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# Truncated generalized averaged Gauss quadrature rules

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ABSTRACT

Generalized averaged Gaussian quadrature formulas may yield higher accuracy than Gauss quadrature formulas that use the same moment information. This makes them attractive to use when moments or modified moments are cumbersome to evaluate. However, generalized averaged Gaussian quadrature formulas may have nodes outside the convex hull of the support of the measure defining the associated Gauss rules. It may therefore not be possible to use generalized averaged Gaussian quadrature formulas with integrands that only are defined on the convex hull of the support of the measure. Generalized averaged Gaussian quadrature formulas are determined by symmetric tridiagonal matrices. This paper investigates whether removing some of the last rows and columns of these matrices gives quadrature rules whose nodes live in the convex hull of the support of the measure.



## 1. Introduction

Let $d\sigma$ be a nonnegative measure with infinitely many points of support. The smallest closed interval that contains the support of $d\sigma$ is denoted by $[a, b]$ with $-\infty \leq a < b \leq \infty$, and we assume that the distribution function $\sigma$ has infinitely many points of increase in this interval. If $\sigma$ is an absolutely continuous function, then $d\sigma(x) = w(x)\,dx$ on $\mathrm{supp}(d\sigma)$, where $w(x) \geq 0$ is a weight function. Let $\mathbb{P}_k$ denote the set of all polynomials of degree at most $k$ and introduce the quadrature formula (abbreviated q.f.)

$$Q_n[f] = \sum_{j=1}^{n} \omega_j f(x_j)$$

with real distinct nodes $x_1 < x_2 < \cdots < x_n$ and real weights $\omega_j$. We say that $Q_n$ is a $(2n-m-1, n, d\sigma)$ q.f. if the remainder term $R_n[f]$, defined by

$$\int f(x)\,d\sigma(x) = Q_n[f] + R_n[f],$$

satisfies $R_n[f] = 0$ for all $f \in \mathbb{P}_{2n-m-1}$. The rule $Q_n$ then is said to have *algebraic degree of precision* $2n-m-1$. Here $m$ is an integer such that $0 \leq m \leq n$. If in addition all quadrature weights $\omega_j$ are positive, then $Q_n$ is said to be a *positive* $(2n-m-1, n, d\sigma)$ q.f. Furthermore, we say that a polynomial $t_n = \prod_{j=1}^{n}(x - x_j)$ generates a $(2n-m-1, n, d\sigma)$ q.f. if its zeros $x_j$ are real and simple, and the q.f. with nodes $x_1, x_2, \ldots, x_n$ is a $(2n-m-1, n, d\sigma)$ q.f. A $(2n-m-1, n, d\sigma)$ q.f. is *internal* if all its nodes are in the closed interval $[a, b]$. A node not belonging to the interval $[a, b]$ is said to be *external*.

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It is well known that an $\ell$-node Gauss quadrature rule associated with the measure $d\sigma$ can be represented by an $\ell \times \ell$ real symmetric tridiagonal matrix $J_\ell^G(d\sigma)$ determined by the recursion coefficients of the first $\ell$ orthogonal polynomials associated with the measure $d\sigma$; see, e.g., Gautschi [1] or below. Spalević [2] proposed that the leading $(\ell-1)\times(\ell-1)$ tridiagonal submatrix of $J_\ell^G(d\sigma)$ be flipped right–left and upside-down, and appended to $J_\ell^G(d\sigma)$ to obtain a new symmetric tridiagonal matrix $J_{2\ell-1,\ell-1}$ of order $2\ell-1$. The latter matrix defines a $(2\ell-1)$-node quadrature formula referred to as a *generalized averaged Gaussian quadrature formula*. Spalević showed in [3] that these quadrature rules may yield a smaller quadrature error than what can be explained by just considering their algebraic degree of precision. This makes the generalized averaged Gaussian quadrature formulas attractive to use when it is inexpensive to evaluate the integrand at the nodes, but it is expensive or cumbersome to compute the moment information needed to determine the Gauss rule. Applications of generalized averaged Gaussian quadrature rules to problems of this kind are described in [4], where the quadrature rules are used to estimate quantities of interest in network analysis. In this application, the computation of each row and column of the matrix $J_\ell^G(d\sigma)$ requires the evaluation of a matrix–vector product with the adjacency matrix that defines the graph. The evaluation of matrix–vector products is expensive when the adjacency matrix is large. Gautschi describes in [1, Section 2.2], as well as in [5], other applications with measures $d\sigma$, for which the recursion coefficients for the associated orthogonal polynomials are not explicitly known and therefore have to be computed in order to determine Gaussian quadrature formulas. Gautschi proposed to compute approximations of the recursion coefficients by discretizing the measure $d\sigma$ and applying a Stieltjes procedure using the approximations of the required inner products determined by the discretized measure. These computations may be cumbersome if a fine discretization is required and a Gauss rule of high order is desired. It may then be attractive to use generalized averaged Gaussian quadrature formulas instead of standard Gauss rules, because the former often give higher accuracy when the same recursion coefficients are available for their construction; see Section 5 for computed examples.

It is the purpose of the present paper to describe extensions of the generalized averaged Gaussian quadrature formulas introduced in [2]. Section 2 discusses the extension of the real symmetric tridiagonal $\ell \times \ell$ matrix $J_\ell^G(d\sigma)$ associated with an $\ell$-node Gauss quadrature rule with respect to the measure $d\sigma$ to a real symmetric tridiagonal matrix $J_{k+\ell,k}$ of order $k+\ell$ by appending a fairly arbitrary real symmetric tridiagonal matrix of order $k$ to $J_\ell^G(d\sigma)$. Similarly as the generalized averaged Gaussian formulas introduced by Spalević [2], these extensions may yield a smaller quadrature error than the underlying $\ell$-node Gaussian quadrature formula. Section 3 is concerned with the possible presence of exterior nodes of generalized averaged Gaussian quadrature formulas. It is well known that the nodes of (standard) Gaussian quadrature formulas live in the convex hull of the support of the measure that determines the formulas. Spalević showed that the generalized averaged Gaussian quadrature formulas in [2] may have one node to the right or to the left of the convex hull of the support of the measure. It therefore may not be possible to apply these quadrature rules when the integrand is defined on the convex hull of the support of the measure only. To remedy this shortcoming, truncated generalized averaged Gaussian quadrature rules were introduced in [4]. These rules are obtained by removing the last few rows and columns of the real symmetric tridiagonal matrix $J_{2\ell-1,\ell-1}$ associated with the generalized averaged Gaussian quadrature rules described in [2]. These truncated generalized averaged Gaussian quadrature rules have the same algebraic degree of precision as the non-truncated ones. We investigate these rules by using results by Peherstorfer [6] on positive quadrature rules. Section 4 presents a detailed analysis of truncated generalized averaged Gaussian quadrature rules obtained by appending only one row and column to the matrix $J_\ell^G(d\sigma)$, and investigates for classical measures $d\sigma$ when these rules are internal. Section 5 presents a few computed examples and Section 6 contains concluding remarks.

## 2. Generalized averaged Gaussian quadrature formulas

The following result by Peherstorfer [6, Lemma 1.1] is important for the investigation of generalized averaged Gaussian quadrature rules. The lemma uses properties of so-called associated polynomials. These polynomials are defined below.

**Lemma 2.1.** *Let $n, m \in \mathbb{N}_0$. Then $t_n \in \mathbb{P}_n$ determines a positive $(2n-1-m, n, d\sigma)$ q.f. if and only if $t_n$ is orthogonal to $\mathbb{P}_{n-m-1}$ with respect to $d\sigma$, $t_n$ has $n$ simple zeros in the open interval $(a, b)$, and the zeros of $t_n$ and $t_{n-1}^{(1)}$ interlace, where $t_{n-1}^{(1)}$ denotes the associated polynomial to $t_n$.*

Let $p_k$ denote the monic polynomial of degree $k$ that is orthogonal to $\mathbb{P}_{k-1}$ with respect to $d\sigma$, i.e.,

$$\int_a^b x^j p_k(x)d\sigma(x) = 0, \quad j = 0, 1, \ldots, k-1.$$

Recall that the polynomials $\{p_k\}_{k=0}^{\infty}$ satisfy a three-term recurrence relation of the form

$$p_{k+1}(x) = (x-\alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k = 0, 1, \ldots, \tag{2.1}$$

where $p_{-1}(x) \equiv 0$, $p_0(x) \equiv 1$, $\alpha_k \in \mathbb{R}$, and $\beta_k > 0$ for all $k$; see, e.g., Gautschi [1] for details. The $\ell$-node Gaussian rule

$$Q_\ell^G[f] = \sum_{j=1}^{\ell} \omega_j^G f\left(x_j^G\right) \tag{2.2}$$

is the unique $\ell$-node quadrature formula of algebraic degree of precision $2\ell - 1$. This is the highest possible algebraic degree of precision $2\ell - 1$ of a quadrature rule with $\ell$ nodes. The nodes are the zeros of $p_\ell$. Both the nodes and weights can be conveniently computed by the Golub–Welsch algorithm [7], which is based on the observation that the nodes are the eigenvalues and the weights are proportional to the squares of the first components of the eigenvectors of the symmetric tridiagonal matrix

$$J_\ell^G(d\sigma) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\beta_{\ell-1}} \\ 0 & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-1} \end{bmatrix} \in \mathbb{R}^{\ell\times\ell}$$

determined by the recursion coefficients (2.1). This is discussed, e.g., by Wilf [8]. The algorithm computes the nodes and weights of the Gaussian quadrature rule (2.2) from the matrix $J_\ell^G(d\sigma)$ in only $\mathcal{O}(\ell^2)$ arithmetic floating point operations. A nice recent discussion of the Golub–Welsch algorithm is provided by Golub and Meurant [9].

The polynomials $p_k^{(j)},\ k, j \in \mathbb{N}_0$, known as the associated polynomials to the monic orthogonal polynomials $p_k$, play an important role in the sequel. They are defined by the shifted recurrence relation

$$p_{k+1}^{(j)}(x) = (x - \alpha_{k+j})p_k^{(j)}(x) - \beta_{k+j}p_{k-1}^{(j)}(x), \quad k = 0, 1, \ldots,$$

where $p_{-1}^{(j)}(x) \equiv 0$ and $p_0^{(j)}(x) \equiv 1$.

Peherstorfer [6] showed that a polynomial $t_n$ generates a positive $(2n-1-m, n, d\sigma)$ q.f. $(0 \le m \le n)$ if and only if $t_n$ can be generated by a three-term recurrence relation of the form

$$t_{j+1}(x) = (x - \tilde{\alpha}_j)t_j(x) - \tilde{\beta}_j t_{j-1}(x), \quad j = 0, 1, \ldots, n-1,$$

where $t_{-1}(x) \equiv 0,\ t_0(x) \equiv 1, \tilde{\alpha}_j \in \mathbb{R}, \tilde{\beta}_j > 0$, and

$$\tilde{\alpha}_j = \alpha_j \quad \text{for } j = 0, 1, \ldots, n-1-\left[\frac{m+1}{2}\right],$$
$$\tilde{\beta}_j = \beta_j \quad \text{for } j = 0, 1, \ldots, n-1-\left[\frac{m}{2}\right],$$

are such that

$$\operatorname{sgn} t_j(a) = (-1)^j, \qquad t_j(b) > 0, \quad j = 1, 2, \ldots, n.$$

Here $[\alpha]$ denotes the integer part of $\alpha \ge 0$. The properties of the polynomials $t_j$ are equivalent to that $t_n$ can be represented in the form $(\ell := [(m+1)/2],\ n \ge 2\ell$, i.e., $n - \ell \ge \ell)$

$$t_n = g_\ell p_{n-\ell} - \tilde{\beta}_{n-\ell} g_{\ell-1} p_{n-\ell-1}, \tag{2.3}$$

where $g_{\ell-1}$ and $g_\ell$ are generated by a three-term recurrence relation

$$g_{j+1}(x) = (x - \tilde{\alpha}_{n-1-j})g_j(x) - \tilde{\beta}_{n-j} g_{j-1}(x), \quad j = 0, 1, \ldots, \ell-1,$$

and $g_{-1}(x) \equiv 0, g_0(x) \equiv 1$, with $\tilde{\alpha}_{n-1-j} \in \mathbb{R}$ and $\tilde{\beta}_{n-j} > 0$ for $j = 0, 1, \ldots, \ell-1$; $\tilde{\beta}_{n-\ell} > 0, \tilde{\beta}_{n-\ell} = \beta_{n-\ell}$ if $m = 2\ell - 1$, are such that

$$\operatorname{sgn} g_j(a) = (-1)^j, \qquad g_j(b) > 0, \quad j = 1, 2, \ldots, \ell;$$

see the proof of [6, Theorem 3.2], in particular $(d) \Longrightarrow (a)$.

We may define quadrature formulas of the kind discussed as follows. Let $d\mu$ be a nonnegative measure with the same support as $d\sigma$. In particular, $\mu$ has infinitely many points of increase. Let $\tilde{p}_k$ denote the monic polynomial of degree $k$ that is orthogonal to $\mathbb{P}_{k-1}$ with respect to $d\mu$, i.e.,

$$\int x^j \tilde{p}_k(x) d\mu(x) = 0, \quad j = 0, 1, \ldots, k-1.$$

Then the polynomials $\{\tilde{p}_k\}_{k=0}^\infty$ satisfy a three-term recurrence relation of the form

$$\tilde{p}_{k+1}(x) = (x - \gamma_k)\tilde{p}_k(x) - \lambda_k \tilde{p}_{k-1}(x), \quad k = 0, 1, \ldots,$$

where $\tilde{p}_{-1}(x) \equiv 0, \tilde{p}_0(x) \equiv 1, \gamma_k \in \mathbb{R}$ and $\lambda_k \ge 0$.

Consider the positive quadrature formula determined by the symmetric tridiagonal matrix with nontrivial entries

$$\begin{aligned} \tilde{\alpha}_{n-1-j} &= \gamma_j \quad \text{and} \quad \tilde{\beta}_{n-j} = \lambda_j \quad \text{for } j = 0, 1, \ldots, \ell-1, \\ \tilde{\beta}_{n-\ell} &= \beta_{n-\ell}\ (m = 2\ell - 1), \quad \text{i.e., } \tilde{\beta}_{n-\ell} = \lambda_\ell\ (m = 2\ell). \end{aligned} \tag{2.4}$$

We then obtain

$$g_j \equiv \tilde{p}_j, \quad j = 1, 2, \ldots, \ell.$$

**Remark 2.1.** The choice (2.4) of the recursion coefficients $\tilde{\alpha}_{n-1-j}$ and $\tilde{\beta}_{n-j}$ was proposed in [2]. It gives quadrature formulas with several desirable properties. However, other choices are possible; see Section 5 for an illustration.

Conversely, letting

$$g_\ell \equiv \tilde{p}_\ell \quad \text{and} \quad g_{\ell-1} \equiv \tilde{p}_{\ell-1}, \tag{2.5}$$

we obtain the relations (2.4). Hence, if (2.5) or (2.4) hold, then (2.3) is reduced to

$$t_n = \tilde{p}_\ell \cdot p_{n-\ell} - \tilde{\beta}_{n-\ell}\, \tilde{p}_{\ell-1} \cdot p_{n-\ell-1},$$

and $t_n$ generates a positive quadrature formula, which we denote by $(2n-m-1, n, d\sigma, d\mu)$. The associated symmetric tridiagonal matrix $J_{n,\ell}(d\sigma, d\mu) \in \mathbb{R}^{n\times n}$ is given by

$$\begin{bmatrix}
\alpha_0 & \sqrt{\beta_1} & & & & & & & & \mathbf{0} \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
 & & \sqrt{\beta_{n-\ell-2}} & \alpha_{n-\ell-2} & \sqrt{\beta_{n-\ell-1}} & & & & & \\
 & & & \sqrt{\beta_{n-\ell-1}} & \alpha_{n-\ell-1} & \boxed{\sqrt{\tilde{\beta}_{n-\ell}}} & & & & \\
 & & & & \boxed{\sqrt{\tilde{\beta}_{n-\ell}}} & \gamma_{\ell-1} & \sqrt{\lambda_{\ell-1}} & & & \\
 & & & & & \sqrt{\lambda_{\ell-1}} & \gamma_{\ell-2} & \ddots & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \gamma_1 & \sqrt{\lambda_1} \\
\mathbf{0} & & & & & & & & \sqrt{\lambda_1} & \gamma_0
\end{bmatrix},$$

where we circumscribe the last entries determined by the measure $d\sigma$ by rectangles.

**Remark 2.2.** The special case $d\mu = d\sigma$ and $n = 2k-1$, $\ell = k-1$ is analyzed in [2,3].

## 3. Quadrature rules determined by truncation of $J_{n,\ell}(d\sigma, d\mu)$

We may remove the last $i$ ($i \in \{1, \dots, \ell-1\}$) rows and columns of the tridiagonal $J_{n,\ell}(d\sigma, d\mu) \in \mathbb{R}^{n\times n}$ defined above. The following theorem shows that the new positive quadrature rule $(2n-m-1, n-i, d\sigma, d\mu)$ with $n-i$ nodes so obtained has the same algebraic degree of precision as the original quadrature formula.

**Theorem 3.1.** *The $(2n_i - m_i - 1, n_i, d\sigma, d\mu)$ q.f. obtained by removing the last $i$ rows and columns from the matrix $J_{n,\ell}(d\sigma, d\mu)$, where $n_i = n - i$, $\ell_i = \ell - i := [(m_i+1)/2]$, has the same algebraic degree of precision as the $(2n-m-1, n, d\sigma, d\mu)$ q.f.*

**Proof.** Consider first the case when $m$ is odd, i.e., $m = 2\ell - 1$. Then we have $\tilde{\beta}_{n-\ell} = \beta_{n-\ell}$. The algebraic degree of precision of the $(2n-m-1, n, d\sigma, d\mu)$ q.f. is $d = 2n-m-1 = 2n-2\ell$. In the quadrature formula $(2n_i - m_i - 1, n_i, d\sigma, d\mu)$, we have that $m_i$ is odd, i.e., $m_i = 2\ell_i - 1 = 2(\ell - i) - 1$, since in this case

$$\tilde{\beta}_{n-\ell} = \beta_{n-\ell} = \beta_{n-i-(\ell-i)} = \beta_{n_i-\ell_i} = \tilde{\beta}_{n_i-\ell_i}.$$

Therefore, the algebraic degree of precision of the $(2n_i - m_i - 1, n_i, d\sigma, d\mu)$ q.f. is $d_i = 2n_i - m_i - 1 = 2n - 2\ell$. This implies that $d_i = d$.

We turn to the case when $m$ is even, i.e., $m = 2\ell$. Then $\tilde{\beta}_{n-\ell} = \lambda_\ell$. The algebraic degree of precision of the $(2n-m-1, n, d\sigma, d\mu)$ q.f. is $d = 2n-m-1 = 2n-2\ell-1$. In the quadrature formula $(2n_i - m_i - 1, n_i, d\sigma, d\mu)$, we have that $m_i$ is even, i.e., $m_i = 2\ell_i = 2(\ell - i)$, since in this case

$$\tilde{\beta}_{n-\ell} = \lambda_\ell = \tilde{\beta}_{n-i-(\ell-i)} = \tilde{\beta}_{n_i-\ell_i}.$$

Therefore, the algebraic degree of precision of the $(2n_i - m_i - 1, n_i, d\sigma, d\mu)$ q.f. is $d_i = 2n_i - m_i - 1 = 2n - 2\ell - 1$. Hence, $d_i = d$. $\square$

Using results of Peherstorfer [6], it easily can be shown that the $(2n-m-1, n-i, d\sigma, d\mu)$ q.f. is generated by the monic polynomial $t_{n-i}$ given by

$$t_{n-i} = \tilde{p}^{(i)}_{\ell-i} p_{n-\ell} - \tilde{\beta}_{n-\ell}\, \tilde{p}^{(i)}_{\ell-i-1} p_{n-\ell-1}, \tag{3.1}$$

where $\tilde{p}^{(j)}_k$ is the polynomial of order $j$ associated to $\tilde{p}_k$. For example, if $i = \ell - 1$, then we have the quadrature formula $(2n-m-1, n-\ell+1, d\sigma, d\mu)$. Since $\tilde{p}^{(\ell-1)}_1 = x - \gamma_{\ell-1}$, this q.f. is generated by the monic polynomial $t_{n-\ell+1}$ ($i = \ell - 1$ in (3.1)) given by

$$t_{n-\ell+1}(x) = (x - \gamma_{\ell-1}) p_{n-\ell} - \tilde{\beta}_{n-\ell}\, p_{n-\ell-1}. \tag{3.2}$$

**Remark 3.1.** The analysis and results of Peherstorfer [6] apply to the $(2n-m-1, n, d\sigma, d\mu)$ q.f. when $n-\ell \geq \ell$. We obtain the new $(2n_i-m_i-1, n_i, d\sigma, d\mu)$ q.f. for which $n_i-\ell_i \geq \ell_i$. This follows from the fact that $n-\ell \geq \ell-i$, i.e., $n-i-(\ell-i) \geq \ell-i$.

We will in Section 5 illustrate some properties of quadrature rules of this section.

## 4. Special generalized averaged Gaussian quadrature rules

We will consider generalized averaged Gaussian quadrature formulas that are determined by appending one row and one column to the matrix associated with the $(\ell+1)$-node Gaussian rule for the measure $d\sigma$.

Consider the special case of (2.4),

$$\begin{aligned} \tilde{\alpha}_{n-1-j} &= \alpha_j \quad \text{and} \quad \tilde{\beta}_{n-j} = \beta_j \quad \text{for } j = 0, 1, \ldots, \ell-1, \\ \tilde{\beta}_{n-\ell} &= \beta_{n-\ell}\ (m = 2\ell-1), \quad \text{i.e., } \tilde{\beta}_{n-\ell} = \beta_\ell\ (m = 2\ell), \end{aligned}$$

obtained by letting $n = 2\ell+1$. These formulas give the averaged Gaussian quadrature formulas introduced in [2] when $\tilde{\beta}_{\ell+1} = \beta_{\ell+1}$ with algebraic degree of precision $2\ell+2$. Truncated versions of these quadrature rules, with the same algebraic degree of precision, were considered in [4]. In this section, we will investigate when truncated averaged Gaussian quadrature rules determined by a classical weight function $w$ with only one more node than the associated Gaussian rule are internal. Computed examples in Section 5 show that the results do not carry over to generalized averaged Gaussian quadrature rules that have two more nodes than the underlying Gaussian rule.

It follows from (3.2) that the simplest truncated generalized averaged Gaussian quadrature formula

$$\int f(x)\,d\sigma(x) = Q^{(1)}_{\ell+2}[f] + R^{(1)}_{\ell+2}[f], \quad Q^{(1)}_{\ell+2}[f] = \sum_{j=1}^{\ell+2} \omega_j f(\tau_j), \tag{4.1}$$

is determined by the zeros $\tau_j = \tau_j^{(\ell+2)}$ $(j = 1, 2, \ldots, \ell+2)$ of the polynomial

$$t_{\ell+2}(x) = (x-\alpha_{\ell-1})p_{\ell+1}(x) - \beta_{\ell+1}\,p_\ell(x), \tag{4.2}$$

and is associated with the symmetric tridiagonal matrix

$$\hat{J}^{(1)}_{\ell+2}(d\sigma) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-1} & \sqrt{\beta_\ell} & & \\ & & & \sqrt{\beta_\ell} & \alpha_\ell & \sqrt{\beta_{\ell+1}} \\ 0 & & & & \sqrt{\beta_{\ell+1}} & \alpha_{\ell-1} \end{bmatrix}. \tag{4.3}$$

Note that the matrix $\hat{J}^{(1)}_{\ell+2}(d\sigma)$ is obtained from $J^G_{\ell+2}(d\sigma)$ by replacing the entry $\alpha_{\ell+1}$ in the latter by $\alpha_{\ell-1}$, or equivalently from $J^G_{\ell+1}(d\sigma)$ by appending a suitable row and column. Due to the interlacing property of the zeros of $t_{\ell+2}$ and $p_{\ell+1}$, only the smallest and largest zeros of $t_{\ell+2}$, denoted by $\tau_1 = \tau_1^{(\ell+2)}$ and $\tau_{\ell+2} = \tau_{\ell+2}^{(\ell+2)}$, respectively, may be outside the interval $[a, b]$. We are interested in investigating when the quadrature rule (4.1) is internal.

**Theorem 4.1.** *If the coefficients $\alpha_{\ell-1}$ and $\alpha_{\ell+1}$ in the three-term recurrence relation (2.1) satisfy $\alpha_{\ell-1} = \alpha_{\ell+1}$, then the quadrature formula (4.1) is internal. If $\alpha_{\ell-1} < \alpha_{\ell+1}$, then $(a <)\ \tau_{\ell+2} < b$, and if $\alpha_{\ell-1} > \alpha_{\ell+1}$, then $a < \tau_1\ (< b)$.*

**Proof.** Assume that $k = \ell+1$ in (2.1). Then

$$p_{\ell+2}(x) = (x-\alpha_{\ell+1})p_{\ell+1}(x) - \beta_{\ell+1}p_\ell(x). \tag{4.4}$$

Subtracting (4.4) from (4.2) gives

$$t_{\ell+2}(x) - p_{\ell+2}(x) = (\alpha_{\ell+1}-\alpha_{\ell-1})p_{\ell+1}(x). \tag{4.5}$$

If $\alpha_{\ell-1} = \alpha_{\ell+1}$, then all zeros of $t_{\ell+2}(x) \equiv p_{\ell+2}(x)$ live in $(a, b)$. Hence, the quadrature formula (4.1) is internal.

If instead $\alpha_{\ell-1} < \alpha_{\ell+1}$, then we obtain from (4.5) that

$$t_{\ell+2}(b) - p_{\ell+2}(b) = (\alpha_{\ell+1}-\alpha_{\ell-1})p_{\ell+1}(b) > 0.$$

This implies that $t_{\ell+2}(b) > 0$, because $p_{\ell+2}(b) > 0$. Therefore, $\tau_{\ell+2} < b$.

If $\alpha_{\ell-1} > \alpha_{\ell+1}$, then it follows from (4.5) that

$$t_{\ell+2}(a) - p_{\ell+2}(a) = (\alpha_{\ell+1}-\alpha_{\ell-1})p_{\ell+1}(a). \tag{4.6}$$

If $\ell$ is even, then we obtain from (4.6) that $t_{\ell+2}(a) > 0$, since $t_{\ell+2}(a) - p_{\ell+2}(a) > 0$ and $p_{\ell+2}(a) > 0$. This implies that $\tau_1 > a$. If $\ell$ is odd, then it follows from (4.6) that $t_{\ell+2}(a) < 0$, since $t_{\ell+2}(a) - p_{\ell+2}(a) < 0$ and $p_{\ell+2}(a) < 0$, which implies that $\tau_1 > a$. $\square$

**Corollary 4.2.** *Let the measure $d\sigma$ be symmetric, i.e. $d\sigma(-x) = d\sigma(x)$ for $x \in [a, b] = [-c, c]$, $c > 0$. Then the quadrature formula (4.1) is internal. If the sequence of the coefficients $\alpha_k$ $(k = 0, 1, \ldots)$ in the three-term recurrence relation (2.1) is increasing, then $(a <)\ \tau_{\ell+2} < b$. Conversely, if the sequence of the coefficients $\alpha_k$ $(k = 0, 1, \ldots)$ is decreasing, then $a < \tau_1$ $(< b)$.*

**Proof.** If the measure $d\sigma$ is symmetric, then the coefficients $\alpha_k$ $(k = 0, 1, \ldots)$ in the three-term recurrence relation (2.1) vanish. $\square$

In the sequel we will analyze when the quadrature formula (4.1) is internal for measures of the form $d\sigma(x) = w(x)\,dx$, where $w$ is one of the classical weight functions. By Corollary 4.2 the quadrature formula (4.1) is internal when $w$ is an even weight function, and in particular for the Gegenbauer weight function $w(x) = (1 - x^2)^{\alpha}$, $-1 < x < 1$, with $\alpha > -1$. This weight function includes the important special cases:

(a) $w(x) = 1$ over $[-1, 1]$ (Legendre weight function),
(b) $w(x) = (1 - x^2)^{-\frac{1}{2}}$ over $[-1, 1]$ (Chebyshev weight function of the first kind),
(c) $w(x) = (1 - x^2)^{\frac{1}{2}}$ over $[-1, 1]$ (Chebyshev weight function of the second kind).

We now turn to some nonsymmetric weight functions.

### *4.1. Generalized Laguerre weight functions*

Let $w(x) = x^s e^{-x}$, $s > -1$, on $[0, \infty)$. For this weight function, we have

$$\begin{aligned} \alpha_\ell &= 2\ell + s + 1, \quad \beta_\ell = \ell(\ell + s), \\ p_\ell(0) &= (-1)^\ell\, \ell! \binom{\ell + s}{\ell} \quad (s > -1); \end{aligned} \tag{4.7}$$

see, e.g., [10].

Assume first that $\ell$ is odd. The quadrature formula (4.1) is internal, i.e., the first zero $\tau_1$ of $t_{\ell+2}$ satisfies $\tau_1 \geq 0$, if (cf. (4.2))

$$t_{\ell+2}(0) = -\alpha_{\ell-1} p_{\ell+1}(0) - \beta_{\ell+1}\, p_\ell(0) \leq 0,$$

i.e., if

$$-\alpha_{\ell-1} p_{\ell+1}(0) \leq \beta_{\ell+1}\, p_\ell(0).$$

Dividing the last inequality by $\beta_{\ell+1}\, p_\ell(0)$ $(< 0)$, we get

$$\frac{-\alpha_{\ell-1} p_{\ell+1}(0)}{\beta_{\ell+1}\, p_\ell(0)} \geq 1. \tag{4.8}$$

Substituting (4.7) into (4.8) yields

$$\ell \geq 2 - s. \tag{4.9}$$

A similar analysis in the case when $\ell$ is even also gives the condition (4.9). The inequality (4.9) holds for $s \geq 0$ and for all $\ell \geq 2$, as well as for $s \in (-1, 0)$ and for all $\ell \geq 3$. The condition (4.9) does not hold for $\ell = 2$ and $s \in (-1, 0)$. We have shown the following result.

**Theorem 4.3.** *The quadrature formula (4.1) for the generalized Laguerre weight function $w(x) = x^s e^{-x}$, $s > -1$, on $[0, \infty)$, is internal for $s \geq 0$, $\ell \geq 2$, and for $s \in (-1, 0)$, $\ell \geq 3$. The quadrature formula is external for $s \in (-1, 0)$, $\ell = 2$.*

We remark that the corresponding (non-truncated) averaged generalized Gaussian quadrature formula defined by $J_{2\ell-1,\ell-1}(d\sigma, d\sigma)$ and discussed in [2] is internal for the generalized Laguerre weight function when $s \geq 1$ and external for $s \in (-1, 1)$ for any $\ell \geq 1$; see [11].

### *4.2. Jacobi weight functions*

Let $w^{(\alpha,\beta)}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $\alpha, \beta > -1$, and $-1 < x < 1$. We may assume that $\alpha \neq \beta$, since the case $\alpha = \beta$ has been discussed above. The recursion coefficients $\alpha_\ell$ and $\beta_\ell$, and the values of the monic orthogonal polynomials $p_\ell^{(\alpha,\beta)}$ at the interval endpoints are explicitly known; see cf. [10]. We have

$$
\begin{aligned}
\alpha_\ell &= \frac{\beta^2-\alpha^2}{(2\ell+\alpha+\beta)(2\ell+\alpha+\beta+2)},\\
\beta_\ell &= \frac{4\ell(\ell+\alpha)(\ell+\beta)(\ell+\alpha+\beta)}{(2\ell+\alpha+\beta)^2\left((2\ell+\alpha+\beta)^2-1\right)},\\
p_\ell^{(\alpha,\beta)}(1) &= \frac{2^\ell\binom{\ell+\alpha}{\ell}}{\binom{2\ell+\alpha+\beta}{\ell}},\\
p_\ell^{(\alpha,\beta)}(-1) &= (-1)^\ell p_\ell^{(\beta,\alpha)}(1).
\end{aligned}
\tag{4.10}
$$

Assume that $\alpha^2 > \beta^2$. Then the sequence $\{\alpha_\ell\}_{\ell=0}^{\infty}$ is increasing. It follows from Corollary 4.2 that $\tau_{\ell+2} \le 1$. It remains to investigate under which conditions the inequality $-1 \le \tau_1$ holds. We first consider the situation when $\ell$ is odd. Then the quadrature formula (4.1) is internal, i.e., the first zero $\tau_1$ of $t_{\ell+2}$ satisfies $\tau_1 \ge -1$ if (cf. (4.2))

$$t_{\ell+2}(-1) = (-1-\alpha_{\ell-1})p_{\ell+1}(-1) - \beta_{\ell+1}\,p_\ell(-1) \le 0,$$

i.e., if

$$-(1+\alpha_{\ell-1})p_{\ell+1}(-1) \le \beta_{\ell+1}\,p_\ell(-1).$$

Dividing the last inequality by $\beta_{\ell+1}\,p_\ell(-1)$ $(< 0)$, we get

$$\frac{-(1+\alpha_{\ell-1})p_{\ell+1}(-1)}{\beta_{\ell+1}\,p_\ell(-1)} \ge 1. \tag{4.11}$$

Define $g(\alpha,\beta) := (\alpha+\beta+2\ell+2)(\alpha+\beta+2\ell+3)$. Substituting (4.10) into (4.11) gives the inequality

$$\frac{\left[(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)+\beta^2-\alpha^2\right]g(\alpha,\beta)}{2(\ell+1)(\ell+1+\alpha)(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)} \ge 1. \tag{4.12}$$

Proceeding in a similar manner when $\ell$ is even, we also obtain the condition (4.12).

Now assume that $\alpha^2 < \beta^2$. Then the sequence $\{\alpha_\ell\}_{\ell=0}^{\infty}$ is decreasing. Corollary 4.2 shows that $\tau_1 \ge -1$. It remains to study when the inequality $\tau_{\ell+2} \le 1$ is valid. This inequality holds if

$$\frac{\left[(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)-(\beta^2-\alpha^2)\right]g(\alpha,\beta)}{2(\ell+1)(\ell+1+\beta)(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)} \ge 1, \tag{4.13}$$

where $g(\alpha,\beta)$ is defined as above.

Note that by interchanging $\alpha$ by $\beta$, the conditions $\alpha^2 > \beta^2$ and (4.12) turn into the conditions $\alpha^2 < \beta^2$ and (4.13). Therefore, in the sequel, it suffices to consider the conditions $\alpha^2 > \beta^2$ and (4.12) only.

**Theorem 4.4.** *The quadrature formula (4.1) with the Jacobi weight function $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ for $\alpha,\beta > -1$ $(\alpha \ne \beta)$ and $-1 < x < 1$ is internal for all $\ell \ge 3$.*

**Proof.** Let $\alpha+\beta = s$ $(s > -2)$ and $\alpha-\beta = d$. Then the inequality (4.12) can be expressed as

$$(s+2\ell-2)(s+2\ell+2)(s+2\ell)(s+\ell+2) \ge dP(s), \tag{4.14}$$

where

$$
\begin{aligned}
P(s) &= s(s+2\ell+2)(s+2\ell+3)+(\ell+1)(s+2\ell-2)(s+2\ell)\\
&= s^3+(5\ell+6)s^2+(8\ell^2+12\ell+4)s+4\ell^3-4\ell,
\end{aligned}
$$

which can be written in the form

$$P(s) = (s+\ell+2)(s+2\ell-2)(s+2\ell+6)-8(\ell-1)(s+2\ell+3). \tag{4.15}$$

Assume for the moment that $P(s) > 0$. Since

$$P(s) < (s+\ell+2)(s+2\ell-2)(s+2\ell+6),$$

a sufficient condition for (4.12) to hold is that

$$d = \alpha-\beta \le \frac{(s+2\ell+2)(s+2\ell)}{s+2\ell+6} = s+2\ell-4+\frac{24}{s+2\ell+6},$$

i.e.,

$$\beta + \frac{12}{\alpha + \beta + 2\ell + 6} \geq 2 - \ell.$$

When $\ell \geq 3$ this inequality obviously holds for all $\alpha, \beta > -1$. For $\ell = 2$ it becomes

$$\beta + \frac{12}{\alpha + \beta + 10} \geq 0. \tag{4.16}$$

It remains to investigate the sign of $P(s)$ for $s > -2$. We consider several cases:

(1°) Case $\ell \geq 6$. Multiplying the inequalities $s + 2\ell + 6 \geq s + 2\ell + 3$, $s + 2\ell - 2 \geq \frac{8}{5}(\ell - 1)$ and $s + \ell + 2 \geq 6$ yields

$$(s + \ell + 2)(s + 2\ell - 2)(s + 2\ell + 6) \geq \frac{48}{5}(\ell - 1)(s + 2\ell + 3) > 8(\ell - 1)(s + 2\ell + 3).$$

It now follows from (4.15) that $P(s) > 0$.

(2°) Case $\ell = 5$. All zeros of $P(x) = x^3 + 31x^2 + 264x + 480$ are smaller than $-2$. Therefore $P(s) > 0$ for all $\alpha, \beta$.

(3°) Case $\ell = 4$. The inequality (4.14) becomes

$$(s + 6)(s + 6)(s + 8)(s + 10) \geq d(s^3 + 26s^2 + 180s + 240).$$

Letting $d = s - 2\beta$, this inequality can be written as

$$s^3 + 26s^2 + 180s + 240 + \frac{12}{\beta + 2}(2s^2 + 27s + 80) \geq 0. \tag{4.17}$$

If $P(s) = s^3 + 26s^2 + 180s + 240 < 0$, then it follows that $s < -1.74$. Thus, $\beta < -0.74$ and $\frac{12}{\beta+2} > 9.52$. Since $2s^2 + 27s + 80 > 0$ and the zeros of $s^3 + 26s^2 + 180s + 240 + 9.52 \cdot (2s^2 + 27s + 80)$ are smaller than $-2$, the inequality (4.17) must hold.

(4°) Case $\ell = 3$. Similarly to case (3°), the inequality (4.14) can be expressed as

$$s^3 + 21s^2 + 112s + 96 + \frac{4}{\beta + 1}(5s^2 + 49s + 96) \geq 0, \tag{4.18}$$

while $P(s) = s^3 + 21s^2 + 112s + 96 < 0$ implies $s < -1.05$, $\beta < -0.05$ and $\frac{4}{\beta+1} > 4.21$. Now (4.18) holds because $5s^2 + 49s + 96 > 0$ and $s^3 + 21s^2 + 112s + 96 + 4.21 \cdot (5s^2 + 49s + 96) > 0$ for $s > -2$. □

We conclude this section with a discussion of the case $\ell = 2$. The inequality (4.14) can be written as

$$f(\beta, s) = \beta(s^3 + 16s^2 + 60s + 24) + 4(4s^2 + 25s + 24) \geq 0. \tag{4.19}$$

Let $\beta \geq 0$. Then $s > -1$ and $\beta < s + 1$. Since the left-hand side of (4.19) is linear in $\beta$, we have for $s > -1$ that

$$f(0, s) = 4(4s^2 + 25s + 24) > 0, \quad f(s + 1, s) = s^4 + 17s^3 + 92s^2 + 184s + 120 > 0,$$

and it follows that (4.19) holds.

Next, if $s > 0$, then

$$P(s) = (s + 2)(s + 4)(s + 10) - 8(s + 7) \geq 8(s + 10) - 8(s + 7) > 0.$$

Therefore the inequality (4.16) is valid.

Finally, assume that $-1 < \beta < 0$ and $-2 < s < 0$. The zeros of

$$\frac{\partial f(\beta, s)}{\partial s} = 3\beta s^2 + 32(\beta + 1)s + 20(3\beta + 5)$$

are given by

$$s_1 = \frac{16(\beta + 1) + 2\sqrt{19\beta^2 + 53\beta + 64}}{-3\beta},$$

$$s_2 = \frac{16(\beta + 1) - 2\sqrt{19\beta^2 + 53\beta + 64}}{-3\beta},$$

where we note that $s_1 > 0$. The condition $s_2 > -2$ is equivalent to $5\beta + 8 > \sqrt{19\beta^2 + 53\beta + 64}$, which in turn is equivalent to $3\beta(2\beta + 9) > 0$. The last inequality is impossible. Hence $s_2 \leq -2$, which implies that $\frac{\partial f(\beta,s)}{\partial s} > 0$ and $f(\beta, s)$ is strictly increasing in $s$ on $(-2, 0)$. It follows that

$$f(\beta, s) > f(\beta, \beta - 1) = \beta^4 + 13\beta^3 + 47\beta^2 + 47\beta + 12 = (\beta + 1)(\beta + 4)(\beta^2 + 8\beta + 3).$$

Thus, $f(\beta, s) > 0$ for $\beta \geq \sqrt{13} - 4 \approx -0.3944487245$.

The discriminant of the polynomial in $\beta$,

$$f(\beta,\alpha+\beta)=\beta^4+(3\alpha+16)\beta^3+(3\alpha^2+32\alpha+76)\beta^2+(\alpha^3+16\alpha^2+92\alpha+124)\beta+(16\alpha^2+100\alpha+96),$$

equals

$$Q(\alpha)=64(993\alpha^6+24228\alpha^5+113200\alpha^4-10400\alpha^3-1021212\alpha^2-1982200\alpha-1041580)$$

and has a unique zero $\alpha_0\approx -0.9419540398$ in the interval $(-1,0)$. Since the polynomial $f(\beta,\beta+\alpha_0)$ only has one zero of multiplicity two in $(-1,0)$, it follows that $f(\beta,s)>0$ for $\beta\in(-1,0)$ and $\alpha_0<\alpha<0$. We now are in a position to give sufficient conditions for the quadrature formula (4.1), with $\ell=2$, to be internal.

If $\alpha^2>\beta^2$ $(\alpha\neq\beta)$ and $\ell=2$, then we conclude that the inequality (4.12) holds whenever

$$\beta+\frac{12}{\alpha+\beta+10}>0\quad\text{and}\quad(\beta>\sqrt{13}-4\text{ or }\alpha>\alpha_0).$$

If $\beta^2>\alpha^2$ $(\alpha\neq\beta)$ and $\ell=2$, then we conclude that the inequality (4.13) is valid when

$$\alpha+\frac{12}{\alpha+\beta+10}>0\quad\text{and}\quad(\alpha>\sqrt{13}-4\text{ or }\beta>\alpha_0).$$

## 5. Computed examples

This section illustrates some properties and the performance of some of the quadrature formulas discussed in the preceding sections. All examples were implemented in MATLAB and executed with about 15 significant decimal digits.

**Example 5.1.** This example illustrates the possibility that generalized averaged Gauss rules are external, but the associated truncated quadrature formulas are interior. Consider the Jacobi weight function

$$w(x)=(1-x)^\alpha(1+x)^\beta,\quad -1<x<1,\ \alpha=-1/2,\ \beta=1.$$

First let $\ell=3$. The Gaussian rule $Q_\ell^G$ has all nodes in the open interval $(-1,1)$; however, the generalized averaged rule determined by the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$ with $d\sigma(x)=w(x)dx$, proposed in [2], has a node at about 1.003.

The quadrature rule $Q_{\ell+1}^{(1)}$ given by (4.1) and defined by the matrix (4.3) has all nodes in the open interval $(-1,1)$ in agreement with Theorem 4.4. We remark that we also can define this quadrature rule by removing the last $\ell-2$ rows and columns of the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$

We turn to the quadrature rule $Q_{\ell+2}^{(2)}$ with $\ell+2$ nodes. The nodes and weights are determined by the symmetric tridiagonal matrix obtained by removing the last $\ell-3$ rows and columns of the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$. The rule $Q_{\ell+2}^{(2)}$ has a node larger than unity. Hence, Theorem 4.4 cannot be extended to quadrature rules $Q_{\ell+2}^{(2)}$.

It is easy to construct examples for which the generalized averaged quadrature rule defined by the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$ is not interior. However, often interior rules can be determined by removing only a few of the last rows and columns. For instance, consider the weight function

$$w(x)=(1-x)^\alpha(1+x)^\beta,\quad \alpha=-3/4,\ \beta=3/4,$$

and let $\ell=4$. Then the generalized averaged rule defined by the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$ has one exterior node at about 1.006. Truncated rules obtained by removing the last $k$ rows and columns of $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$ are interior for both $k=1$ and $k=2$. Similarly, when $\ell=8$, the generalized averaged rule defined by the matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$ has one exterior node at about 1.001. Truncated rules obtained by removing the last $k$ rows and columns are interior for $k=5$ and $k=6$. □

**Example 5.2.** We show the accuracy achieved when applying quadrature rules of the previous example to the approximation of the integral

$$\int_{-1}^{1}f(x)dx,\quad f(x)=(5-10x)\exp(x-x^2).\tag{5.1}$$

Thus, $d\sigma(x)=dx$. The value of the integral is $1-e^{-10}$. Table 5.1 displays quadrature errors for the Gauss rule $Q_\ell^G[f]$ (2.2), for the generalized averaged Gauss rule $Q_{2\ell-1,\ell-1}[f]$ determined by the tridiagonal matrix $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$, and for the rules $Q_{\ell+1}^{(1)}[f]$ and $Q_{\ell+2}^{(2)}[f]$ defined by the symmetric tridiagonal matrices obtained by removing the last $\ell-2$ and $\ell-3$ rows and columns of $J_{2\ell-1,\ell-1}(d\sigma,d\sigma)$, respectively. The table shows the magnitude of the quadrature error for the rule $Q_{2\ell-1,\ell-1}[f]$ to be about the square of the error for $Q_\ell^G[f]$ for all $\ell$-values. The truncated rules $Q_{\ell+1}^{(1)}[f]$ and $Q_{\ell+2}^{(2)}[f]$ are seen to give lower accuracy than $Q_{2\ell-1,\ell-1}[f]$, but higher accuracy than the Gaussian rule $Q_\ell^G[f]$ for all $\ell$. We conclude that the rule $Q_{2\ell-1,\ell-1}[f]$ only should be truncated if it is important that the quadrature rule be interior. □

**Table 5.1**
Errors in computed approximations of the integral (5.1).

| $\ell$ | $Q_\ell^G[f]$ | $Q_{2\ell-1,\ell-1}[f]$ | $Q_{\ell+1}^{(1)}[f]$ | $Q_{\ell+2}^{(2)}[f]$ |
|---|---|---|---|---|
| 4 | $4.84 \cdot 10^{-1}$ | $-1.16 \cdot 10^{-2}$ | $-1.86 \cdot 10^{-1}$ | $5.19 \cdot 10^{-2}$ |
| 5 | $-1.86 \cdot 10^{-1}$ | $-6.66 \cdot 10^{-4}$ | $4.20 \cdot 10^{-2}$ | $-7.29 \cdot 10^{-3}$ |
| 6 | $4.20 \cdot 10^{-2}$ | $5.19 \cdot 10^{-5}$ | $-6.41 \cdot 10^{-3}$ | $7.09 \cdot 10^{-4}$ |
| 7 | $-6.41 \cdot 10^{-3}$ | $-3.27 \cdot 10^{-6}$ | $6.41 \cdot 10^{-4}$ | $-2.90 \cdot 10^{-5}$ |
| 8 | $6.41 \cdot 10^{-4}$ | $1.29 \cdot 10^{-7}$ | $-2.40 \cdot 10^{-5}$ | $-5.48 \cdot 10^{-6}$ |

**Table 5.2**
Errors in computed approximations of the integral (5.1).

| $\ell$ | $Q_{2\ell-1,\ell-1}[f]$ | $\widetilde{Q}_{2\ell-1,\ell-1}[f]$ |
|---|---|---|
| 4 | $-1.16 \cdot 10^{-2}$ | $-5.95 \cdot 10^{-3}$ |
| 5 | $-6.66 \cdot 10^{-4}$ | $-1.11 \cdot 10^{-4}$ |
| 6 | $5.19 \cdot 10^{-5}$ | $1.05 \cdot 10^{-5}$ |
| 7 | $-3.27 \cdot 10^{-6}$ | $-6.44 \cdot 10^{-7}$ |
| 8 | $1.29 \cdot 10^{-7}$ | $2.75 \cdot 10^{-8}$ |
| 9 | $6.10 \cdot 10^{-9}$ | $1.40 \cdot 10^{-9}$ |
| 10 | $-1.85 \cdot 10^{-9}$ | $-4.37 \cdot 10^{-10}$ |
| 11 | $2.30 \cdot 10^{-10}$ | $5.61 \cdot 10^{-11}$ |
| 12 | $-2.14 \cdot 10^{-11}$ | $-5.35 \cdot 10^{-12}$ |

**Table 5.3**
Errors in computed approximations of the integral (5.1) using the rule $\widetilde{Q}_{3\ell-1,\ell-1}[f]$.

| $\widetilde{Q}_{11,3}[f]$ | $\widetilde{Q}_{14,4}[f]$ | $\widetilde{Q}_{17,5}[f]$ |
|---|---|---|
| $5.92 \cdot 10^{-4}$ | $-8.64 \cdot 10^{-5}$ | $9.00 \cdot 10^{-6}$ |

**Table 5.4**
Errors in computed approximations of the integral (5.1) using the rule $\widetilde{Q}_{\ell+5,5}[f]$.

| $\widetilde{Q}_{14,5}[f]$ | $\widetilde{Q}_{15,5}[f]$ | $\widetilde{Q}_{16,5}[f]$ |
|---|---|---|
| $-1.18 \cdot 10^{-9}$ | $-2.29 \cdot 10^{-10}$ | $4.18 \cdot 10^{-11}$ |

**Example 5.3.** Starting with a symmetric tridiagonal matrix $J_\ell^G$ associated with an $\ell$-node Gaussian quadrature rule $Q_\ell^G$, and the last subdiagonal entries of the symmetric tridiagonal matrix $J_{\ell+1}^G$, the extended symmetric matrix $J_{2\ell-1,\ell-1}(d\sigma, d\mu)$ can be defined in a variety of ways. In this example, we define $J_{2\ell-1,\ell-1}(d\sigma, d\mu)$ by letting all subdiagonal entries in rows $\ell+2, \ell+3, \ldots, 2\ell-1$ be the same as the last subdiagonal entry of $J_{\ell+1}^G$; the diagonal entries in rows $\ell+1, \ell+2, \ldots, 2\ell-1$ are chosen the same as the last diagonal entry of $J_\ell^G$. We denote this matrix by $\widetilde{J}_{2\ell-1,\ell-1}$ and the associated quadrature rule by $\widetilde{Q}_{2\ell-1,\ell-1}$. The same measure as in Example 5.2 is used. Table 5.2 displays quadrature errors. The rule $\widetilde{Q}_{2\ell-1,\ell-1}$ is seen to give a smaller error than $Q_{2\ell-1,\ell-1}$ for all values of $\ell$.

It may be meaningful to append the matrix $J_\ell^G$ with more rows and columns than in the computations for Table 5.2. For instance, Table 5.3 reports results for quadrature rules with $3\ell - 1$ nodes obtained by extending the tridiagonal matrix $\widetilde{J}_{2\ell-1,\ell-1}$ by $\ell$ more rows and columns. The new rows and columns are analogous to the last row and column of $\widetilde{J}_{2\ell-1,\ell-1}$. We refer to the associated quadrature rule as $\widetilde{Q}_{3\ell-1,\ell-1}$. This rule gives a smaller error than $\widetilde{Q}_{2\ell-1,\ell-1}$ for small values of $\ell$.

Finally, we found that truncating the matrix $\widetilde{J}_{2\ell-1,\ell-1}$ by removing the last rows may give quadrature rules that yield higher accuracy than $\widetilde{Q}_{2\ell-1,\ell-1}[f]$ when $\ell$ is large. Specifically, we removed the $\ell - 6$ last rows and columns of the matrix $\widetilde{J}_{2\ell-1,\ell-1}$ to obtain $\widetilde{J}_{\ell+5,5}$ and the associated quadrature rule $\widetilde{Q}_{\ell+5,5}$. Quadrature errors for this rule are displayed in Table 5.4.

Tables 5.3 and 5.4 illustrate that other extensions of Gaussian quadrature formulas than the generalized averaged Gaussian rules discussed in [11,2] may be of interest. We are presently investigating these rules. □

## 6. Conclusion

An analysis of truncated generalized averaged Gaussian quadrature formulas is presented that sheds light on whether these formulas are interior. Computed examples show that the analysis is sharp in the sense that it cannot be generalized to quadrature rules that are extended more than $Q_{\ell+1}^{(1)}$. Further examples illustrate the performance of generalized averaged Gaussian quadrature formulas and their truncations.

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