. 83, no. 4, pp. 569–646, 1977.

[24] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Functional Analysis (Ulianowsk. Gos. Ped. Inst.)Functional Analysis (Ulianowsk. Gos. Ped. Inst.)*, vol. 30, pp. 26–37, 1989.

[25] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[26] S.-H. Cho, G.-Y. Lee, and J.-S. Bae, "On coincidence and fixed-point theorems in symmetric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 562130, 10 pages, 2008.

[27] N. Shahzad, M. Ali Alghamdi, S. Alshehri, and I. Arandjelović, "Semi-metric spaces and fixed points of α-φ-contractive maps," *ournal of Nonlinear Sciences and Applications*, vol. 9, no. 5, pp. 3147–3156, 2016.

[28] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.