Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **13** (3) (2019), 733–745.
https://doi.org/10.2298/AADM190315030M

THE ERROR BOUNDS OF GAUSS-LOBATTO QUADRATURES FOR WEIGHTS OF BERNSTEIN-SZEGŐ TYPE

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In this paper, we consider the Gauss-Lobatto quadrature formulas for the Bernstein-Szegő weights, i.e., any of the four Chebyshev weights divided by a polynomial of the form $\rho(t) = 1 - \frac{4\gamma}{(1+\gamma)^2}t^2$, where $t \in (-1,1)$ and $\gamma \in (-1,0]$. Our objective is to study the kernel in the contour integral representation of the remainder term and to locate the points on elliptic contours where the modulus of the kernel is maximal. We use this to derive the error bounds for mentioned quadrature formulas.

1. INTRODUCTION

The Gauss-Lobatto quadrature formula for a (nonnegative) weight function w on the interval $[-1,1]$ is given by

$$\int_{-1}^{1} f(t)w(t)dt = \lambda_0 f(-1) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + \lambda_{n+1} f(1) + R_n(f), \tag{1}$$

where τ_ν are the zeros of the n-th degree (monic) orthogonal polynomial $\pi_n^L(\cdot) = \pi_n^L(\cdot; w^L)$ relative to the weight function $w^L(t) = (1-t^2)w(t)$, and $R_n(f)$ is the

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2010 Mathematics Subject Classification. 65D32, 41A55
Keywords and Phrases. Gauss-Lobatto quadrature formula, Bernstein-Szegő weight functions, contour integral representation, remainder term for analytic functions, error bound.

error term. It is known that the formula (1) has all the weights positive and the degree of precision $d = 2n + 1$.

In this paper, w is one of the weight functions of Bernstein-Szegő type:

$$w(t) \equiv w_\gamma^{(\mp 1/2)}(t) = \frac{(1-t^2)^{\mp 1/2}}{1 - \frac{4\gamma}{(1+\gamma)^2}t^2}, \quad t \in (-1,1), \quad \gamma \in (-1,0], \tag{2}$$

$$w(t) \equiv w_\gamma^{(\mp 1/2, \pm 1/2)}(t) = \frac{(1-t)^{\mp 1/2}(1+t)^{\pm 1/2}}{1 - \frac{4\gamma}{(1+\gamma)^2}t^2}, \quad t \in (-1,1), \quad \gamma \in (-1,0]. \tag{3}$$

The four Chebyshev weight functions are special cases of (2), (3) with $\gamma = 0$.

Denote the (monic) orthogonal polynomials relative to the weight functions (2) and (3) by $\pi_n^{(\mp 1/2)}(\cdot)$ and $\pi_n^{(\mp 1/2, \pm 1/2)}(\cdot)$, respectively.

Let f be an analytic function in a domain D which contains the interval $[-1,1]$, and let Γ be a simple closed curve in D surrounding $[-1,1]$. Then the remainder term in (1) admits the contour integral representation

$$R_n(f) = \frac{1}{2\pi i} \oint_\Gamma K_n(z) f(z) dz, \tag{4}$$

where the kernel is given by

$$K_n(z) = \frac{\varrho_n(z)}{(1-z^2)\pi_n^L(z)}, \quad z \notin [-1,1],$$

and

$$\varrho_n(z) = \int_{-1}^{1} \frac{\pi_n^L(t)}{z-t} w^L(t) dt.$$

The integral representation (4) leads to the error estimate

$$|R_n(f)| \leqslant \frac{l(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_n(z)| \right) \left(\max_{z\in\Gamma} |f(z)| \right), \tag{5}$$

where $l(\Gamma)$ is the length of the contour Γ. We take the contour Γ to be an ellipse $\mathcal{E}_\rho$ with foci at the points ± 1 and the sum of the semi-axes $\rho > 1$:

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid \ z = \frac{1}{2}\left(u + u^{-1}\right), \ 0 \le \theta \le 2\pi \right\}, \quad u = \rho\, e^{i\theta}.$$

The advantage of the elliptical contours, compared to the circular ones, is that such a choice requires the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1.

The modulus of the kernel is symmetric with respect to the real axis. When the weight function is even, i.e., given by (2), this modulus is symmetric with respect to both axes, so it is sufficient to consider the interval $[0, \frac{\pi}{2}]$.

In this paper we will determine the locations on the ellipses where the kernel has maximal modulus. Hence, we derive the error bounds (5) for quadrature (1) with respect to the weight functions $w_\gamma^{(\mp 1/2)}$ and $w_\gamma^{(\mp 1/2, \pm 1/2)}$.

The estimate (5) for the Gaussian quadratures with respect to the Chebyshev weight functions was considered in [**2**] and [**3**]. Later it has been extended by Schira in [**10**] to symmetric weight functions assuming monotonicity, and by Gautschi in [**1**] and Hunter and Nikolov in [**6**] for the Gauss-Radau and Gauss-Lobatto quadratures. For a different approach to the estimation of $R_n(f)$, see [**5**]. An alternative approach, which has initiated this research, has been proposed by S. Notaris [**7**].

Recently, we found some sufficient conditions for the modulus of the kernel to be maximized on one of the axes for each ρ large enough (see [**12**], [**13**], [**14**]) for Gaussian quadrature formulas relative to the weight functions of Bernstein-Szegő type. This issue has been examined for the Gauss-Radau quadratures in [**9**], and for the Gauss-Kronrod quadratures in [**4**]. A more general class of the Bernstein-Szegő weight functions has been considered in [**8**].

1.1 Gauss-Lobatto quadrature with the weight $w_\gamma^{(-1/2)}$

Consider the formula (1) with $w = w_\gamma^{(-1/2)}$ given by (2):

$$\int_{-1}^{1} f(t) w_\gamma^{(-1/2)}(t)dt = Q_n^{(-1/2)}(f) + R_n^{(-1/2)}(f),$$

$$Q_n^{(-1/2)}(f) = \lambda_0^{(-1/2)} f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^{(-1/2)} f(\tau_\nu^{(-1/2)}) + \lambda_{n+1}^{(-1/2)} f(1).$$

Since $w_\gamma^{L(-1/2)}(t) = (1-t^2) w_\gamma^{(-1/2)} = w_\gamma^{(1/2)}(t)$, where $w_\gamma^{(1/2)}(t)$ is given by (2), we have (cf. [**7**], p. 5)

$$\varrho_n^{(-1/2)}(z) = \int_{-1}^{1} \frac{\pi_n^{(1/2)}(t)}{z-t} w_\gamma^{(1/2)}(t)dt, \qquad K_n^{(-1/2)}(z) = \frac{\varrho_n^{(-1/2)}(z)}{(1-z^2)\pi_n^{(1/2)}(z)}, \tag{6}$$

where

$$\pi_n^{(1/2)}(z) = \frac{1}{2^n}(U_n(z) - \gamma U_{n-2}(z)), \quad n \geqslant 1, \tag{7}$$

and U_n is the Chebyshev polynomial of the second kind.

We write $z = (u+u^{-1})/2$, where $u = z + \sqrt{z^2-1}$ and $u^{-1} = z - \sqrt{z^2-1}$. Then we have

$$U_n(z) = \frac{u^{n+1} - u^{-n-1}}{u - u^{-1}}. \tag{8}$$

From (6) and (7) we get

$$\begin{aligned} \varrho_n^{(-1/2)}(z) &= \frac{1}{2^n} \int_{-1}^{1} \frac{U_n \sqrt{1-t^2}}{(z-t)(1-\frac{4\gamma}{(1+\gamma)^2}t^2)} dt - \frac{\gamma}{2^n} \int_{-1}^{1} \frac{U_{n-2}\sqrt{1-t^2}}{(z-t)(1-\frac{4\gamma}{(1+\gamma)^2}t^2)} dt \\ &= -\frac{(1+\gamma)^2}{2^{n+2}\gamma} \left(A_n - \gamma A_{n-2}\right), \end{aligned}$$

where (see formula (12) in [**4**])

$$A_k = \int_{-1}^{1} \frac{U_k\sqrt{1-t^2}}{(z-t)(t^2-\frac{(1+\gamma)^2}{4\gamma})}dt = \begin{cases} \pi\frac{(z-\sqrt{z^2-1})^{k+1}-\gamma^{\frac{k+1}{2}}}{z^2-\frac{(1+\gamma)^2}{4\gamma}}, & k \text{ odd}, \\ \pi\frac{(z-\sqrt{z^2-1})^{k+1}-\frac{2z}{1+\gamma}\gamma^{\frac{k+2}{2}}}{z^2-\frac{(1+\gamma)^2}{4\gamma}}, & k \text{ even}. \end{cases}$$

It is easy to find that

$$\varrho_n^{(-1/2)}(z) = \frac{1}{2^n}\frac{\pi(1+\gamma)^2}{u^{n-1}(u^2-\gamma)} \quad \text{for} \quad n \geqslant 1. \tag{9}$$

Hence

$$(1-z^2)\pi_n^{(1/2)}(z) = -\frac{1}{2^{n+2}}(u^{n+1}-u^{-n-1}-\gamma(u^{n-1}-u^{-n+1}))(u-u^{-1}).$$

Therefore, the expression for the kernel in (6) becomes

$$K_n^{(-1/2)}(z) = \frac{4\pi(1+\gamma)^2}{u^{n-1}(\gamma-u^2)(u^{n+1}-u^{-n-1}-\gamma(u^{n-1}-u^{-n+1}))(u-u^{-1})}. \tag{10}$$

In order to derive an upper bound for the modulus of K_n, we write

$$\begin{aligned} A(\theta) &= |\gamma-u^2|^2 = \rho^4-2\gamma\rho^2\cos 2\theta+\gamma^2, \\ B(\theta) &= |u-u^{-1}|^2 = 2(a_2-\cos 2\theta), \\ C(\theta) &= |u^{n+1}-u^{-n-1}-\gamma(u^{n-1}-u^{-n+1})|^2 = 2(a_{2n+2}-\cos(2n+2)\theta) \\ &\quad +2\gamma^2(a_{2n-2}-\cos(2n-2)\theta)-4\gamma(a_{2n}\cos 2\theta-a_2\cos 2n\theta), \end{aligned} \tag{11}$$

where $a_j = (\rho^j+\rho^{-j})/2$. Thus (10) can be written as

$$\left|K_n^{(-1/2)}(z)\right| = \frac{4\pi(1+\gamma)^2}{\rho^{n-1}} \cdot \frac{1}{\sqrt{A(\theta)B(\theta)C(\theta)}}. \tag{12}$$

In the main theorems in this paper we investigate asymptotic by inspecting the leading coefficients only. This does not work in general, but in these cases it does and there is a formal proof for that.

Theorem 1.1. *For the Gauss-Lobatto quadrature formula (1) with the weight function $w_\gamma^{(-1/2)}$, where $n > 1$ and $\gamma \in (-1,0)$, for each ρ large enough, the modulus of the kernel (12) is maximal on the real axis ($\theta = 0$) if $\gamma \geqslant -\frac{1}{2}$ and on the positive imaginary semi-axis ($\theta = \pi/2$) if $\gamma < -\frac{1}{2}$, i.e.*

$$\max_{z\in\mathcal{E}_\rho}\left|K_n^{(-1/2)}(z)\right| = \begin{cases} \left|K_n^{(-1/2)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|, & \text{for } \gamma \geqslant -\frac{1}{2}, \\ \left|K_n^{(-1/2)}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right|, & \text{for } \gamma < -\frac{1}{2}, \end{cases} \quad \text{whenever} \quad \rho \geqslant \rho^*.$$

If $n = 1$, the modulus of the kernel (12) attains its maximum value on the real axis ($\theta = 0$).

Proof. Using the notation from (11), we need to show that for $\theta_0 = 0$ or $\theta_0 = \pi/2$ we have $\frac{1}{A(\theta)B(\theta)C(\theta)} \leqslant \frac{1}{A(\theta_0)B(\theta_0)C(\theta_0)}$, i.e. that

$$I_{\theta,\theta_0}(\rho) = A(\theta)B(\theta)C(\theta) - A(\theta_0)B(\theta_0)C(\theta_0) \geqslant 0$$

holds for sufficiently large ρ and each $\theta \in [0, \frac{\pi}{2}]$. This will hold if and only if the leading coefficient of $I_{\theta,\theta_0}(\rho)$, written as a power series in ρ, is positive.

For $\theta = 0$ and $\theta = \pi/2$ we have

$$\begin{aligned}
&A(0) = \rho^4 - 2\gamma\rho^2 + \gamma^2, && A(\tfrac{\pi}{2}) = \rho^4 + 2\gamma\rho^2 + \gamma^2,\\
&B(0) = 2(a_2 - 1), && B(\tfrac{\pi}{2}) = 2(a_2 + 1),\\
&C(0) = 2(a_{2n+2} - 1) + 2\gamma^2(a_{2n-2} - 1) - 4\gamma(a_{2n} - a_2),\\
&C(\tfrac{\pi}{2}) = 2a_{2n+2} + 2\gamma^2 a_{2n-2} + 4\gamma a_{2n} + (-1)^n(2 + 2\gamma^2 - 4\gamma a_2).
\end{aligned}$$

For $n = 1$ the leading coefficient in $I_{\theta,0}(\rho)$ is $2(1+\gamma)(1-\cos 2\theta)$, which is positive for $\gamma \in (-1, 0)$ and each $\theta \in (0, \frac{\pi}{2}]$.

Now we assume that $n > 1$.

The degree of $I_{\theta,0}(\rho)$ in ρ is at most $2n+6$ in ρ, and its coefficient at ρ^{2n+6} is $2\,(1+2\gamma)\,(1-\cos 2\theta)$, which is positive for $\gamma > -\frac{1}{2}$ and $\theta \in (0, \frac{\pi}{2}]$.

Similarly, $I_{\theta,\pi/2}(\rho)$ has degree at most $2n+6$ in ρ, and its coefficient at ρ^{2n+6} is $-2\,(1+2\gamma)\,(1+\cos 2\theta)$, which is positive for $\gamma < -\frac{1}{2}$ and $\theta \in [0, \frac{\pi}{2})$.

Next, if $\gamma = -\frac{1}{2}$, the part of $I_{\theta,0}(\rho)$ containing two highest powers of ρ is

$$\begin{aligned}
&5\sin^2 2\theta\rho^{2n+4} + (2 + \tfrac{3}{2}\cos 2\theta - 2\cos^3 2\theta - \tfrac{3}{2}\cos 6\theta)\rho^{2n+2}, && \text{for } n = 2,\\
&3\sin^2 2\theta\rho^{2n+4} + (\tfrac{3}{2} + \tfrac{3}{2}\cos 2\theta - 2\cos^3 2\theta - \cos 6\theta)\rho^{2n+2}, && \text{for } n = 3,\\
&3\sin^2 2\theta\rho^{2n+4} + (\tfrac{1}{2} + \tfrac{3}{2}\cos 2\theta - 2\cos^3 2\theta)\rho^{2n+2}, && \text{for } n > 3.
\end{aligned}$$

Thus the leading coefficient is trivially positive if $\theta \in (0, \frac{\pi}{2})$, whereas for $\theta = \frac{\pi}{2}$ it equals 4, 3 and 1, respectively. □

1.2 Gauss-Lobatto quadrature with the weight $w_\gamma^{(1/2)}$

Consider the formula (1) with $w = w_\gamma^{(1/2)}$ given by (2):

$$\int_{-1}^{1} f(t) w_\gamma^{(1/2)}(t)dt = Q_n^{(1/2)}(f) + R_n^{(1/2)}(f),$$

$$Q_n^{(1/2)}(f) = \lambda_0^{(1/2)} f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^{(1/2)} f(\tau_\nu^{(1/2)}) + \lambda_{n+1}^{(1/2)} f(1).$$

We have

$$\varrho_n^{(1/2)}(z) = \int_{-1}^{1} \frac{\pi_n^{L(1/2)}(t)}{z-t} w_\gamma^{L(1/2)}(t)dt, \qquad K_n^{(1/2)}(z) = \frac{\varrho_n^{(1/2)}(z)}{(1-z^2)\pi_n^{L(1/2)}(z)},$$

where $w_\gamma^{L(1/2)} = (1-t^2)w_\gamma^{(1/2)}$ and $\pi_n^{L(1/2)}(t)$ is the n-th (monic) orthogonal polynomial relative to the weight function $w_\gamma^{L(1/2)}$. Thus, referring to Theorem 3.4 in [**7**], we have

$$\pi_n^{L(1/2)}(t) = \frac{1}{t^2-1}\left(\pi_{n+2}^{(1/2)}(t) - \alpha\pi_n^{(1/2)}(t)\right), \quad n \geqslant 1, \tag{13}$$

where

$$\alpha = \frac{n-\gamma n+3-\gamma}{4(n-\gamma n+1+\gamma)} > 0. \tag{14}$$

Now it follows that

$$\varrho_n^{(1/2)}(z) = \int_{-1}^{1} \frac{\pi_{n+2}^{(1/2)}(t) - \alpha\pi_n^{(1/2)}(t)}{(t^2-1)(z-t)}(1-t^2)w_\gamma^{(1/2)}(t)dt = \alpha\varrho_n^{(-1/2)}(z) - \varrho_{n+2}^{(-1/2)}(z).$$

Therefore, from (9) we get

$$\varrho_n^{(1/2)}(z) = \frac{\pi(4\alpha u^2-1)(1+\gamma)^2}{2^{n+2}u^{n+1}(u^2-\gamma)}.$$

Moreover, substituting (7) and (8) in (13) yields

$$(1-z^2)\pi_n^{L(1/2)}(z) = -\frac{Q}{2^{n+2}(u-u^{-1})},$$

where $Q = u^{n+3} - u^{-n-3} - (\gamma+4\alpha)(u^{n+1} - u^{-n-1}) + 4\alpha\gamma(u^{n-1} - u^{-n+1})$. Hence

$$K_n^{(1/2)}(z) = \frac{\pi(1+\gamma)^2(4\alpha u^2-1)(u-u^{-1})}{u^{n+1}(\gamma-u^2)\cdot Q}. \tag{15}$$

In order to derive the upper bound for the modulus of the kernel, we write

$$\begin{aligned} D(\theta) &= |4\alpha u^2-1|^2 = 16\alpha^2\rho^4 - 8\alpha\rho^2\cos 2\theta + 1, \\ E(\theta) &= |Q|^2 = 2(a_{2n+6} - \cos(2n+6)\theta) + 2(\gamma+4\alpha)^2(a_{2n+2} - \cos(2n+2)\theta) \\ &\quad +32\alpha^2\gamma^2(a_{2n-2} - \cos(2n-2)\theta \\ &\quad -4(\gamma+4\alpha)(a_{2n+4}\cos 2\theta - a_2\cos(2n+4)\theta) \\ &\quad +16\alpha\gamma(a_{2n+2}\cos 4\theta - a_4\cos(2n+2)\theta) \\ &\quad -16\alpha\gamma(\gamma+4\alpha)(a_{2n}\cos 2\theta - a_2\cos 2n\theta). \end{aligned}$$

Then (15) yields

$$\left|K_n^{(1/2)}(z)\right| = \frac{\pi(1+\gamma)^2}{\rho^{n+1}}\sqrt{\frac{B(\theta)D(\theta)}{A(\theta)E(\theta)}}, \tag{16}$$

where A and B are defined by the first two equations in (11).

Theorem 1.2. *For the Gauss-Lobatto quadrature formula (1) with the weight function $w_\gamma^{(1/2)}$, where $n > 2$ and $\gamma \in (-1,0)$, the modulus of the kernel (16) is maximal on the positive imaginary semi-axis ($\theta = \pi/2$) for each ρ large enough, i.e.*

$$\max_{z\in\mathcal{E}_\rho}\left|K_n^{(1/2)}(z)\right| = \left|K_n^{(1/2)}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| \quad \textit{whenever} \quad \rho \geqslant \rho^*.$$

If $n = 1$, the modulus of the kernel is maximal on the positive imaginary semi axis ($\theta = \pi/2$) for $\gamma < \frac{1-\sqrt{5}}{2}$ and on the real axis ($\theta = 0$) for $\gamma \geqslant \frac{1-\sqrt{5}}{2}$.

If $n = 2$, the modulus of the kernel is maximal on the positive imaginary semi axis ($\theta = \pi/2$) for $\gamma < g_1$, and on the real axis ($\theta = 0$) for $\gamma \geqslant g_1$, where $g_1 \approx -0.0474$ is the smallest zero of the polynomial $-6\gamma^3 + 23\gamma^2 - 20\gamma - 1$.

Proof. We need to show that for $\theta_0 = 0$ or $\theta_0 = \pi/2$ we have $\frac{B(\theta)D(\theta)}{A(\theta)E(\theta)} \leqslant \frac{B(\theta_0)D(\theta_0)}{A(\theta_0)E(\theta_0)}$, i.e.

$$I_{\theta,\theta_0}(\rho) = B(\theta_0)D(\theta_0)A(\theta)E(\theta) - B(\theta)D(\theta)A(\theta_0)E(\theta_0) \geqslant 0$$

holds for sufficiently large ρ and each $\theta \in [0, \frac{\pi}{2}]$. This will hold if and only if the leading coefficient of $I_{\theta,\theta_0}(\rho)$, written as a power series in ρ, is positive.

For $\theta = 0$ and $\theta = \pi/2$ we have

$$\begin{aligned}
D(0) &= 16\alpha^2\rho^4 - 8\alpha\rho^2 + 1, \qquad\qquad D(\tfrac{\pi}{2}) = 16\alpha^2\rho^4 + 8\alpha\rho^2 + 1,\\
E(0) &= 2(a_{2n+6} - 1) + 2(\gamma + 4\alpha)^2(a_{2n+2} - 1) + 32\alpha^2\gamma^2(a_{2n-2} - 1)\\
&\quad - 4(\gamma + 4\alpha)(a_{2n+4} - a_2) + 16\alpha\gamma(a_{2n+2} - a_4) - 16\alpha\gamma(\gamma + 4\alpha)(a_{2n} - a_2),\\
E(\tfrac{\pi}{2}) &= 2a_{2n+6} + 4(\gamma + 4\alpha)a_{2n+4} + (16\alpha\gamma + 2(\gamma + 4\alpha)^2)a_{2n+2}\\
&\quad + 16\alpha\gamma(\gamma + 4\alpha)a_{2n} + 32\alpha^2\gamma^2 a_{2n-2}\\
&\quad + (-1)^n[2+2(\gamma + 4\alpha)^2+32\alpha^2\gamma^2+4(\gamma+4\alpha)(1+4\alpha\gamma)a_2+16\alpha\gamma a_4].
\end{aligned}$$

It is easily obtained that the degree of $I_{\theta,\pi/2}(\rho)$ in ρ is at most $2n + 14$, and the coefficient at ρ^{2n+14} is

$$8\alpha(-16\alpha^2 - 8\gamma\alpha + 4\alpha + 1)(1 + \cos 2\theta),$$

which by (14) transforms into

$$\frac{8\alpha(-2\gamma^3(n^2 - 1) + \gamma^2(5n^2 + 4n - 5) - 4\gamma(n^2 + n - 1) + n^2 - 5)}{(1 + \gamma + n - \gamma n)^2}(1 + \cos 2\theta).$$

If $n > 2$ and $\gamma \in (-1, 0)$, this is obviously positive for all $\theta \in [0, \frac{\pi}{2})$. Moreover, for $n = 1$ and $n = 2$ this coefficient simplifies to

$$8\alpha(\gamma^2 - \gamma - 1)(1 + \cos 2\theta) \quad \text{and} \quad \frac{8\alpha(-6\gamma^3 + 23\gamma^2 - 20\gamma - 1)}{(\gamma - 3)^2}(1 + \cos 2\theta),$$

which is positive on $\theta \in [0, \frac{\pi}{2})$ for $\gamma < \frac{1-\sqrt{5}}{2}$ and $\gamma < g_1$, respectively.

Next, the coefficient in $I_{\theta,0}(\rho)$ at ρ^{2n+14} is

$$8\alpha(16\alpha^2 + 8\gamma\alpha - 4\alpha - 1)(1 - \cos 2\theta),$$

which is positive on $\theta \in (0, \frac{\pi}{2}]$ for $\gamma > \frac{1-\sqrt{5}}{2}$ if $n = 1$, and for $\gamma > g_1$ if $n = 2$.

Finally, for $(n, \gamma) = \left(1, \frac{1-\sqrt{5}}{2}\right)$ and $(n, \gamma) = (2, g_1)$ the expression $I_{\theta,0}(\rho)$ has degree $2n + 12$ in ρ. The leading coefficient is then

$$(11+5\sqrt{5})(1-\cos 4\theta) \quad \text{and} \quad \frac{2(5-3g_1)\left(-3g_1^4-4g_1^3+46g_1^2-56g_1+9\right)}{(3-g_1)^3}(1-\cos 4\theta),$$

respectively, which is positive for each $\theta \in (0, \frac{\pi}{2}]$. ∎

1.3 Gauss-Lobatto quadrature with the weight $w_\gamma^{(\mp 1/2,\pm 1/2)}$

Since $w_\gamma^{(-1/2,1/2)}(t) = w_\gamma^{(1/2,-1/2)}(-t)$, it is enough to consider only the weight function $w_\gamma^{(1/2,-1/2)}$.

Consider the formula (1) with $w = w_\gamma^{(1/2,-1/2)}$ given by (3):

$$\int_{-1}^{1} f(t) w_\gamma^{(1/2,-1/2)}(t)dt = Q_n^{(1/2,-1/2)}(f) + R_n^{(1/2,-1/2)}(f),$$

$$Q_n^{(1/2,-1/2)}(f) = \lambda_0^{(1/2,-1/2)} f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^{(1/2,-1/2)} f(\tau_\nu^{(1/2,-1/2)}) + \lambda_{n+1}^{(1/2,-1/2)} f(1).$$

We have

$$\varrho_n^{(1/2,-1/2)}(z) = \int_{-1}^{1} \frac{\pi_n^{L(1/2,-1/2)}(t)}{z-t} w_\gamma^{L(1/2,-1/2)}(t)dt$$

and

$$K_n^{(1/2,-1/2)}(z) = \frac{\varrho_n^{(1/2,-1/2)}(z)}{(1-z^2)\pi_n^{L(1/2,-1/2)}(z)},$$

where $w_\gamma^{L(1/2,-1/2)} = (1-t^2) w_\gamma^{(1/2,-1/2)} = (1-t) w_\gamma^{(1/2)}(t)$ and $\pi_n^{L(1/2,-1/2)}(t)$ is the n-th (monic) orthogonal polynomial relative to the weight function $w_\gamma^{L(1/2,-1/2)}$.

From (3.23) and (3.21) in [**7**] we have

$$\pi_n^{L(1/2,-1/2)}(t) = \frac{1}{t-1}(\pi_{n+1}^{(1/2)}(t) - \beta \pi_n^{(1/2)}(t)) \quad \text{for} \quad n \geqslant 1, \tag{17}$$

where

$$\beta = \frac{1}{2} \frac{(1-\gamma)n+2}{(1-\gamma)n+1+\gamma} > 0.$$

Now it follows from (9) that

$$\begin{aligned} \varrho_n^{(1/2,-1/2)}(z) &= \int_{-1}^{1} \frac{\pi_{n+1}^{(1/2)}(t) - \beta\pi_n^{(1/2)}(t)}{(t-1)(z-t)} (1-t) w_\gamma^{(1/2)}(t)dt = \beta \varrho_n^{(-1/2)} - \varrho_{n+1}^{(-1/2)} \\ &= \frac{\pi(2\beta u - 1)(1+\gamma)^2}{2^{n+1} u^n (u^2-\gamma)}. \end{aligned}$$

Moreover, substituting (7) and (8) into (17) yields

$$\begin{aligned} (1-z^2)\pi_n^{L(1/2,-1/2)}(z) &= -(z+1)(\pi_{n+1}^{(1/2)}(z) - \beta \pi_n^{(1/2)}(z)) \\ &= -\frac{1}{2^{n+2}} \frac{u+1}{u-1} (u^{n+2} - u^{-n-2} - \gamma(u^n - u^{-n}) \\ &\quad -2\beta(u^{n+1} - u^{-n-1}) + 2\beta\gamma(u^{n-1} - u^{-n-1})). \end{aligned}$$

Hence

$$K_n^{(1/2,-1/2)}(z) = \frac{2\pi(2\beta u - 1)(1+\gamma)^2}{u^n(\gamma - u^2)\cdot P} \cdot \frac{u-1}{u+1}, \tag{18}$$

where

$$P = u^{n+2} - u^{-n-2} - \gamma(u^n - u^{-n}) - 2\beta(u^{n+1} - u^{-n-1}) + 2\beta\gamma(u^{n-1} - u^{-n-1}).$$

In order to derive an upper bound for the modulus of the kernel, we write

$$\begin{aligned}
F(\theta) &= |2\beta u - 1|^2 = 4\beta^2\rho^2 - 4\beta\rho\cos\theta + 1,\\
G(\theta) &= |u - 1|^2 = \rho^2 - 2\rho\cos\theta + 1,\\
H(\theta) &= |u + 1|^2 = \rho^2 + 2\rho\cos\theta + 1,\\
J(\theta) = |P|^2 &= 2(a_{2n+4} - \cos(2n+4)\theta) + 2\gamma^2(a_{2n} - \cos 2n\theta)\\
&\quad +8\beta^2(a_{2n+2} - \cos(2n+2)\theta) + 8\beta^2\gamma^2(a_{2n-2} - \cos(2n-2)\theta)\\
&\quad -4\gamma(a_{2n+2}\cos 2\theta - a_2\cos(2n+2)\theta)\\
&\quad -8\beta(a_{2n+3}\cos\theta - a_1\cos(2n+3)\theta)\\
&\quad +8\beta\gamma(a_{2n+1}\cos 3\theta - a_3\cos(2n+1)\theta)\\
&\quad +8\beta\gamma(a_{2n+1}\cos\theta - a_1\cos(2n+1)\theta)\\
&\quad -8\beta\gamma^2(a_{2n-1}\cos\theta - a_1\cos(2n-1)\theta)\\
&\quad -16\beta^2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta).
\end{aligned}$$

Then (18) yields

$$\left|K_n^{(1/2,-1/2)}(z)\right| = \frac{2\pi(1+\gamma)^2}{\rho^n}\sqrt{\frac{F(\theta)G(\theta)}{A(\theta)H(\theta)J(\theta)}}, \tag{19}$$

where A is defined by first equation in (11).

Theorem 1.3. *For the Gauss-Lobatto quadrature formula (1) with the weight function $w_\gamma^{(1/2,-1/2)}$, where $\gamma \in (-1,0)$, the modulus of the kernel (19) is maximal on the negative real semi-axis ($\theta = \pi$) for each ρ large enough, i.e.*

$$\max_{z\in\mathcal{E}_\rho}\left|K_n^{(1/2,-1/2)}(z)\right| = \left|K_n^{(1/2,-1/2)}\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right| \quad \textit{whenever} \quad \rho \geqslant \rho^*.$$

Proof. We need to show that $\frac{F(\theta)G(\theta)}{A(\theta)H(\theta)J(\theta)} \leqslant \frac{F(\pi)G(\pi)}{A(\pi)H(\pi)J(\pi)}$, i.e. that

$$I_\theta(\rho) = F(\pi)G(\pi)A(\theta)H(\theta)J(\theta) - F(\theta)G(\theta)A(\pi)H(\pi)J(\pi) \geqslant 0$$

holds for sufficiently large ρ and each $\theta \in [0,\pi]$.

The degree of $I_\theta(\rho)$ in ρ is $2n+13$, and the coefficient at ρ^{2n+13} is

$$\frac{4\beta(\gamma^2\left(2n^2-4n+1\right)+\gamma\left(6-4n^2\right)+2n^2+4n+1)}{(-n\gamma+\gamma+n+1)^2}(1+\cos\theta).$$

This is obviously positive for all $n > 1$, $\gamma \in (-1,0)$ and $\theta \in [0,\pi)$. One can show that it is also positive for $n = 1$, $\gamma \in (-1,0)$ and $\theta \in [0,\pi)$. □

Now the statement for the weight function $w_\gamma^{(-1/2,1/2)}$ follows by symmetry.

Theorem 1.4. *For the Gauss-Lobatto quadrature formula (1) with the weight function $w_\gamma^{(-1/2,1/2)}$, where $\gamma \in (-1,0)$, the modulus of the kernel (19) is maximal on the positive real semi-axis ($\theta = 0$) for each ρ large enough, i.e.*

$$\max_{z\in\mathcal{E}_\rho}\left|K_n^{(-1/2,1/2)}(z)\right| = \left|K_n^{(-1/2,1/2)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right| \quad \textit{whenever} \quad \rho \geqslant \rho^*.$$

2. NUMERICAL RESULTS

In theorems 1.1, 1.2 and 1.3 we proved that, given n and γ, the argument θ of the maximum point z of the function $|K_n(z)|$ on $\mathcal{E}_\rho$ eventually stabilizes at some value $\theta_0 \in \{0, \pi/2, \pi\}$ as ρ increases. The smallest ρ for which this happens was denoted by ρ^*. The tables below show, in each case, the value of ρ^* obtained experimentally, with four significant digits.

Table 1: The smallest possible corresponding value of ρ^* from Theorem 1.1 in the case of the weight function $w_\gamma^{(-1/2)}$ for $n = 10$.

γ	-0.99	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
ρ^*	1.006	1.056	1.097	1.133	1.183	1.001	1.001	1.001	1.001	1.001

Table 2: The smallest possible corresponding value of ρ^* from Theorem 1.2 in the case of the weight function $w_\gamma^{(1/2)}$ for $n = 10$.

γ	-0.99	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
ρ^*	1.006	1.054	1.092	1.124	1.152	1.179	1.204	1.229	1.255	1.282

Table 3: The smallest possible corresponding value of ρ^* from Theorem 1.3 in the case of the weight function $w_\gamma^{(-1/2,1/2)}$ for $n = 10$.

γ	-0.99	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
ρ^*	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001

Example 1. Suppose that we want to approximate the integral

$$\int_{-1}^{1}\cos(t)w_\gamma^{(-1/2)}(t)dt$$

using the Gauss-Lobatto formula (1), where $w_\gamma^{(-1/2)}(t)$ is given by (2). The function $f_1(z) = \cos(z)$ is entire and we have

$$\max_{z\in\mathcal{E}_\rho}|\cos(z)| = \cosh(b_1), \quad \text{where} \quad b_1 = \frac{1}{2}(\rho-\rho^{-1}).$$

The length of the ellipse in (5) can be estimated by (cf. [**11**, Eq. (2.2)])

$$l(\mathcal{E}_\rho) \leqslant L(\rho) = 2\pi a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right),$$

where $a_1 = \frac{1}{2}(\rho + \rho^{-1})$. Thus we get the error bound

$$|R_n(f)| \leqslant r(f_1) = \inf_{\rho_n^* < \rho < +\infty} \left(\frac{4\pi(1+\gamma)^2}{\rho^{n-1}} \cdot L(\rho) \cdot \frac{\cosh(b_1)}{\sqrt{A(\theta_0)B(\theta_0)C(\theta_0)}} \right),$$

where A, B, C and θ_0 are described in Theorem 1.1.

In Table 4, this estimate is compared with the estimates (1.7) and (4.2) from [**7**] and the actual error, denoted by $\hat{r}_1^{(Not)}$, $\hat{r}_2^{(Not)}$ and *Error*, respectively.

Table 4: The values of $\hat{r}_1^{(Not)}(f_1), \hat{r}_2^{(Not)}(f_1), r(f_1)$, *Error* for some n and $\gamma \in (-1, 0)$ in the case of the weight function $w_\gamma^{(-1/2)}$.

n	γ	$\hat{r}_1^{(Not)}(f_1)$	$\hat{r}_2^{(Not)}(f_1)$	$r(f_1)$	*Error*
2	-0.9	1.36(−6)	4.43(−6)	4.09(−6)	1.33(−6)
5	-0.9	3.20(−14)	1.43(−13)	1.37(−13)	3.15(−14)
10	-0.9	1.33(−29)	7.96(−29)	7.78(−29)	1.32(−29)
20	-0.9	1.02(−65)	8.33(−65)	8.23(−65)	1.01(−65)
2	-0.5	3.41(−5)	1.11(−4)	1.02(−4)	3.30(−5)
5	-0.5	8.01(−13)	3.58(−12)	3.43(−12)	7.86(−13)
10	-0.5	3.33(−28)	1.99(−27)	1.94(−27)	3.30(−28)
20	-0.5	2.54(−64)	2.08(−63)	2.06(−63)	2.53(−64)
2	-0.1	1.11(−4)	3.63(−4)	3.32(−4)	1.07(−4)
5	-0.1	2.59(−12)	1.16(−11)	1.11(−11)	2.55(−12)
10	-0.1	1.08(−27)	6.45(−27)	6.30(−27)	1.07(−27)
20	-0.1	8.24(−64)	6.74(−63)	6.66(−63)	8.19(−64)

3. CONCLUDING REMARKS

For one class of Bernstein-Szegő weights for analytic integrands, explicit error bounds were obtained in [**12**], [**13**], [**14**] for Gaussian quadrature formulas, and in [**9**] and [**4**] for Gauss-Radau and Gauss-Kronrod quadrature formulas. In this paper, we continue with the analogous analysis for Gauss-Lobatto quadrature and obtain the error bounds. Numerical examples are also included.

Acknowledgements This article was supported by the Serbian Ministry of Education, Science and Technological Development (Research Project: "Methods of numerical and nonlinear analysis with applications" (#174002)).

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(Received 15.03.2019)
(Revised 30.07.2019)

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