Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **11** (2017), 451–469.
https://doi.org/10.2298/AADM1702451P

ERROR ESTIMATES OF GAUSSIAN QUADRATURE FORMULAE WITH THE THIRD CLASS OF BERNSTEIN-SZEGŐ WEIGHTS

Aleksandar Pejčev

For analytic functions we study the remainder terms of Gauss quadrature rules with respect to Bernstein-Szegő weight functions

$$w(t) = w_{\alpha,\beta,\delta}(t) = \frac{\sqrt{\dfrac{1+t}{1-t}}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2 + \delta^2}, \quad t \in (-1,1),$$

where $0 < \alpha < \beta$, $\beta \neq 2\alpha$, $|\delta| < \beta - \alpha$, and whose denominator is an arbitrary polynomial of exact degree 2 that remains positive on $[-1,1]$. The subcase $\alpha = 1$, $\beta = 2/(1+\gamma)$, $-1 < \gamma < 0$ and $\delta = 0$ has been considered recently by M. M. Spalević, Error bounds of Gaussian quadrature formulae for one class of Bernstein-Szegő weights, Math. Comp., 82 (2013), 1037-1056.

1. INTRODUCTION

We study the kernels $K_n(z)$ in the remainder term $R_n(f)$ of the Gaussian quadrature formula

$$\int_{-1}^{1} f(t)w(t)\,dt = G_n[f] + R_n(f), \quad G_n[f] = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) \quad (n \in \mathbb{N}) \tag{1}$$

for analytic functions on elliptical contours with foci at ∓ 1 and the sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid \; z = \frac{1}{2}\left(\xi + \xi^{-1}\right), \; 0 \leq \theta \leq 2\pi \right\}, \quad \xi = \rho\, e^{i\theta}, \tag{2}$$

2010 Mathematics Subject Classification: 65D32, 65D30, 41A55.
Keywords and Phrases. remainder term, Gauss quadrature, analytic function, elliptic contour, error bound.

where w is a nonnegative and integrable function on the interval $(-1,1)$. This formula is exact for all algebraic polynomials of degree at most $2n-1$. The nodes τ_ν in (1) are zeros of the orthogonal polynomials π_n with respect to the weight function w.

When $\rho \to 1$ the ellipse (2) shrinks to the interval $[-1,1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice requires the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1.

In this paper w represents the third class of Bernstein-Szegő weight functions

$$w(t) = \frac{\sqrt{\dfrac{1+t}{1-t}}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2+\delta^2}, \quad t \in (-1,1), \tag{3}$$

where $0 < \alpha < \beta$, $\beta \neq 2\alpha$, $|\delta| < \beta - \alpha$, whose denominator is an arbitrary polynomial of exact degree 2 that remains positive on $[-1,1]$. The weight function (3) has been studied extensively in [**1**].

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $\mathcal{D} = \operatorname{int}\Gamma$ its interior. If the integrand f is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term $R_n(f)$ in (1) admits the contour integral representation

$$R_n(f) = \frac{1}{2\pi i}\oint_\Gamma K_n(z)f(z)dz. \tag{4}$$

The *kernel* is given by

$$K_n(z) \equiv K_n(z,w) = \frac{\varrho_n(z)}{\pi_n(z)}, \quad z \notin [-1,1],$$

where

$$\varrho_n(z) \equiv \varrho_{n,w}(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\,w(t)dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_n(\overline{z})| = |K_n(z)|$. If the weight function w in (1) is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_n(-\overline{z})| = |K_n(z)|$ (see [**2**]).

The integral representation (4) leads to the error estimate

$$|R_n(f)| \le \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{5}$$

where $\ell(\Gamma)$ is the length of the contour Γ. For a different approach to the estimation of $R_n(f)$ see [**3**].

Since here we take $\Gamma = \mathcal{E}_\rho$, where the ellipse $\mathcal{E}_\rho$ is given by (2), the estimate (5) reduces to

$$|R_n(f)| \le \frac{\ell(\mathcal{E}_\rho)}{2\pi}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right). \tag{6}$$

The derivation of adequate bounds for $|R_n(f)|$ on the basis of (6) is possible only if good estimates for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$ are available, especially if we know the location of the extremal point $\eta \in \mathcal{E}_\rho$ at which $|K_n|$ attains its maximum. In such a case, instead of looking for upper bounds for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$, one can simply try to calculate $|K_n(\eta, w)|$. In general, this may not be an easy task, but in the case of the Gauss-type quadrature formula (1) there exist effective algorithms for the calculation of $K_n(z)$ at any point z outside of $[-1,1]$ (see [**2**]). In [**1**] the analytic form of the corresponding orthogonal polynomials is determined for the weight-functions under consideration; in addition, analytical expressions for the coefficients of the corresponding three-term recurrence relation are derived, which enabled a numerically stable construction of the corresponding Gaussian quadrature formulas.

In [**10**], in the case of the weight function

$$w_\gamma(t) = \frac{\sqrt{\frac{1+t}{1-t}}}{1 - \frac{4\gamma}{(1+\gamma)^2}\, t^2},\ t \in (-1,1),\ \gamma \in (-1,0),$$

sufficient conditions were found to ensure the existence of a $\rho_{\min} = \rho_{\min}(n,\gamma)$ such that for each $\rho \geq \rho_{\min}$ the kernel attains its maximal absolute value at the intersection points of the ellipse with the coordinate axes. For this specialized case, much smaller values for $\rho = \rho_{\min}$ than the ones obtained by [**9**] (except for γ close to 0 and n even), especially for large values of n, were found.

In this paper, with respect to the more general class of the weight functions (3), sufficient conditions are found, which ensure the existence of a $\rho_{\min} = \rho_{\min}(n,\alpha,\beta,\delta)$ such that for each $\rho \geq \rho_{\min}$ the kernel attains its maximum absolute value at the intersection points of the ellipse with the real axis. This leads to effective error bounds for the corresponding Gaussian quadratures. The approach we use in this paper ensures that the values $\rho_{\min} = \rho_{\min}(n,\alpha,\beta,\delta)$ are very well estimated. The quality of the derived bounds is analyzed by a comparison with other error bounds proposed in the literature for the same class of integrands, such as in the cases of the second and the first class of Bernstein-Szegő weight functions

$$\frac{(1-t^2)^{1/2}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2+\delta^2},\quad \frac{(1-t^2)^{-1/2}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2+\delta^2},$$

$$t \in (-1,1),\quad 0<\alpha<\beta,\ \beta \neq 2\alpha,\ |\delta| < \beta-\alpha,$$

which were considered in [**6**] and [**5**] respectively. The results obtained by Schira [**9**] cannot be applied in general for all Bernstein-Szegő weight functions, since those are not symmetric.

2. MAXIMUM OF THE MODULUS OF KERNEL FOR GAUSSIAN QUADRATURE FORMULAS WITH THE WEIGHT FUNCTION (3)

For the weight function (3) under consideration here, the corresponding (monic) orthogonal polynomial $\pi_n(t)$ of degree n has the form (see [**1**])
$\pi_n(t) = \frac{1}{2^n}\left[V_n(t) + \frac{2\delta}{\beta}V_{n-1}(t) + \left(1-\frac{2\alpha}{\beta}\right)V_{n-2}(t)\right]$, $n \geq 2$, where V_n denotes the Chebyshev polynomial of the third kind, characterized by $V_n(\cos\varphi) = \frac{\cos(n+\frac{1}{2})\varphi}{\cos\frac{\varphi}{2}}$, $n \in \mathbb{N}_0$. After standard substitution $z = \frac{1}{2}(\xi+\xi^{-1})$, $\xi = \rho\, e^{i\theta}$, using the expressions for $V_n(z)$ and $\int_{-1}^{1}\frac{V_n(t)}{z-t}\sqrt{\frac{1+t}{1-t}}\,dt$ from ([**10**], p. 4), in the analogous way to ([**5**], p. 5) we obtain by direct calculation that for $n \geq 2$ the kernel can be expressed as

(7)
$$K_n(z) = \frac{8\pi}{\xi^n}\cdot\frac{(\xi+1)(\beta+2\delta\xi+(\beta-2\alpha)\xi^2)}{(\xi-1)\left[\beta(\beta-2\alpha)(\xi+\xi^{-1})^2+4\delta(\beta-\alpha)(\xi+\xi^{-1})+4(\alpha^2+\delta^2)\right]}$$
$$\cdot\frac{1}{\beta(\xi^{n+1}+\xi^{-n})+2\delta(\xi^n+\xi^{-(n-1)})+(\beta-2\alpha)(\xi^{n-1}+\xi^{-(n-2)})}.$$

The similar result can be obtained for $n = 1$ by direct calculations, using some results from [**1**], but this case doesn't have practical importance.

Employing the usual notation $a_j = \frac{1}{2}(\rho^j+\rho^{-j})$, $j \in \mathbb{N}$ and using (7), in conjunction with the identities

$$\begin{aligned}
&\left|\beta+2\delta\xi+(\beta-2\alpha)\xi^2\right| = \left[\beta^2+4\delta^2\rho^2+(\beta-2\alpha)^2\rho^4\right.\\
&\left.+\,4\rho\delta(\rho^2(\beta-2\alpha)+\beta)\cos\theta+2\beta(\beta-2\alpha)\rho^2\cos 2\theta\right]^{1/2} = \sqrt{a},\\
&|\xi-1| = \sqrt{2\rho}\,(a_1-\cos\theta)^{1/2} = \sqrt{2\rho b}\\
&\left|\beta(\beta-2\alpha)(\xi+\xi^{-1})^2+4\delta(\beta-\alpha)(\xi+\xi^{-1})+4(\alpha^2+\delta^2)\right|\\
=\;&\left[2\beta^2(\beta-2\alpha)^2(a_4+\cos 4\theta)\; +(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))^2\right.\\
&+4(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))(\beta(\beta-2\alpha)a_2\cos 2\theta+4\delta(\beta-\alpha)a_1\cos\theta)\\
&\left.+16\beta\delta(\beta-\alpha)(\beta-2\alpha)(a_3\cos\theta+a_1\cos 3\theta)+32\delta^2(\beta-\alpha)^2(a_2+\cos 2\theta)\right]^{1/2}\\
=\;&\sqrt{c},\\
&\left|\beta(\xi^{n+1}+\xi^{-n})+2\delta(\xi^n+\xi^{-(n-1)})+(\beta-2\alpha)(\xi^{n-1}+\xi^{-(n-2)})\right|\\
=\;&\rho^{\frac{1}{2}}\left[2\beta^2(a_{2n+1}+\cos(2n+1)\theta)+8\delta^2(a_{2n-1}+\cos(2n-1)\theta)\right.\\
&+2(\beta-2\alpha)^2(a_{2n-3}+\cos(2n-3)\theta)+8\beta\delta(a_{2n}\cos\theta+a_1\cos 2n\theta)\\
&+4\beta(\beta-2\alpha)(a_{2n-1}\cos 2\theta+a_2\cos(2n-1)\theta)\\
&\left.+8\delta(\beta-2\alpha)(a_{2n-2}\cos\theta+a_1\cos(2n-2)\theta)\right]^{1/2} = \sqrt{\rho d}
\end{aligned}$$

$$|\xi+1| = \sqrt{2\rho}\,(a_1+\cos\theta)^{1/2} = \sqrt{2\rho e},$$

we obtain (for $n \in \mathbb{N}\backslash\{1\}$) that

$$|K_n(z)|^2 = \frac{64\pi^2}{\rho^{2n+1}}\frac{ae}{bcd}. \tag{8}$$

Numerical experiments show that there exists a $\rho_{\min} = \rho_{\min}(n,\alpha,\beta,\delta) > 1$ such that $|K_n(z)|$ attains its maximum value on the real axis, i.e., on the positive real semi-axis for $\delta < \beta/2$, and on the negative real semi-axis for $\delta > \beta/2$, for each $\rho \geq \rho_{\min}$. We state the following theorem.

Theorem 1: *For the Gauss quadrature formula (1) with the weight function (3),* $0 < \alpha < \beta$, $\beta \neq 2\alpha$, $|\delta| < \beta - \alpha$, *there exists a* $\rho_{\min} \in (1, +\infty)$ ($\rho_{\min} = \rho_{\min}(n, \alpha, \beta, \delta)$) *such that for each* $\rho \geq \rho_{\min}$ *the modulus of the kernel* $|K_{n,\alpha,\beta,\delta}(z)|$ *from the expression (7) attains its maximum value on the positive real semi axis* ($\theta = 0$) *if* $\delta < \beta/2$ *and on the negative real semi axis* ($\theta = \pi$) *if* $\delta > \beta/2$, *i. e.,*

$$\begin{array}{ll} \max_{z\in\mathcal{E}_\rho} |K_{n,\alpha,\beta,\delta}(z)| = & \left|K_{n,\alpha,\beta,\delta}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|, \quad \text{for} \quad \delta < \beta/2, \\ \max_{z\in\mathcal{E}_\rho} |K_{n,\alpha,\beta,\delta}(z)| = & \left|K_{n,\alpha,\beta,\delta}\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right|, \quad \text{for} \quad \delta > \beta/2. \end{array}$$

Proof: We, actually, have to show that there exists a large enough $\rho_{\min}$ such that for each $\rho > \rho_{\min}$, $\theta \in [0, \pi]$, $\delta < \beta/2$ ($\delta > \beta/2$), where α, β, δ satisfy the Bernstein–Szegő conditions mentioned previously, we have that:

$$\frac{ae}{bcd} \leq \frac{AE}{BCD}, \tag{9}$$

where A, B, C, D, E are the values of a, b, c, d, e for $\theta = 0$ ($\theta = \pi$).

Let us define $A_1 = a - A$, $B_1 = b - B$, $C_1 = c - C$, $D_1 = d - D$, $E_1 = e - E$. The inequality (9) can be rewritten in the form (the same expression as (2.7) in [**10**])

$$\begin{array}{l} EA(B+B_1)C_1D + EA(B+B_1)(C+C_1)D_1 + ACD(EB_1 - E_1B) \\ -BCDA_1(E+E_1) \geq 0. \end{array} \tag{10}$$

When $\delta < \beta/2$, i.e. when A, B, C, D, E are the values of a, b, c, d, e for $\theta = 0$, we get that (see p. 7) the coefficient of the highest degree of ρ in this expression, ρ^{2n+10}, is equal to $H^0 \sin^2 \frac{\theta}{2}$, where

$$\begin{aligned} H^0 \;=\; & \frac{1}{2}(\beta-2\alpha)^2 \cdot \frac{1}{2}\left(-32\beta\delta(\beta-\alpha)(\beta-2\alpha)\right) \cdot \frac{1}{2}2\beta^2 \cdot \frac{1}{2}2\beta^2 \cdot \frac{1}{2} \\ & + \frac{1}{2}(\beta-2\alpha)^2 \cdot \frac{1}{2} \cdot 2\beta^2(\beta-2\alpha)^2 \cdot \frac{1}{2}(-16\beta\delta) \cdot \frac{1}{2} \\ & + (\beta-2\alpha)^2 \cdot 2\beta^2(\beta-2\alpha)^2 \cdot \frac{1}{2} \cdot 2\beta^2 \cdot \frac{1}{2} \cdot 2 \\ & - \frac{1}{2} \cdot 2\beta^2(\beta-2\alpha)^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2\beta^2(-8\delta(\beta-2\alpha)) \cdot \frac{1}{2} \\ =\; & \beta^3(\beta-2\alpha)^3\left(-4\delta(\beta-\alpha) - 2(\beta-2\alpha)\delta + 2\beta(\beta-2\alpha) + 2\beta\delta\right) \\ =\; & 2(\beta-2\alpha)^4(\beta-2\delta) \end{aligned}$$

and (for $\theta \neq 0$) it is positive exactly if $\delta < \beta/2$. When $2\alpha > \beta$, that condition always holds because $\delta < \beta - \alpha < \beta/2$.

When $\delta > \beta/2$, i.e. when A, B, C, D, E are the values of a, b, c, d, e for $\theta = \pi$, we analogously get (see p. 9/10) the corresponding highest-degree coefficient of ρ is equal to $H^\pi \cos^2 \frac{\theta}{2}$, where $H^\pi = -H^0 = (\beta-2\alpha)^4(2\delta-\beta) > 0$, which obviously finishes our proof.

3. THE DETERMINATION OF $\rho_{\min}$

From the practical point of view, our aim is to precisely determine the (minimal possible) value of $\rho_{\min}$ defined in the Theorem 1 in the purpose of getting stronger error-estimates. Hence all of our considerations concerning $\rho_{\min}$ will be in relation to the domain $(1, +\infty)$. Numerical experiments showed that the case $\beta > 2\alpha$, $\delta = \beta/2$ is not interesting from that point of view and that's why we didn't analyse it even theoretically.

The whole of this problem, for given α, β, δ, can be reduced to the case of $\alpha = 1$ because

$$\begin{aligned} K_n(z) = & \frac{1}{\pi_n(z)} \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\, w(t)dt \\ = & \frac{1}{\alpha^2} \frac{1}{\pi_n(z)} \int_{-1}^{1} \frac{\pi_n(t)}{z-t} \frac{\sqrt{\frac{1+t}{1-t}}}{\beta_1(\beta_1-2)t^2 + 2\delta_1(\beta_1-1)t + 1 + \delta_1^2}\, dt, \end{aligned}$$

where $\beta_1 = \beta/\alpha$ and $\delta_1 = \delta/\alpha$.

Referring to the work [**10**], we deduce that for $\delta = 0$, if $2\alpha < \beta$, then the locations of values of $\rho_{\min}$ obtained are different. Numerical experiments we have performed additionally show that in the subcase $2\alpha < \beta$, $\delta \neq 0$ $\rho_{\min}$ is close to 1 when δ is positive and very close to $\beta - \alpha$, or when $\delta/(\beta - \alpha) < -0.5$. Numerical experiments also show that if $2\alpha > \beta$, then $\rho_{\min}$ is always close to 1. Regardless of the location of the actual value of $\rho_{\min}$, which cannot be calculated exactly and can only be determined by experiments, for each choice of the parameters α, β and δ (for a sufficiently large value of n) - the proposed methods yield its estimation $\rho^* \geq \rho_{\min}$, which is very close to the corresponding value of $\rho_{\min}$ (that is found experimentally).

The expressions A_1, B_1, C_1 and E_1 are the polynomials in $x = \cos^2 \frac{\theta}{2}$ $(\sin^2 \frac{\theta}{2})$ of the constant degree (which doesn't depend on n), while D_1 is not such a expression. The idea (which works in all the cases except $2\alpha < \beta$, $\delta < \beta/2$) is to find a polynomial T in $x = \cos^2 \frac{\theta}{2}$ $(\sin^2 \frac{\theta}{2})$ of the constant degree, such that $D_1 \geq T$ for each $\theta \in [0, \pi]$. In the expression from (10) D_1 appears only in the summand $EA(B + B_1)(C + C_1)D_1$ and since $EA(B + B_1)(C + C_1)$ can't be negative (each of its factors is the modulus of some expression), we will have that this expression is greater than or equal to $P(x)$ for each $\theta \in [0, \pi]$ $(x \in [0, 1])$, where

(11)
$$\begin{aligned} P(x) = & EA(B + B_1)C_1D + EA(B + B_1)(C + C_1)T + ACD(EB_1 - E_1B) \\ & -BCDA_1(E + E_1) \end{aligned}$$

is also going to be the polynomial in $x = \cos^2 \frac{\theta}{2}$ or $\sin^2 \frac{\theta}{2}$ of the constant degree, with the coefficients dependent on ρ. Finally, the correponding value of ρ^* will be obtained as the minimal value of ρ such that $P(x) \geq 0$ for each $x \in [0, 1]$ $(\theta \in [0, \pi])$ for all $\rho > \rho^*$ (this will be easier to do precisely in the case of that expression and, clearly, $P(x) \geq 0$ implies positivity of the expression from (10)).

1. Let us first suppose that $\delta < \beta/2$. Since this case is concerned with the value of the expression for the modulus of the kernel at $\theta = 0$, all the expressions which appear in (10) will be denoted by 0 in superscripts. By denoting $\cos^2 \frac{\theta}{2} = x$, we get

$$
\begin{aligned}
A_1^0 &= -8\rho\delta(\rho^2(\beta-2\alpha)+\beta)\sin^2\frac{\theta}{2} - 4\beta(\beta-2\alpha)\rho^2\sin^2\theta \\
&= \sin^2\frac{\theta}{2}\left(-8\rho\delta(\rho^2(\beta-2\alpha)+\beta) - 16\beta(\beta-2\alpha)\rho^2\cos^2\frac{\theta}{2}\right) = (1-x)A_2^0(x), \\
B_1^0 &= 2\sin^2\frac{\theta}{2} = (1-x)B_2^0(x), \\
C_1^0 &= -4\beta^2(\beta-2\alpha)^2\sin^2 2\theta - 64\delta^2(\beta-\alpha)^2\sin^2\theta \\
&\quad -8(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))\left(\beta(\beta-2\alpha)a_2\sin^2\theta + 4\delta(\beta-\alpha)a_1\sin^2\frac{\theta}{2}\right) \\
&\quad -32\beta\delta(\beta-\alpha)(\beta-2\alpha)\left(a_3\sin^2\frac{\theta}{2} + a_1\sin^2\frac{3\theta}{2}\right) \\
&= \sin^2\frac{\theta}{2}\Bigg[-64\beta^2(\beta-2\alpha)^2\cos^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)^2 - 256\delta^2(\beta-\alpha)^2\cos^2\frac{\theta}{2} \\
&\quad - 32(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))\left(\beta(\beta-2\alpha)a_2\cos^2\frac{\theta}{2} + \delta(\beta-\alpha)a_1\right) \\
&\quad - 32\beta\delta(\beta-\alpha)(\beta-2\alpha)\left(a_3 + a_1\left(4\cos^2\frac{\theta}{2}-1\right)^2\right)\Bigg] = (1-x)C_2^0(x), \\
E_1^0 &= -2\sin^2\frac{\theta}{2} = (1-x)E_2^0(x), \quad \left(E_2^0(x) \equiv -2\right) \\
D_1^0 &= -4\beta^2\sin^2\frac{(2n+1)\theta}{2} - 16\delta^2\sin^2\frac{(2n-1)\theta}{2} \\
&\quad -4(\beta-2\alpha)^2\sin^2\frac{(2n-3)\theta}{2} - 16\beta\delta\left(a_{2n}\sin^2\frac{\theta}{2} + a_1\sin^2 n\theta\right) \\
&\quad -8\beta(\beta-2\alpha)\left(a_{2n-1}\sin^2\theta + a_2\sin^2\frac{(2n-1)\theta)}{2}\right) \\
&\quad -16\delta(\beta-2\alpha)\left(a_{2n-2}\sin^2\frac{\theta}{2} + a_1\sin^2(n-1)\theta\right).
\end{aligned}
$$

We will estimate the expression D_1^0 (and then, also, the whole expression from (10)) using the well-known inequality

$$
\left|\frac{\sin n\theta}{\sin\theta}\right| \le n, \quad n \in \mathbb{N}. \tag{12}
$$

1. a) Let first suppose that $\beta < 2\alpha$. If $\delta \ge 0$, we have

$$\begin{aligned} D_1^0 &\geq -4\beta^2(2n+1)^2\sin^2\frac{\theta}{2} - 16\delta^2(2n-1)^2\sin^2\frac{\theta}{2} \\ &\quad -4(\beta-2\alpha)^2(2n-3)^2\sin^2\frac{\theta}{2} - 16\beta\delta\left(a_{2n}\sin^2\frac{\theta}{2} + a_1 n^2\sin^2\theta\right) \\ &\quad -8\beta(\beta-2\alpha)a_{2n-1}\sin^2\theta - 16\delta(\beta-2\alpha)a_{2n-2}\sin^2\frac{\theta}{2} \\ &= \sin^2\frac{\theta}{2}\left[-4\beta^2(2n+1)^2 - 16\delta^2(2n-1)^2 - 4(\beta-2\alpha)^2(2n-3)^2\right. \\ &\quad - 16\beta\delta\left(a_{2n} + 4a_1 n^2\cos^2\frac{\theta}{2}\right) - 32\beta(\beta-2\alpha)a_{2n-1}\cos^2\frac{\theta}{2} \\ &\quad \left. - 16\delta(\beta-2\alpha)a_{2n-2}\right] = (1-x)D_2^{0,a,+}(x), \end{aligned}$$

and if $\delta \leq 0$, we have

$$\begin{aligned} D_1^0 &\geq -4\beta^2(2n+1)^2\sin^2\frac{\theta}{2} - 16\delta^2(2n-1)^2\sin^2\frac{\theta}{2} \\ &\quad -4(\beta-2\alpha)^2(2n-3)^2\sin^2\frac{\theta}{2} - 8\beta(\beta-2\alpha)a_{2n-1}\sin^2\theta \\ &\quad -16\beta\delta a_{2n}\sin^2\frac{\theta}{2} - 16\delta(\beta-2\alpha)\left(a_{2n-2}\sin^2\frac{\theta}{2} + a_1(n-1)^2\sin^2\theta\right) \\ &= \sin^2\frac{\theta}{2}\left[-4\beta^2(2n+1)^2 - 16\delta^2(2n-1)^2 - 4(\beta-2\alpha)^2(2n-3)^2\right. \\ &\quad - 16\beta\delta a_{2n} - 32\beta(\beta-2\alpha)a_{2n-1}\cos^2\frac{\theta}{2} \\ &\quad \left. - 16\delta(\beta-2\alpha)\left(a_{2n-2} + 4a_1(n-1)^2\cos^2\frac{\theta}{2}\right)\right] = (1-x)D_2^{0,a,-}(x). \end{aligned}$$

1. b) Let now suppose that $2\alpha < \beta$. In this case, if $0 \leq \delta < \beta/2$, we have

$$\begin{aligned} D_1^0 &\geq -4\beta^2(2n+1)^2\sin^2\frac{\theta}{2} - 16\delta^2(2n-1)^2\sin^2\frac{\theta}{2} \\ &\quad -4(\beta-2\alpha)^2(2n-3)^2\sin^2\frac{\theta}{2} - 16\beta\delta\left(a_{2n}\sin^2\frac{\theta}{2} + a_1 n^2\sin^2\theta\right) \\ &\quad -8\beta(\beta-2\alpha)\left(a_{2n-1}\sin^2\theta + a_2(2n-1)^2\sin^2\frac{\theta}{2}\right) \\ &\quad -16\delta(\beta-2\alpha)\left(a_{2n-2}\sin^2\frac{\theta}{2} + a_1(n-1)^2\sin^2\theta\right) \end{aligned}$$

$$
\begin{aligned}
&= \sin^2\frac{\theta}{2}\Big[-4\beta^2(2n+1)^2-16\delta^2(2n-1)^2-4(\beta-2\alpha)^2(2n-3)^2\\
&\quad -16\beta\delta\left(a_{2n}+4a_1n^2\cos^2\frac{\theta}{2}\right)\\
&\quad -8\beta(\beta-2\alpha)\left(4a_{2n-1}\cos^2\frac{\theta}{2}+a_2(2n-1)^2\right)\\
&\quad -16\delta(\beta-2\alpha)\left(a_{2n-2}+4a_1(n-1)^2\cos^2\frac{\theta}{2}\right)\Big]=(1-x)D_2^{0,b,+}(x),
\end{aligned}
$$

and for $\delta\le 0$ we have

$$
\begin{aligned}
D_1^0 &\ge -4\beta^2(2n+1)^2\sin^2\frac{\theta}{2}-16\delta^2(2n-1)^2\sin^2\frac{\theta}{2}\\
&\quad -4(\beta-2\alpha)^2(2n-3)^2\sin^2\frac{\theta}{2}-16\beta\delta a_{2n}\sin^2\frac{\theta}{2}\\
&\quad -8\beta(\beta-2\alpha)\left(a_{2n-1}\sin^2\theta+a_2(2n-1)^2\sin^2\frac{\theta}{2}\right)\\
&\quad -16\delta(\beta-2\alpha)a_{2n-2}\sin^2\frac{\theta}{2}\\
&= \sin^2\frac{\theta}{2}\Big[-4\beta^2(2n+1)^2-16\delta^2(2n-1)^2-4(\beta-2\alpha)^2(2n-3)^2\\
&\quad -16\beta\delta a_{2n}-8\beta(\beta-2\alpha)\left(4a_{2n-1}\cos^2\frac{\theta}{2}+a_2(2n-1)^2\right)\\
&\quad -16\delta(\beta-2\alpha)a_{2n-2}\Big]=(1-x)D_2^{0,b,-}(x).
\end{aligned}
$$

2. If $\delta>\beta/2$, all the expressions which appear in (10) will be denoted by π in superscripts and, also, after denoting $x=\sin^2\frac{\theta}{2}$:

$$
\begin{aligned}
A_1^\pi &= 8\rho\delta(\rho^2(\beta-2\alpha)+\beta)\cos^2\frac{\theta}{2}-4\beta(\beta-2\alpha)\rho^2\sin^2\theta\\
&= \cos^2\frac{\theta}{2}\left(8\rho\delta(\rho^2(\beta-2\alpha)+\beta)-16\beta(\beta-2\alpha)\rho^2\sin^2\frac{\theta}{2}\right)=(1-x)A_2^\pi(x)\\
B_1^\pi &= -2\cos^2\frac{\theta}{2}=(1-x)B_2^\pi(x),\\
C_1^\pi &= -4\beta^2(\beta-2\alpha)^2\sin^2 2\theta-64\delta^2(\beta-\alpha)^2\sin^2\theta\\
&\quad +8(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))\left(-\beta(\beta-2\alpha)a_2\sin^2\theta+4\delta(\beta-\alpha)a_1\cos^2\frac{\theta}{2}\right)\\
&\quad +32\beta\delta(\beta-\alpha)(\beta-2\alpha)\left(a_3\cos^2\frac{\theta}{2}+a_1\cos^2\frac{3\theta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \cos^2\frac{\theta}{2}\left[-64\beta^2(\beta-2\alpha)^2\sin^2\frac{\theta}{2}\left(2\sin^2\frac{\theta}{2}-1\right)^2-256\delta^2(\beta-\alpha)^2\sin^2\frac{\theta}{2}\right.\\
&\quad+32(2\beta(\beta-2\alpha)+4(\alpha^2+\delta^2))\left(-\beta(\beta-2\alpha)a_2\sin^2\frac{\theta}{2}+\delta(\beta-\alpha)a_1\right)\\
&\quad\left.+32\beta\delta(\beta-\alpha)(\beta-2\alpha)\left(a_3+a_1\left(4\sin^2\frac{\theta}{2}-3\right)^2\right)\right]=(1-x)C_2^\pi(x),
\end{aligned}
$$

$$
\begin{aligned}
D_1^\pi &= 4\beta^2\cos^2\frac{(2n+1)\theta}{2}+16\delta^2\cos^2\frac{(2n-1)\theta}{2}\\
&\quad+4(\beta-2\alpha)^2\cos^2\frac{(2n-3)\theta}{2}+16\beta\delta\left(a_{2n}\cos^2\frac{\theta}{2}-a_1\sin^2 n\theta\right)\\
&\quad+8\beta(\beta-2\alpha)\left(-a_{2n-1}\sin^2\theta+a_2\cos^2\frac{(2n-1)\theta}{2}\right)\\
&\quad+16\delta(\beta-2\alpha)\left(a_{2n-2}\cos^2\frac{\theta}{2}-a_1\sin^2(n-1)\theta\right),
\end{aligned}
$$

$$
E_1^\pi = 2\cos^2\frac{\theta}{2}=(1-x)E_2^\pi(x),\quad (E_2^\pi(x)\equiv 2)
$$

and using $p^2+q^2+r^2\ge |pq|+|qr|+|rp|\ge \pm pq\pm qr\pm rp$, we obtain

$$
\begin{aligned}
D_1^\pi &\ge -8\beta\delta\cos\frac{(2n+1)\theta}{2}\cos\frac{(2n-1)\theta}{2}\\
&\quad-8\delta(\beta-2\alpha)\cos\frac{(2n-1)\theta}{2}\cos\frac{(2n-3)\theta}{2}\\
&\quad-4\beta(\beta-2\alpha)\cos\frac{(2n+1)\theta}{2}\cos\frac{(2n-3)\theta}{2}\\
&\quad+16\beta\delta\left(a_{2n}\cos^2\frac{\theta}{2}-a_1\sin^2 n\theta\right)\\
&\quad+8\beta(\beta-2\alpha)\left(-a_{2n-1}\sin^2\theta+a_2\cos^2\frac{(2n-1)\theta}{2}\right)\\
&\quad+16\delta(\beta-2\alpha)\left(a_{2n-2}\cos^2\frac{\theta}{2}-a_1\sin^2(n-1)\theta\right)\\
&= -8\beta\delta\left(\cos^2\frac{\theta}{2}-\sin^2 n\theta\right)-8\delta(\beta-2\alpha)\left(\cos^2\frac{\theta}{2}-\sin^2(n-1)\theta\right)\\
&\quad-4\beta(\beta-2\alpha)\left(\cos^2\frac{(2n-1)\theta}{2}-\sin^2\frac{\theta}{2}\right)\\
&\quad+16\beta\delta\left(a_{2n}\cos^2\frac{\theta}{2}-a_1\sin^2 n\theta\right)\\
&\quad+8\beta(\beta-2\alpha)\left(-a_{2n-1}\sin^2\theta+a_2\cos^2\frac{(2n-1)\theta}{2}\right)\\
&\quad+16\delta(\beta-2\alpha)\left(a_{2n-2}\cos^2\frac{\theta}{2}-a_1\sin^2(n-1)\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 8\beta\delta\left((2a_{2n}-1)\cos^2\frac{\theta}{2}-(2a_1-1)\sin^2 n\theta\right)\\
&\quad +4\beta(\beta-2\alpha)\left(-(2a_{2n-1}-1)\sin^2\theta+(2a_2-1)\cos^2\frac{(2n-1)}{2}\right)\\
&\quad +8\delta(\beta-2\alpha)\left((2a_{2n-2}-1)\cos^2\frac{\theta}{2}-(2a_1-1)\sin^2(n-1)\theta\right).
\end{aligned}
$$

Using (12), since $2\alpha < \beta$ and $\delta > 0$ ($\delta > \beta/2$), we get the further estimates for the last expression, which are also estimates for D_1^π:

$$
\begin{aligned}
D_1^\pi &\geq 8\beta\delta\left((2a_{2n}-1)\cos^2\frac{\theta}{2}-n^2(2a_1-1)\sin^2\theta\right)\\
&\quad +4\beta(\beta-2\alpha)\left(-(2a_{2n-1}-1)\sin^2\theta\right)\\
&\quad +8\delta(\beta-2\alpha)\left((2a_{2n-2}-1)\cos^2\frac{\theta}{2}-(n-1)^2(2a_1-1)\sin^2\theta\right)\\
&= \cos^2\frac{\theta}{2}\left[8\beta\delta\left((2a_{2n}-1)-4n^2(2a_1-1)\sin^2\frac{\theta}{2}\right)\right.\\
&\quad -16\beta(\beta-2\alpha)(2a_{2n-1}-1)\sin^2\frac{\theta}{2}\\
&\quad \left.+8\delta(\beta-2\alpha)\left((2a_{2n-2}-1)-4(n-1)^2(2a_1-1)\sin^2\frac{\theta}{2}\right)\right]\\
&= (1-x)D_2^\pi(x).
\end{aligned}
$$

As we can see, in all of these cases we made the estimate of the form $D_1 \geq (1-x)D_2(x)\,(=T(x))$, which means that for $P(x)$ from (11) we can take $(1-x)I(x)$ ($x=\cos^2\frac{\theta}{2}$ when $\beta > 2\delta$ and $x=\sin^2\frac{\theta}{2}$ when $2\alpha < \beta < 2\delta$) , where

$$
\begin{aligned}
I(x) = \;& EA\left(B+(1-x)B_2(x)\right)C_2(x)D\\
&+EA\left(B+(1-x)B_2(x)\right)\left(C+(1-x)C_2(x)\right)D_2(x)\\
&+ACD\left(EB_2(x)-E_2(x)B\right)-BCDA_2(x)\left(E+(1-x)E_2(x)\right).
\end{aligned}
\tag{13}
$$

I Thus in the cases when $1 < \beta/\alpha < 2$ and $\beta/\alpha > 2$, $\delta > \beta/2$, we analyze the expression (13) in the purpose of determining as precisely as possible the minimal value of ρ^* such that for each $\rho > \rho^*$, $x \in [0,1]$ holds $I(x) > 0$. This will be done using the same method as in ([**6**], p. 11/12) or in ([**5**], p. 16/17). There we had the polynomial of the 4-th or the 3-th degree and here we have the polynomial of the 6-th degree, but the procedure gives equally satisfactory results, which are presented in the Tables 1 and 2. Hence, in this case, for each choice of the parameters α, β, δ we have to treat the 7 corresponding expressions depending on ρ and this doesn't cost algorithmically. Namely, after substitution $x=\frac{1}{y+1}$ for some unique $y \geq 0$, the polynomial

$$
P(x)=p_6(\rho)\cdot x^6+p_5(\rho)\cdot x^5+p_4(\rho)\cdot x^4+p_3(\rho)\cdot x^3+p_2(\rho)\cdot x^2+p_1(\rho)\cdot x+p_0(\rho),
$$

becomes

$$\begin{aligned} P(x) = {} & \frac{1}{(y+1)^6}\big[p_0(\rho)\cdot(y+1)^6 + p_1(\rho)\cdot(y+1)^5 + p_2(\rho)\cdot(y+1)^4 \\ & \qquad + p_3(\rho)\cdot(y+1)^3 + p_4(\rho)\cdot(y+1)^2 + p_5(\rho)\cdot(y+1) + p_6(\rho)\big] \\ = {} & \frac{1}{(y+1)^6}\big[q_6(\rho)\cdot y^6 + q_5(\rho)\cdot y^5 + q_4(\rho)\cdot y^4 + q_3(\rho)\cdot y^3 \\ & \qquad + q_2(\rho)\cdot y^2 + q_1(\rho)\cdot y + q_0(\rho)\big] \\ = {} & \frac{Q(y)}{(y+1)^6}, \end{aligned}$$

where

$$\begin{aligned} q_6(\rho) &= p_0(\rho), \\ q_5(\rho) &= 6p_0(\rho) + p_1(\rho), \\ q_4(\rho) &= 15p_0(\rho) + 5p_1(\rho) + p_2(\rho), \\ q_3(\rho) &= 20p_0(\rho) + 10p_1(\rho) + 4p_2(\rho) + p_3(\rho), \\ q_2(\rho) &= 15p_0(\rho) + 10p_1(\rho) + 6p_2(\rho) + 3p_3(\rho) + p_4(\rho), \\ q_1(\rho) &= 6p_0(\rho) + 5p_1(\rho) + 4p_2(\rho) + 3p_3(\rho) + 2p_4(\rho) + p_5(\rho), \\ q_0(\rho) &= p_0(\rho) + p_1(\rho) + p_2(\rho) + p_3(\rho) + p_4(\rho) + p_5(\rho) + p_6(\rho). \end{aligned}$$

It is obvious that $P(x)$ is nonnegative on $[0, 1]$ if and only if $Q(y)$ is nonnegative on $[0, \infty)$.

The nonnegativity of the q-coefficients, $q_0, q_1, q_2, q_3, q_4, q_5, q_6$ is a sufficient condition for the nonnegativity of the polynomial $Q(y)$ on the interval $[0, \infty)$.

As we can see, $p_0(\rho)$ contains the strictly highest degree of ρ of all the p-coefficients, because in each of the polynomials $A_2(x), B_2(x), C_2(x), D_2(x)$, $E_2(x)$, also, the free term contains the highest degree of ρ. Thus we easily conclude that the highest degree coefficient in the expression $p_0(\rho)$ will be the same as in the expression I (in all of the cases), and then the term with the highest degree of ρ in all the q-coefficients will be $2(\beta - 2\alpha)^4\,|\beta - 2\delta|\,\rho^{2n+10}$ (eventually multiplied by 6, 15 or 20, but this does not change the sign of the corresponding coefficient). This means that there exist values ρ_i, $i \in \{0, 1, 2, 3, 4, 5, 6\}$ such that for $\rho > \rho_i$ we have $q_i(\rho) \geq 0$. Then we can take

$$\rho^* = \max\{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\, \rho_5, \rho_6\}.$$

It is necessary to investigate the functions $q_i(\rho)$, which depend on ρ alone (all of the other parameters — α, β, δ, n — being fixed). Determining the value of ρ_i in Matlab doesn't represent a problem (nor does, for that matter, the creation of the expressions $q_i(\rho)$ in Matlab present a difficulty, for given α, β, δ, n, although their explicit expressions are huge and are therefore omitted here for the sake of brevity). In the interest of economy of space, the corresponding tables (1 and 2) contain only the final value ρ^*.

II In the case when $2\alpha < \beta$ and $\delta < \beta/2,\ \delta \neq 0$ the previous method doesn't give enough good results and we will treat the whole expression from (10). Let us call it $J(\rho)$. It is easily seen that $J_1(\rho) = \rho^{2n+6}J(\rho)$ is a polynomial in ρ. Its degree is equal to $4n+16$, i.e. $J_1(\rho) = \sum_{i=0}^{4n+16} r_i(\theta)\cdot\rho^i$. In order to ensure that J_1 is non-negative for each ρ greater than ρ' (for each $\theta \in [0,\pi]$), we will use the same method as in the (2.b) in ([**5**], p. 19). This method was also used in the purpose of proving the last Gautschi's conjecture ([**7**]). Hence, the method we will use in this case is to find the minimal ρ' such that if we rewrite the polynomial $J_1(\rho)$ as a polynomial in $\rho - \rho'$, then all of its coefficients (depending only on θ when we fix ρ') are non-negative for all $\theta \in [0,\pi]$. Such a value ρ' is found by using Matlab (by trial and error). The resulting value of ρ' will then be our ρ^*.

Numerical experiments show that here it gives precise results when $2\alpha < \beta$, $\delta < \beta/2,\ \delta \neq 0$ (Table 3), maybe also in some of the other cases, but the number of calculations we have to do here is $O(n)$, so we use the method just in this case where the previous one doesn't work.

Near each value of ρ^* displayed in the Tables 1, 2 and 3 there is the minimal value $\rho_{\min}$ found experimentally. The two values are very close to each other in most the cases. Observe that the results become particularly satisfactory as n increases. The experiments are limited to the case of $\alpha = 1$ in view of what was mentioned before. In all the tables in the paper (this and the next section), the values of β and δ were chosen randomly in a way to be uniformly distributed in their domain.

4. NUMERICAL RESULTS

The same examples as in [**10**] are tested. According to the results from that paper, in the case of numerical calculation of the integral

$$I(f) = \int_{-1}^{1} f(t)\,\frac{\sqrt{\frac{1+t}{1-t}}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2 + \delta^2}\,dt, \tag{14}$$

with $f(t) = f_0(t) = e^{t^2}$ we analogously get

$$|R_n(f_0)| \leq r_n(f_0), \tag{15}$$

where

$$r_n(f_0) \;=\; \inf_{\rho_n^* < \rho < +\infty} \left\{ \frac{1}{\rho^{n+\frac{1}{2}}} 8\pi a_1 \sqrt{\frac{AE}{BCD}} \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right) e^{a_1^2} \right\},$$

$\left(a_1 = \dfrac{\rho+\rho^{-1}}{2}\right)$ and A, B, C, D, E are defined above (we get the values of ρ_n^* using the methods proposed in the previous section). Also, reffering to ([**11**], Th. 1) and

Table 1: Values ρ^*, ρ_{min} for some $1 < \beta/\alpha < 2$, $\delta < |\beta - \alpha|$

$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$
$(10, 1.01, 0)$	$1.538, 1.001$	$(10, 1.1, 0.8)$	$1.537, 1.001$	$(10, 1.5, -0.25)$	$1.429, 1.001$
$(30, 1.01, 0)$	$1.225, 1.001$	$(30, 1.1, 0.08)$	$1.223, 1.001$	$(30, 1.5, -0.25)$	$1.173, 1.001$
$(100, 1.01, 0)$	$1.085, 1.001$	$(100, 1.1, 0.08)$	$1.084, 1.001$	$(100, 1.5, -0.25)$	$1.063, 1.001$
$(200, 1.01, 0)$	$1.047, 1.001$	$(200, 1.1, 0.08)$	$1.047, 1.001$	$(200, 1.5, -0.25)$	$1.035, 1.001$
$(10, 1.1, 0)$	$1.515, 1.001$	$(10, 1.1 - 0.03)$	$1.508, 1.001$	$(10, 1.7, 0.5)$	$1.538, 1.001$
$(30, 1.1, 0)$	$1.212, 1.001$	$(30, 1.1, -0.03)$	$1.208, 1.001$	$(30, 1.7, 0.5)$	$1.213, 1.001$
$(100, 1.1, 0)$	$1.078, 1.001$	$(100, 1.1, -0.03)$	$1.076, 1.001$	$(100, 1.7, 05)$	$1.076, 1.001$
$(200, 1.1, 0)$	$1.043, 1.001$	$(200, 1.1, -0.03)$	$1.042, 1.001$	$(200, 1.7, 0.5)$	$1.042, 1.001$
$(10, 1.35, 0)$	$1.473, 1.001$	$(10, 1.3, 0.15)$	$1.512, 1.001$	$(10, 1.7, -0.2)$	$1.422, 1.001$
$(30, 1.35, 0)$	$1.191, 1.001$	$(30, 1.3, 0.15)$	$1.208, 1.001$	$(30, 1.7, -0.2)$	$1.170, 1.001$
$(100, 1.35, 0)$	$1.069, 1.001$	$(100, 1.3, 0.15)$	$1.076, 1.001$	$(100, 1.7, -02)$	$1.062, 1.001$
$(200, 1.35, 0)$	$1.038, 1.001$	$(200, 1.3, 0.15)$	$1.042, 1.001$	$(200, 1.7, -0.2)$	$1.034, 1.001$
$(10, 1.6, 0)$	$1.448, 1.001$	$(10, 1.4, -0.3)$	$1.433, 1.001$	$(10, 1.8, -0.75)$	$1.401, 1.001$
$(30, 1.6, 0)$	$1.180, 1.001$	$(30, 1.4, -0.3)$	$1.175, 1.001$	$(30, 1.8, -0.75)$	$1.162, 1.001$
$(100, 1.6, 0)$	$1.065, 1.001$	$(100, 1.4, -0.3)$	$1.063, 1.001$	$(100, 1.8, -0.75)$	$1.059, 1.001$
$(200, 1.6, 0)$	$1.036, 1.001$	$(200, 1.4, -0.3)$	$1.035, 1.001$	$(200, 1.8, -0.75)$	$1.033, 1.001$
$(10, 1.85, 0)$	$1.434, 1.001$	$(10, 1.4, 0.1)$	$1.485, 1.001$	$(10, 1.8, 0.3)$	$1.480, 1.001$
$(30, 1.85, 0)$	$1.174, 1.001$	$(30, 1.4, 0.1)$	$1.195, 1.001$	$(30, 1.8, 0.3)$	$1.190, 1.001$
$(100, 1.85, 0)$	$1.063, 1.001$	$(100, 1.4, 0.1)$	$1.070, 1.001$	$(100, 1.8, 0.3)$	$1.068, 1.001$
$(200, 1.85, 0)$	$1.035, 1.001$	$(200, 1.4, 0.1)$	$1.039, 1.001$	$(200, 1.8, 0.3)$	$1.038, 1.001$
$(10, 1.99, 0)$	$1.428, 1.001$	$(10, 1.5, 0.48)$	$1.575, 1.001$	$(10, 1.95, 0.5)$	$1.504, 1.001$
$(30, 1.99, 0)$	$1.171, 1.001$	$(30, 1.5, 0.48)$	$1.233, 1.001$	$(30, 1.1.95, 0.5)$	$1.198, 1.001$
$(100, 1.99, 0)$	$1.062, 1.001$	$(100, 1.5, 0.48)$	$1.086, 1.001$	$(100, 1.95, 0.5)$	$1.071, 1.001$
$(200, 1.99, 0)$	$1.034, 1.001$	$(200, 1.5, 0.48)$	$1.048, 1.001$	$(200, 1.99, 0.5)$	$1.039, 1.001$

([**4**], Eq. (3.28)) we get

$$\hat{r}_n^{(\mathrm{Syd})}(f_0) = \inf_{1<\rho<+\infty} \left\{ \frac{4\mu_0}{(1-\rho^{-2})\rho^{2n}} \cdot e^{a_1^2} \right\},$$

where $\mu_0 = \pi/(\alpha(\beta - \alpha + \delta))$ and

$$\hat{r}_n^{(\mathrm{Not})}(f_0) \quad = \quad \inf_{1<r<+\infty} \left\{ \frac{X}{Y} \cdot e^{r^2} \right\},$$

where
(16)

$$\tau = r - \sqrt{r^2 - 1}, \; X = 8\pi\tau^{2n+1}\, r\sqrt{\frac{r+1}{r-1}},$$
$$Y = [(\beta - 2\alpha)\tau^2 - 2\delta\tau + \beta][\beta(1 + \tau^{2n+1}) - 2\delta\tau(1 + \tau^{2n-1}) + (\beta - 2\alpha)\tau^2(1 + \tau^{2n-3})].$$

Table 2: Values ρ^*, ρ_{min} for some $2 < \beta/\alpha$, $\delta > \beta/2$

$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$
$(10, 2.1, 1.07)$	$1.583, 1.581$	$(10, 50, 45)$	$1.888, 1.879$	$(10, 500, 400)$	$3.077, 3.075$
$(100, 2.1, 1.07)$	$1.582, 1.582$	$(100, 50, 45)$	$1.880, 1.879$	$(100, 500, 400)$	$3.077, 3.075$
$(10, 2.1, 1.09)$	$1.375, 1.099$	$(10, 50, 48)$	$1.582, 1.325$	$(10, 500, 460)$	$1.870, 1.858$
$(100, 2.1, 1.09)$	$1.119, 1.119$	$(100, 50, 48)$	$1.343, 1.341$	$(100, 500, 460)$	$1.860, 1.859$
$(10, 5, 3.85)$	$1.552, 1.348$	$(10, 50, 48.9)$	$1.547, 1.068$	$(10, 500, 498)$	$1.551, 1.066$
$(100, 5, 3.85)$	$1.378, 1.350$	$(100, 50, 48.9)$	$1.095, 1.085$	$(100, 500, 498)$	$1.099, 1.093$
$(10, 5, 3.96)$	$1.502, 1.084$	$(10, 100, 90)$	$1.969, 1.963$	$(10, 1000, 960)$	$1.641, 1.521$
$(100, 5, 3.96)$	$1.102, 1.097$	$(100, 100, 90)$	$1.965, 1.963$	$(100, 1000, 960)$	$1.528, 1.526$
$(10, 10, 8)$	$2.088, 2.085$	$(10, 100, 95)$	$1.644, 1.529$	$(10, 1000, 996)$	$1.553, 1.085$
$(100, 10, 8)$	$2.087, 2.086$	$(100, 100, 95)$	$1.535, 1.534$	$(100, 1000, 996)$	$1.121, 1.117$
$(10, 10, 8.96)$	$1.529, 1.063$	$(10, 100, 98.95)$	$1.554, 1.121$	$(10, 1000, 997)$	$1.551, 1.066$
$(100, 10, 8.96)$	$1.093, 1.086$	$(100, 100, 98.95)$	$1.156, 1.153$	$(100, 1000, 997)$	$1.100, 1.093$

For $\alpha = 1$ and selected values of β, δ, the obtained values $\hat{r}_{35}^{(\mathrm{Syd})}(f_0)$, $\hat{r}_{35}^{(\mathrm{Not})}(f_0)$, $r_{35}(f_0)$ are displayed in Table 4.

In the case of (14) with

$$f_1(t) = \frac{e^{e^t}}{(a+t)^k (b+t)^\ell (c+t)^m},$$

where $c \le b \le a < -1$; $k \in \mathbb{N}$, $\ell, m \in \mathbb{N}_0$ we have

$$\begin{aligned} r_n(f_1) \quad &= \inf_{\rho_n^* < \rho < \rho_{\max}} \left\{ \frac{1}{\rho^{n+\frac{1}{2}}} 8\pi a_1 \sqrt{\frac{AE}{BCD}} \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6}\right) \right. \\ &\quad \left. \cdot \frac{e^{e^{a_1}}}{|a+a_1|^k \, |b+a_1|^\ell \, |c+a_1|^m} \right\}, \end{aligned}$$

for the same A, B, C, D, E as in the previous case and $|a| = (\rho_{\max} + \rho_{\max}^{-1})/2$. Also,

$$\hat{r}_n(f_1) = \hat{r}_n^{(\mathrm{Syd})}(f_1) = \inf_{1<\rho<\rho_{\max}} \left\{ \frac{4\mu_0}{(1-\rho^{-2})\rho^{2n}} \cdot \frac{e^{e^{a_1}}}{|a+a_1|^k \, |b+a_1|^\ell \, |c+a_1|^m} \right\},$$

$$\hat{r}_n(f_1) \quad = \quad \hat{r}_n^{(\mathrm{Not})}(f_1) = \inf_{1<r<r_{\max}} \left\{ \frac{X}{Y} \cdot \frac{e^{e^r}}{|a+r|^k \, |b+r|^\ell \, |c+r|^m} \right\},$$

where $r_{\max} = |a|$, while X and Y are given by (16).

The calculated values of $\hat{r}_n^{(\mathrm{Syd})}(f_1), \hat{r}_n^{(\mathrm{Not})}(f_1), r_n(f_1)$ for the corresponding integral $I(f_1)$ given by (14) for $k = 1, \ell = 5, m = 10$, and $a = -12.02083333333333$, $b = -17.51428571428572$, $c = -23.01086956521739$, $\alpha = 1$, some values β, δ ($0 < \alpha < \beta$, $|\delta| < \beta - \alpha$) and $n = 15$ and $n = 50$ are displayed in the Tables 5 and 6 respectively. (Numbers in parentheses indicate decimal exponents.)

Table 3: Values ρ^*, ρ_{min} for some $2 < \beta/\alpha$, $\delta < \beta/2$

$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$	$(n, \beta/\alpha, \delta/\alpha)$	$\rho^*, \rho_{\min}$
$(10, 5, 0)$	$1.291, 1.001$	$(10, 5, 1)$	$1.291, 1.001$	$(10, 50, -15)$	$1.532, 1.532$
$(100, 5, 0)$	$1.291, 1.001$	$(100, 5, 1)$	$1.288, 1.001$	$(100, 50, -15)$	$1.531, 1.531$
$(10, 7, 0)$	$1.184, 1.001$	$(10, 5, -3)$	$1.291, 1.001$	$(10, 50, -30)$	$1.301, 1.300$
$(100, 7, 0)$	$1.184, 1.001$	$(100, 5, -3)$	$1.291, 1.001$	$(100, 50, -30)$	$1.292, 1.292$
$(10, 8, 0)$	$1.155, 1.105$	$(10, 7, 2)$	$1.649, 1.648$	$(10, 50, -47)$	$1.103, 1.099$
$(100, 8, 0)$	$1.155, 1.001$	$(100, 7, 2)$	$1.648, 1.648$	$(100, 50, -47)$	$1.060, 1.005$
$(10, 10, 0)$	$1.390, 1.390$	$(10, 7, -2)$	$1.191, 1.001$	$(10, 100, 20)$	$2.731, 2.731$
$(100, 10, 0)$	$1.386, 1.385$	$(100, 7, -2)$	$1.191, 1.001$	$(100, 100, 20)$	$2.731, 2.731$
$(10, 12, 0)$	$1.512, 1.511$	$(10, 10, 4)$	$3.725, 3, 725$	$(10, 100, -90)$	$1.130, 1.129$
$(100, 12, 0)$	$1.511, 1.510$	$(100, 10, 4)$	$3.725, 3.725$	$(100, 100, -90)$	$1.115, 1.113$
$(10, 15, 0)$	$1.627, 1.627$	$(10, 10, -3)$	$1.118, 1.059$	$(10, 100, -98.9)$	$1.195, 1.001$
$(100, 15, 0)$	$1.626, 1.626$	$(100, 10, -3)$	$1.115, 1.001$	$(100, 100, -98.9)$	$1.191, 1.001$
$(10, 25, 0)$	$1.807, 1.807$	$(10, 10, -8.5)$	$1.118, 1.001$	$(10, 500, -250)$	$1.443, 1.442$
$(100, 25, 0)$	$1.807, 1.807$	$(100, 10, -8.5)$	$1.117, 1.001$	$(100, 500, -250)$	$1.442, 1.441$
$(10, 40, 0)$	$1.909, 1.909$	$(10, 20, 5)$	$2.571, 2.570$	$(10, 500, -400)$	$1.236, 1.235$
$(100, 40, 0)$	$1.909, 1.909$	$(100, 20, 5)$	$2.571, 2.570$	$(100, 500, -400)$	$1.224, 1.224$
$10, 100, 0$	$2.013, 2.012$	$(10, 20, -12)$	$1.185, 1.185$	$(10, 500, -475)$	$1.148, 1.147$
$(100, 100, 0)$	$2.013, 2.012$	$(100, 20, -12)$	$1.150, 1.149$	$(100, 500, -475)$	$1.100, 1.096$
$(10, 500, 0)$	$2.069, 2.069$	$(10, 20, -18.9)$	$1.199, 1.001$	$(10, 500, -495)$	$1.175, 1.078$
$(100, 500, 0)$	$2.069, 2.069$	$(100, 20, -18.9)$	$1.184, 1.001$	$(100, 500, -495)$	$1.145, 1.031$

Finally, in the case of numerical calculation of the integral (14) with $f(t) = f_2(t) = e^{e^{\cos \omega t}}$, where $\omega > 0$, for $\alpha = 1$ and selected values of β, δ, the values of $\hat{r}_{29}^{(\mathrm{Syd})}(f_2)$, $\hat{r}_{29}^{(\mathrm{Not})}(f_2), r_{29}(f_2)$ obtained for $\omega = 1.8$, are displayed in the Table 7.

We note that in the cases of the functions f_0 and f_1 this method is no better than the one of Notaris (which is interesting because he uses circular contours), while in the case of the function f_2 the method of Notaris doesn't give satisfactory results. The comment on this fact could be given in a similar way to the *Remark 4* from [**8**]. Namely, the maximum of the modulus of the function $e^{cos\omega z}$ on the circumference $C_r = \{z||z| = r\}$ increases much faster than the maximum of the same function on the ellipse $\mathcal{E}_\rho$ (especially when ρ belongs to the beginning of the domain $(1, +\infty)$), which is aditionally manifested when such an expression appears in the exponent.

Acknowledgments. This work was supported in part by the Serbian Ministry of Education and Science (Research Project: "Methods of numerical and nonlinear analysis with applications" (No. #174002))

Table 4: Values $\hat{r}_{35}^{(\mathrm{Syd})}(f_0), \hat{r}_{35}^{(\mathrm{Not})}(f_0), r_{35}(f_0), Error$ for $\alpha = 1$ and some $\beta,\ \delta$

β, δ	$\hat{r}_{35}^{(\mathrm{Syd})}(f_0)$	$\hat{r}_{35}^{(\mathrm{Not})}(f_0)$	$r_{35}(f_0)$	$Error$
$1.2, 0$	$1.27(-58)$	$2.12(-59)$	$2.10(-59)$	$1.20(-60)$
$1.7, 0$	$3.64(-59)$	$1.05(-59)$	$1.04(-59)$	$5.94(-61)$
$1.3, 0.15$	$1.70(-58)$	$1.87(-59)$	$1.72(-59)$	$1.016(-60)$
$1.5, -0.3$	$3.18(-59)$	$1.26(-59)$	$1.43(-59)$	$7.76(-61)$
$1.6, 0.5$	$2.55(-58)$	$1.32(-59)$	$1.06(-59)$	$6.65(-61)$
$1.6, -0.5$	$2.32(-59)$	$1.07(-59)$	$1.31(-59)$	$6.89(-61)$
$1.8, 0.3$	$5.09(-59)$	$9.87(-60)$	$8.76(-60)$	$5.25(-61)$
$1.9, -0.7$	$1.59(-59)$	$7.41(-60)$	$9.44(-60)$	$4.71(-61)$
$2.1, 1$	$2.55(-58)$	$8.08(-60)$	$5.82(-60)$	$3.85(-61)$
$3, -1.5$	$7.28(-60)$	$2.84(-60)$	$3.95(-60)$	$1.99(-61)$
$5, -2.9$	$3.69(-60)$	$9.93(-61)$	$1.46(-60)$	$7.21(-62)$
$7, 5.8$	$1.27(-58)$	$8.22(-61)$	$5.82(-61)$	$3.50(-62)$
$10, 8.4$	$4.25(-59)$	$4.04(-61)$	$2.86(-61)$	$1.71(-62)$
$10, 3.5$	$4.63(-60)$	$3.37(-61)$	$2.64(-61)$	$1.68(-62)$
$10, -8.7$	$1.44(-60)$	$2.27(-61)$	$4.06(-61)$	$1.89(-62)$
$100, 95$	$6.37(-60)$	$4.21(-63)$	$2.98(-63)$	$1.73(-64)$
$100, 85$	$1.71(-60)$	$4.04(-63)$	$2.86(-63)$	$1.88(-64)$
$100, 27$	$3.54(-61)$	$3.26(-63)$	$2.70(-63)$	$1.68(-64)$
$100, -35$	$1.90(-61)$	$2.65(-63)$	$3.33(-63)$	$1.74(-64)$
$100, -50$	$1.71(-61)$	$2.53(-63)$	$3.52(-63)$	$1.77(-64)$

Table 5: Values $\hat{r}_{15}^{(\mathrm{Syd})}(f_1), \hat{r}_{15}^{(\mathrm{Not})}(f_1), r_{15}(f_1), Error$ for $\alpha = 1$ and some $\beta,\ \delta$

β, δ	$\hat{r}_{15}^{(\mathrm{Syd})}(f_1)$	$\hat{r}_{15}^{(\mathrm{Not})}(f_1)$	$r_{15}(f_1)$	$Error$
$1.2, 0$	$5.17(-34)$	$1.16(-34)$	$1.11(-34)$	$1.85(-36)$
$1.7, 0$	$1.48(-34)$	$5.53(-35)$	$5.31(-35)$	$8.83(-37)$
$1.3, 0.15$	$6.33(-35)$	$1.08(-34)$	$8.51(-35)$	$1.72(-36)$
$1.5, -0.3$	$5.16(-34)$	$6.13(-35)$	$8.24(-35)$	$1.37(-36)$
$1.6, 0.5$	$9.36(-35)$	$8.33(-35)$	$4.71(-35)$	$1.32(-36)$
$1.6, -0.5$	$1.03(-33)$	$4.91(-35)$	$7.99(-35)$	$7.87(-37)$
$1.8, 0.3$	$9.41(-33)$	$5.67(-35)$	$4.11(-35)$	$9.03(-37)$
$1.9, -0.7$	$5.17(-34)$	$3.29(-35)$	$5.86(-35)$	$5.28(-37)$
$2.1, 1$	$4.90(-35)$	$5.52(-35)$	$2.38(-35)$	$8.74(-37)$
$3, -1.5$	$2.06(-34)$	$1.17(-35)$	$2.58(-35)$	$1.88(-37)$
$5, -2.9$	$2.39(-36)$	$3.92(-36)$	$9.80(-36)$	$6.31(-38)$
$7, 5.8$	$8.76(-37)$	$6.80(-36)$	$2.81(-36)$	$1.07(-37)$
$10, 8.4$	$5.93(-36)$	$3.34(-36)$	$1.38(-36)$	$5.27(-38)$
$10, 3.5$	$8.27(-36)$	$1.99(-36)$	$1.08(-36)$	$3.18(-38)$
$10, -8.7$	$3.44(-34)$	$8.07(-37)$	$3.32(-36)$	$1.30(-38)$
$100, 95$	$5.34(-37)$	$3.74(-38)$	$1.55(-38)$	$5.88(-40)$
$100, 85$	$5.62(-37)$	$3.31(-38)$	$1.37(-38)$	$5.22(-40)$
$100, 27$	$8.23(-37)$	$1.82(-38)$	$1.13(-38)$	$2.91(-40)$
$100, -35$	$1.62(-36)$	$1.12(-38)$	$1.88(-38)$	$1.79(-40)$
$100, -50$	$2.11(-36)$	$1.01(-38)$	$2.17(-38)$	$3.60(-40)$

Table 6: Values $\hat{r}_{50}^{(\mathrm{Syd})}(f_1), \hat{r}_{50}^{(\mathrm{Not})}(f_1), r_{50}(f_1), Error$ for $\alpha = 1$ and some $\beta,\ \delta$

β, δ	$\hat{r}_{50}^{(\mathrm{Syd})}(f_1)$	$\hat{r}_{50}^{(\mathrm{Not})}(f_1)$	$r_{50}(f_1)$	$Error$
$1.2, 0$	$8.56(-88)$	$1.66(-88)$	$1.62(-88)$	$1.68(-90)$
$1.7, 0$	$2.45(-88)$	$8.09(-89)$	$7.91(-89)$	$8.19(-91)$
$1.3, 0.15$	$3.80(-88)$	$1.51(-88)$	$1.28(-88)$	$1.53(-90)$
$1.5, -0.3$	$8.56(-88)$	$9.30(-89)$	$1.16(-88)$	$9.43(-91)$
$1.6, 0.5$	$1.56(-88)$	$1.11(-88)$	$7.48(-89)$	$1.13(-90)$
$1.6, -0.5$	$1.72(-87)$	$7.65(-89)$	$1.09(-88)$	$7.75(-91)$
$1.8, 0.3$	$1.56(-88)$	$7.97(-89)$	$6.38(-89)$	$8.07(-91)$
$1.9, -0.7$	$8.56(-88)$	$5.22(-89)$	$7.97(-89)$	$5.29(-91)$
$2.1, 1$	$3.89(-90)$	$7.14(-89)$	$3.93(-89)$	$7.23(-91)$
$3, -1.5$	$3.42(-88)$	$1.92(-89)$	$3.43(-89)$	$1.95(-91)$
$5, -2.9$	$1.56(-88)$	$6.57(-90)$	$1.29(-89)$	$6.67(-92)$
$7, 5.8$	$1.45(-89)$	$8.04(-90)$	$4.29(-90)$	$8.12(-92)$
$10, 8.4$	$9.83(-90)$	$3.96(-90)$	$2.11(-90)$	$4.00(-92)$
$10, 3.5$	$1.36(-89)$	$2.78(-90)$	$1.80(-90)$	$2.82(-93)$
$10, -8.7$	$5.71(-88)$	$1.41(-90)$	$3.96(-90)$	$1.43(-92)$
$100, 95$	$8.82(-91)$	$4.27(-92)$	$2.28(-92)$	$4.32(-94)$
$100, 85$	$9.28(-91)$	$3.95(-92)$	$2.10(-92)$	$3.99(-94)$
$100, 27$	$1.36(-90)$	$2.62(-92)$	$1.86(-92)$	$2.65(-94)$
$100, -35$	$2.68(-90)$	$1.82(-92)$	$2.70(-92)$	$1.85(-94)$
$100, -50$	$3.50(-90)$	$1.68(-92)$	$2.98(-92)$	$1.71(-94)$

Table 7: Values $\hat{r}_{29}^{(\mathrm{Syd})}(f_2), \hat{r}_{29}^{(\mathrm{Not})}(f_2), r_{29}(f_2), Error$ for $\omega = 1.8$, $\alpha = 1$ and some $\beta,\ \delta$

β, δ	$\hat{r}_{29}^{(\mathrm{Syd})}(f_2)$	$\hat{r}_{29}^{(\mathrm{Not})}(f_2)$	$r_{29}(f_2)$	$Error$
$1.2, 0$	$2.22(-13)$	$1.42(+07)$	$9.09(-14)$	$5.26(-16)$
$1.7, 0$	$6.35(-14)$	$3.74(+06)$	$3.60(-14)$	$3.17(-16)$
$1.3, 0.15$	$9.87(-14)$	$1.68(+07)$	$5.80(-14)$	$5.59(-16)$
$1.5, -0.3$	$2.22(-13)$	$3.19(+06)$	$7.72(-14)$	$2.07(-16)$
$1.6, 0.5$	$4.04(-14)$	$1.91(+07)$	$2.47(-14)$	$4.68(-17)$
$1.6, -0.5$	$4.44(-13)$	$2.01(+06)$	$8.67(-14)$	$8.95(-17)$
$1.8, 0.3$	$4.04(-14)$	$5.56(+06)$	$2.33(-14)$	$3.67(-16)$
$1.9, -0.7$	$2.22(-13)$	$1.10(+06)$	$6.33(-14)$	$2.54(-17)$
$2.1, 1$	$2.12(-14)$	$1.69(+07)$	$1.04(-14)$	$3.40(-17)$
$3, -1.5$	$8.89(-14)$	$2.79(+05)$	$2.88(-14)$	$3.11(-17)$
$5, -2.9$	$4.04(-14)$	$7.81(+04)$	$1.10(-14)$	$2.12(-17)$
$7, 5.8$	$3.76(-15)$	$5.45(+06)$	$1.78(-15)$	$1.96(-17)$
$10, 8.4$	$2.56(-15)$	$2.04(+06)$	$8.38(-16)$	$9.45(-18)$
$10, -8.7$	$1.48(-13)$	$1.26(+04)$	$7.10(-15)$	$8.28(-18)$
$100, 95$	$2.29(-16)$	$4.28(+04)$	$1.19(-17)$	$7.54(-20)$
$100, 85$	$2.42(-16)$	$1.09(+04)$	$1.68(-15)$	$9.38(-20)$
$100, -35$	$6.95(-16)$	$2.22(+02)$	$1.23(-17)$	$1.01(-20)$
$100, -50$	$9.06(-16)$	$1.81(+2)$	$1.73(-17)$	$5.79(-20)$

REFERENCES

1. W. Gautschi, S.E. Notaris. *Gauss–Kronrod quadrature formulae for weight function of Bernstein–Szegő type.* J. Comput. Appl. Math., **25** (1989), 199–224; erratum in J. Comput. Appl. Math. 27 (1989) 429.
2. W. Gautschi, R.S. Varga. *Error bounds for Gaussian quadrature of analytic functions.* SIAM J. Numer. Anal., **20** (1983), 1170-1186.
3. D.B. Hunter. *Some error expansions for Gaussian quadrature.* BIT, **35** (1995), 64–82.
4. S.E. Notaris. *The error norm of Gaussian quadrature formulae for weight functions of Bernstein–Szegő type.* Numer. Math., **57** (1990), 271–283.
5. A. Pejčev. *Error bounds for Gauss-type quadratures with Bernstein-Szegö weights.*, Numerical Algorithms, **66** (2014), 569-590.
6. A.V. Pejčev and M.M. Spalević. *Error bounds for Gaussian quadrature formulae with Bernstein–Szegő weights that are rational modifications of Chebyshev weight functions of the second kind.* IMA J.Numer. Anal., **32** (2012), 1733–1754.
7. A.V. Pejčev and M.M. Spalević. *On the remainder term of Gauss-Radau quadrature with Chebyshev weight of the third kind for analytic functions*, Appl. Math. Comput. **219** (2012), 2760–2765.
8. A.V. Pejčev and M.M. Spalević. *The error bounds of Gauss-Radau quadrature formulae with Bernstein–Szegő weight functions.* Numer. Math., **133** (2016) 177–201.
9. T. Schira. *The remainder term for analytic functions of symmetric Gaussian quadratures.* Math. Comp., **66** (1997), 297–310.
10. M.M. Spalević. *Error bounds of Gaussian quadrature formulae for one class of Bernstein-Szegő weights.*, Math. Comp., **82** (2013), 1037-1056.
11. B. von Sydow. *Error estimates for Gaussian quadrature formulae.* Numer. Math., **29** (1977), 59–64.

Aleksandar Pejčev
Department of Mathematics,
University of Beograd,
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120
Belgrade 35,
Serbia;
E-mail: *vpejcev@eunet.rs*

(Received 01.12.2016.)
(Revised 21.09.2017.)