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On the Torque Transmission by a Cardan-Hooke Joint

Kinematics and dynamics of a Cardan-Hooke joint are investigated. Kinematic analysis is based on the kinematic chain rule for angular velocity vectors. Dynamics of the Cardan-Hooke joint is analyzed by means of the Lagrange equations of the second kind. The Cardan-Hooke joint is analysed under varying operating conditions, that is, it is assumed that the input shaft has variable angular velocity. Two cases are considered: (1) the driving yoke plane coincides with the plane of the shafts; (2) the driving yoke plane is normal to the plane of the shafts. An expression for torque transmission in a Cardan-Hooke joint in varying operating conditions was developed. The expression contains terms representing inertia of the shafts and the cross of the Cardan-Hooke joint. Theoretical considerations are accompanied by a numerical example.

Keywords: Cardan-Hooke joint, kinematics, dynamics, torque transmission

1. INTRODUCTION

In literature regarding the dynamics of the Cardan-Hooke joint, the inertial characteristics of shafts and a cross and their influence on the joint torque transmission are often ignored (see e.g. [1]). In [2], the cross inertia and its effect on shafts bearing reactions at the constant angular velocity of the input shaft was considered.

On the other hand, the problem of the torque transmission at a variable angular velocity of the input shaft and in the presence of both structural and error misalignments between the shafts is considered in [3]. Moreover, the inertial characteristic of the cross is not taken into account. In [4], only the inertia of shafts was taken into account.

The aim of this paper is to form a relation between the applied and resisting torques, which will be taken into account inertial characteristics of both the cross and the shafts. In general case, the variable angular velocity of the input shaft is assumed where elasticity effects and friction are ignored.

2. KINEMATICS OF A CARDAN-HOOKE JOINT

Various approaches have been used in kinematic analysis of the Cardan-Hooke joint, such as: the descriptive geometry approach [5,6], the transformation -matrix approach [7], the space analytic geometry approach [8], the approach based on the kinematic chain rule for angular velocity vectors [9] etc. In this paper, the kinematic analysis is based on the approach from reference [9]. In authors' opinion, this approach requires minimal computation effort and is simple in application.

2.1 The case when the driving yoke plane coincides with the plane of the shafts

Figure 1 shows a Cardan-Hooke joint composed of three rigid bodies (V_1), (V_2), and (V_3) where (V_1) is the input shaft with driving yoke, (V_2) is the cross, and (V_3) is the output shaft with driven yoke. Without loss of generality it is assumed that the input shaft is placed along the y-axis of the reference frame $Oxyz$ and that axes of the input and output shaft crossing at the point O_1 lie in the coordinate plane Oyz.
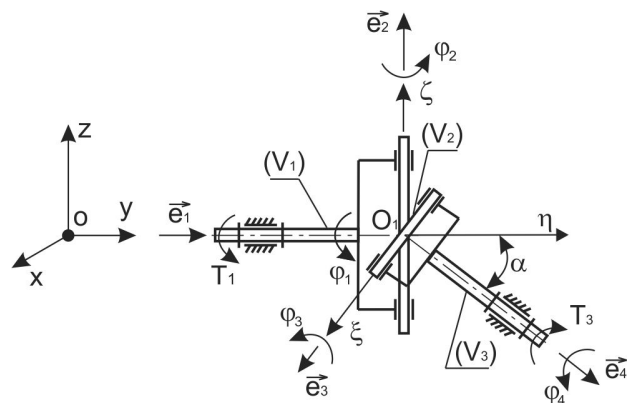

Figure 1. Cardan-Hooke joint – Variant I

The cross is connected to the driving and driven yokes with the revolute joints whose axes are represented with the unit vectors $\vec{e}_2$ and $\vec{e}_3$, respectively. The unit vectors of the input and output shafts are denoted by $\vec{e}_1$ and $\vec{e}_4$, respectively. Let φ_1, φ_2, φ_3, φ_4 and α represent, respectively, the rotating angle of the input shaft, the relative rotating angle of the cross about the axis $\vec{e}_2$ with respect to the input shaft, the relative rotating angle of the cross about the axis $\vec{e}_3$ with respect to the output shaft, the rotating angle of the output shaft, and the angle between axes of the shafts connected. The local coordinate frame $O_1\xi\eta\zeta$ is fixed to the cross in the manner shown in Figure 1. In the

reference configuration of the Cardan-Hooke joint shown in Figure 1 one has that $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi_3 = \alpha$. In this reference configuration, the driving yoke plane coincides with the plane Oyz. As in [9], applying the kinematic chain rule for angular velocity vectors [9] to the bodies (V_1) and (V_2) yields the following expression for the angular velocity of the cross, $\vec{\omega}_2$:

$$\vec{\omega}_2 = \dot{\varphi}_1\vec{e}_1 + \dot{\varphi}_2\vec{e}_2, \qquad (1)$$

where $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are, respectively, the angular speed of the input shaft and the relative angular speed of the cross with respect to the input shaft. Similarly, applying the kinematic chain rule for angular velocity vectors to the bodies (V_2) and (V_3) yields:

$$\vec{\omega}_2 = \dot{\varphi}_4\vec{e}_4 + \dot{\varphi}_3\vec{e}_3, \qquad (2)$$

where $\dot{\varphi}_3$ and $\dot{\varphi}_4$ are, respectively, the relative angular speed of the cross with respect to the output shaft and the angular speed of the output shaft. The vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_4$ may be expressed in the frame $Oxyz$ as follows:

$$\vec{e}_1 = \vec{j}, \quad \vec{e}_2 = \sin\varphi_1\vec{i} + \cos\varphi_1\vec{k} \qquad (3)$$

$$\vec{e}_4 = \cos\alpha\,\vec{j} - \sin\alpha\,\vec{k} \qquad (4)$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors of the axes x, y, and z, respectively. By observing Figure 1, it is obvious that:

$$\vec{e}_3 \cdot \vec{e}_4 = 0, \quad \vec{e}_3 \cdot \vec{e}_2 = 0, \quad \vec{e}_1 \cdot \vec{e}_2 = 0 \qquad (5)$$

Equating (1) with (2) yields:

$$\dot{\varphi}_1\vec{e}_1 + \dot{\varphi}_2\vec{e}_2 = \dot{\varphi}_4\vec{e}_4 + \dot{\varphi}_3\vec{e}_3. \qquad (6)$$

Premultiplying both sides of (6) by $\vec{e}_4$ and $\vec{e}_2$ one has, respectively, that:

$$\dot{\varphi}_1\cos\alpha - \dot{\varphi}_2\sin\alpha\cos\varphi_1 = \dot{\varphi}_4 \qquad (7)$$

$$\dot{\varphi}_2 = -\dot{\varphi}_4\sin\alpha\cos\varphi_1 \qquad (8)$$

Further, solving (7) and (8) for $\dot{\varphi}_4$ and $\dot{\varphi}_2$ gives:

$$\dot{\varphi}_4 = \frac{\cos\alpha}{1 - \sin^2\alpha\cos^2\varphi_1}\dot{\varphi}_1 \equiv f_{\phi_4(I)}\dot{\varphi}_1 \qquad (9)$$

$$\dot{\varphi}_2 = -\frac{\sin\alpha\cos\alpha\cos\varphi_1}{1 - \sin^2\alpha\cos^2\varphi_1}\dot{\varphi}_1 \equiv f_{\phi_2(I)}\dot{\varphi}_1 \qquad (10)$$

Finally, after the integration of (10) it is obtained that:

$$\varphi_2 = -\tan^{-1}(\sin\varphi_1\tan\alpha). \qquad (11)$$

2.2 The case when the driving yoke plane is normal to the plane of the shafts

Let us now consider a Cardan-Hooke joint in which, when the joint is in the reference configuration $\varphi_1 = 0$, $\varphi_2 = -\alpha$, and $\varphi_3 = 0$, the driving yoke plane is normal to the plane of the shafts as it is shown in Figure 2. Now, one has that:

$$\vec{e}_2 = \cos\phi_1\vec{i} - \sin\phi_1\vec{k} \qquad (12)$$

while the vectors $\vec{e}_1$ and $\vec{e}_4$ are determined by the same expressions as in (3) and (4). Applying the similar procedure as in previous section, the following relations are obtained:

$$\dot{\varphi}_1\cos\alpha + \dot{\varphi}_2\sin\alpha\sin\dot{\varphi}_1 = \dot{\varphi}_4 \qquad (13)$$

$$\dot{\varphi}_2 = \dot{\varphi}_4\sin\alpha\sin\varphi_1 \qquad (14)$$

By solving (13) and (14) for $\dot{\varphi}_4$ and $\dot{\varphi}_2$, the following expressions for angular speeds are obtained:

$$\dot{\varphi}_4 = \frac{\cos\alpha}{1 - \sin^2\alpha\sin^2\varphi_1}\dot{\varphi}_1 \equiv f_{\varphi_4(II)}\dot{\varphi}_1, \qquad (15)$$

$$\dot{\varphi}_2 = \frac{\sin\alpha\cos\alpha\sin\varphi_1}{1 - \sin^2\alpha\sin^2\varphi_1}\dot{\varphi}_1 \equiv f_{\varphi_2(II)}\dot{\varphi}_1. \qquad (16)$$

Finally, the integration of (16) yields the following expression for the angle φ_2:

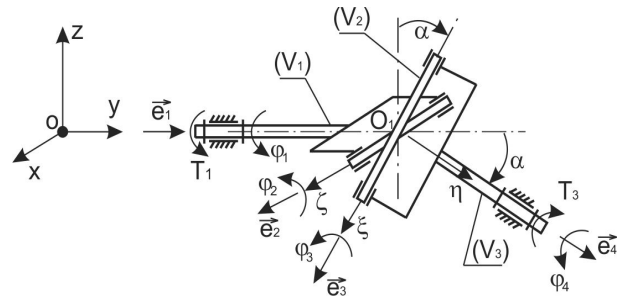$$\varphi_2 = -\tan^{-1}(\cos\varphi_1\tan\alpha) \qquad (17)$$

Figure 2. Cardan-Hooke joint – Variant II

3. DYNAMICS OF A CARDAN-HOOKE JOINT

3.1 The case when the driving yoke plane coincides with the plane of the shafts

During the motion of the Cardan-Hooke joint, the shafts perform the rotation about fixed axes while the cross performs rotation about fixed point O_1. Taking this fact into account, the kinetic energy of the Cardan-Hooke joint is determined by the following expression:

$$E_k = \frac{J_1\dot{\varphi}_1^2}{2} + \frac{J_3\dot{\varphi}_4^2}{2} + \frac{J_{2\xi}\omega_{2\xi}^2 + J_{2\eta}\omega_{2\eta}^2 + J_{2\zeta}\omega_{2\zeta}^2}{2} \qquad (18)$$

where J_1 and J_3 are the mass moments of inertia of the shafts, $J_{2\xi}$, $J_{2\eta}$, and $J_{2\zeta}$ are the mass moments of inertia about the cross principal inertia axes ξ, η and ζ, and $\omega_{2\xi}$, $\omega_{2\eta}$ and $\omega_{2\zeta}$ are the projections of the angular velocity of the cross, $\vec{\omega}_2$, onto the axes of the local coordinate frame $O_1\xi\eta\zeta$. Based on (1), the projections $\omega_{2\xi}$, $\omega_{2\eta}$ and $\omega_{2\zeta}$ are determined by:

$$\omega_{2\xi} = \dot{\varphi}_1\sin\varphi_2, \qquad (19)$$

$$\omega_{2\eta} = \dot{\varphi}_1\cos\varphi_2, \quad \omega_{2\zeta} = \dot{\varphi}_2 \qquad (20)$$

Taking into account (9), (10), (11), (19), and (20), the kinetic energy in expended form reads:

$$E_k = \frac{1}{2}\left(J_1 + J_3 f_{\varphi4(I)}^2 + J_{2\zeta} f_{\varphi2(I)}^2 + \right.$$
$$\left. + J_{2\eta} \cos^2 \varphi_2 + J_{2\xi} \sin^2 \varphi_2 \right) \dot{\varphi}_1^2 \qquad (21)$$

Applying Lagrange's equations of the second kind [10], the differential equation of motion of the Cardan-Hooke joint reads:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\phi}_1} - \frac{\partial E_k}{\partial \phi_1} = Q_{\phi_1} \qquad (22)$$

where Q_{ϕ_1} is the generalized force obtained from the expression for the sum of virtual works of the applied torque T_1 acting on the input shaft and the resisting torque T_3 acting on the output shaft as follows:

$$\delta A = T_1 \delta \varphi_1 + (-T_3 \delta \varphi_4) =$$
$$= \left(T_1 - T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \cos^2 \varphi_1}\right) \delta \varphi_1 \equiv Q_{\varphi_1} \delta \varphi_1 \qquad (23)$$

Now, (22) in developed form reads:

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \cos^2 \varphi_1} + (J_1 + J_3 f_{\varphi4(I)}^2 +$$
$$+ J_{2\zeta} f_{\varphi2(I)}^2 + J_{2\eta} \cos^2 \varphi_2 + J_{2\xi} \sin^2 \varphi_2) \ddot{\varphi}_1 +$$
$$+ \left(J_3 f_{\varphi4(I)} \frac{\partial f_{\varphi4(I)}}{\partial \varphi_1} + J_{2\zeta} f_{\varphi2(I)} \frac{\partial f_{\varphi2(I)}}{\partial \varphi_1}\right.$$
$$\left. + \left(\frac{\partial \varphi_2}{\partial \varphi_1} - 2 f_{\varphi2(I)}\right) \frac{J_{2\eta} - J_{2\xi}}{2} \sin 2\varphi_2 \right) \dot{\varphi}_1^2 \qquad (24)$$

In the case of a constant angular velocity of the input shaft, that is, $\ddot{\varphi}_1 = 0$, one has that:

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \cos^2 \varphi_1} +$$
$$+ \left(J_3 f_{\varphi4(I)} \frac{\partial f_{\varphi4(I)}}{\partial \varphi_1} + J_{2\zeta} f_{\varphi2(I)} \frac{\partial f_{\varphi2(I)}}{\partial \varphi_1} + \right.$$
$$\left. + \left(\frac{\partial \varphi_2}{\partial \varphi_1} - 2 f_{\varphi2(I)}\right) \frac{J_{2\eta} - J_{2\xi}}{2} \sin 2\varphi_2 \right) \dot{\varphi}_1^2 \qquad (25)$$

wherefrom, ignoring the inertial characteristic of cross and output shaft, the following common inter-dependence of the torques is obtained (see [1]):

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \cos^2 \varphi_1} \qquad (26)$$

3.2 The case when the driving yoke plane is normal to the plane of the shafts

In this case one has that:

$$\omega_{2\xi} = -\dot{\varphi}_1 \sin(\alpha - \varphi_2) \qquad (27)$$
$$\omega_{2\eta} = \dot{\varphi}_1 \cos(\alpha - \varphi_2) \qquad (28)$$
$$\omega_{2\zeta} = \dot{\varphi}_2 \qquad (29)$$

$$E_k = \frac{1}{2}\left(J_1 + J_3 f_{\varphi4(II)}^2 + J_{2\zeta} f_{\varphi2(II)}^2 + \right.$$
$$\left. + J_{2\eta} \cos^2(\alpha - \phi_2) + J_{2\xi} \sin^2(\alpha - \varphi_2)\right) \dot{\varphi}_1^2 \qquad (30)$$

$$\delta A = T_1 \delta \phi_1 + (-T_3 \delta \phi_4) =$$
$$= \left(T_1 - T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1}\right) \delta \varphi_1 \equiv Q_{\varphi_1} \delta \varphi_1 \qquad (31)$$

Now, the equation of motion of the Cardan-Hooke joint reads:

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1} +$$
$$+ (J_1 + J_3 f_{\varphi4(II)}^2 + J_{2\zeta} f_{\varphi2(II)}^2 + J_{2\eta} \cos^2(\alpha - \varphi_2) +$$
$$+ J_{2\xi} \sin^2(\alpha - \varphi_2)) \ddot{\varphi}_1 +$$
$$+ \left(J_3 f_{\varphi4(II)} \frac{\partial f_{\varphi4(II)}}{\partial \varphi_1} + J_{2\zeta} f_{\varphi2(II)} \frac{\partial f_{\varphi2(II)}}{\partial \varphi_1} + \right.$$
$$\left. + \left(2 f_{\varphi2(II)} - \frac{\partial \varphi_2}{\partial \varphi_1}\right) \frac{J_{2\eta} - J_{2\xi}}{2} \sin 2(\alpha - \varphi_2) \right) \dot{\varphi}_1^2 \qquad (32)$$

while, for $\ddot{\varphi}_1 = 0$, it is obtained that:

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1} +$$
$$+ \left(J_3 f_{\varphi4(II)} \frac{\partial f_{\varphi4(II)}}{\partial \varphi_1} + J_{2\zeta} f_{\varphi2(II)} \frac{\partial f_{\varphi2(II)}}{\partial \varphi_1} + \right.$$
$$\left. + \left(2 f_{\varphi2(II)} - \frac{\partial \varphi_2}{\partial \varphi_1}\right) \frac{J_{2\eta} - J_{2\xi}}{2} \sin 2(\alpha - \varphi_2) \right) \dot{\varphi}_1^2 \qquad (33)$$

and, finally, for the inertial characteristic of cross and output shaft ignored:

$$T_1 = T_3 \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1} \qquad (34)$$

4. NUMERICAL EXAMPLE

Let us apply everything what is considered in Section 2.1 on the Cardan-Hooke joint with constant angular velocity of the input shaft and let T_{II} and T_{IV} contain terms of (25) that come from the inertia of the cross and the output shaft, respectively, as follows:

$$T_{II} = \left(J_{2\zeta} f_{\varphi2(I)} \frac{\partial f_{\varphi2(I)}}{\partial \varphi_1} + \right.$$
$$\left. + \left(\frac{\partial \varphi_2}{\partial \varphi_1} - 2 f_{\varphi2(I)}\right) \frac{J_{2\eta} - J_{2\xi}}{2} \sin 2\varphi_2 \right) \dot{\varphi}_1^2 \qquad (35)$$

$$T_{IV} = \left(J_3 f_{\phi4(I)} \frac{\partial f_{\phi4(I)}}{\partial \varphi_1}\right) \dot{\varphi}_1^2 \qquad (36)$$

In order to investigate the effect of the inertia of the cross and the output shaft on the torque transmission at

a constant angular velocity of the input shaft, the following numerical values of the joint parameters will be used (see [11]):

$$T_3 = 750\,\text{Nm}, \; J_{2\xi} = J_{2\zeta} = 0.00111\,\text{kgm}^2$$

$$J_{2\eta} = 0.00202\,\text{kgm}^2, \; \alpha = 15^o \qquad (37)$$

$$J_3 = 0.01528\,\text{kgm}^2$$

The plots of the quantities T_{II} and T_{IV} are shown in Figures 3 and 4, respectively, for various angular speed of the input shaft. Also, for the given value of resisting torque and various angular speeds of the input shaft, the plots of the applied torque are shown in Figure 5. Note that the plot of applied torque determined by means of the relation (26) is marked by a dash-dot line.
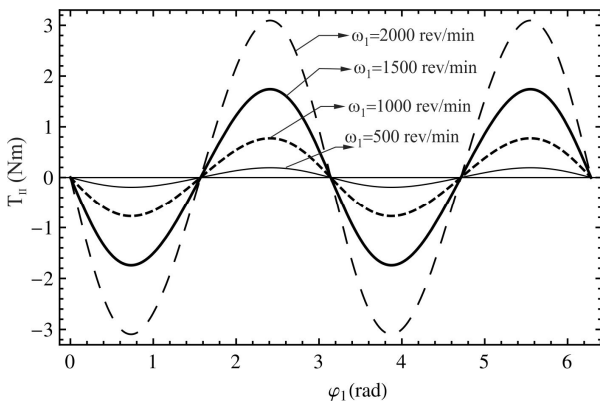

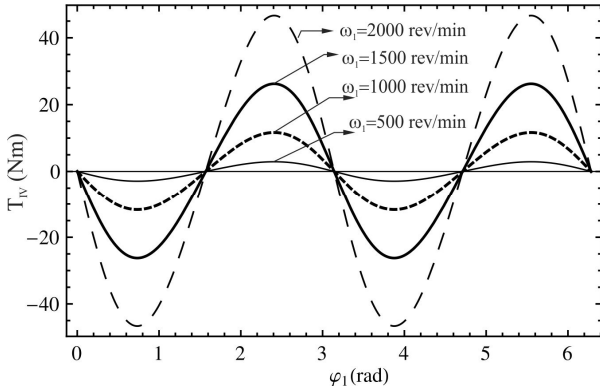Figure 3. Quantity T_{II} versus angular position and various angular speeds

Figure 4. Quantity T_{IV} versus angular position and various angular speeds
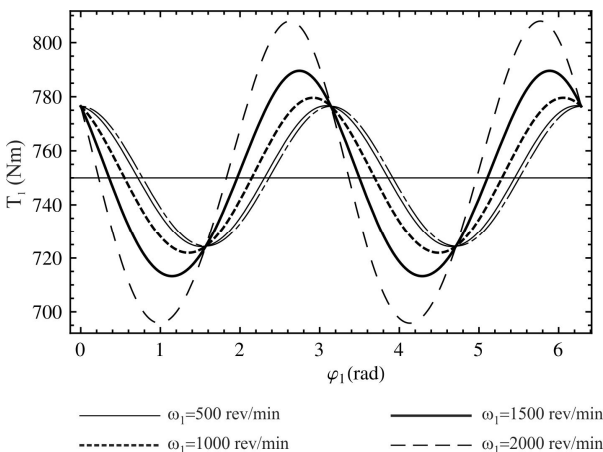

Figure 5. Applied torque T_1 versus angular position and various angular speeds

5. CONCLUSIONS

The relations obtained in this paper give an opportunity to estimate the influence of the inertia of cross and shafts on the torque transmission in Cardan-Hooke joints. By observing Figures 3 and 4 it is noted that at a constant angular speed of the input shaft, the influence of the output shaft inertia is dominant in comparison with the influence of the cross inertia. Also, from Figure 5 it can be observed that at higher values of the input shaft angular speed, the difference between results obtained by the static torque transmission relation (26) and the relation (25) is obvious. So, at higher angular speeds of the input shaft, the effect of inertia of the components of the Cardan-Hooke joint must be taken into account. The results obtained in this paper are valuable for the design as well as the stress and fatigue analysis of Cardan-Hooke joints (see e.g. [12]).

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О ПРЕНОСУ ОБРТНОГ МОМЕНТА КАРДАН-ХУКОВИМ ЗГЛОБОМ

С. Шалинић, А. Вранић, Н. Нешић, А. Томовић

Проучавана је кинемтика и динамика Кардан-Хуковог зглоба. Кинематичка анализа се базира на теорији сложеног кретања крутог тела. Динамичка анализа Кардан-Хуковог зглоба је извршена на бази Лагранжевих једначина друге врсте. Кардан-Хуков зглоб је анализиран у условима променљиве угаоне брзине погонског (улазног) вратила зглоба. Разматране су две варијанте зглоба: (1) раван погонске виљушке се поклапа са равни вратила; (2) раван погонске виљушке је нормална на раван вратила.

Изведен је израз којим се описује пренос обртног момента Кардан-Хуковим зглобом. Изведени израз садржи чланове који потичу од инерције улазног и излазног вратила као и од инерције крста Кардан-Хуковог зглоба. Теоријска разматрања су пропраћена нумеричким примером.