UDK: 025.4.03/06:007.52:519.87
COSATI: 12-01

Fractional-order Iterative Learning Control for Singular Fractional-order System: (P) - PD^{α} Type

Mihailo Lazarević[1)]
Nikola Đurović[1)]
Boško Cvetković[1)]
Petar Mandić[1)]
Ljubiša Bučanović[2)]

Iterative learning control (ILC) is one of the recent topics in control theories and it is suitable for controlling a wider class of mechatronic systems - it is especially suitable for the motion control of robotic systems. This paper addresses the problem of application of fractional order ILC for fractional order singular system. Particularly, we study fractional order singular systems in the pseudo-state space. An closed-loop fractional order PDalpha type ILC of the fractional-order singular system is investigated. Also, open-closed loop of the fractional order P-PDa type ILC is considered. Sufficient conditions for the convergence in the time domain of the proposed ILC schemes are given by the corresponding theorems and proved. Finally, numerical simulations show the feasibility and effectiveness of the proposed approach.

Key words: **control theory, iterative control, learning control, fractional order, singular system, method convergence, robotic system.**

Introduction

ITERATIVE learning control (ILC) is one of the most active fields in control theory and it is a powerful intelligent control concept that iteratively improves behavior of the processes that are repetitive in nature [1-3]. Since the early 80's, ILC [4,5] has been one of the very effective control strategies in dealing with repeated tracking control with the aim of improving tracking performance for the systems that work in a repetitive mode. As opposed to traditional controllers, ILC is a simple and effective control and can progressively reduce tracking errors and improve system performance from iteration to iteration. Namely, ILC is a trajectory tracking improvement technique for control systems, which can perform the same task repetitively in a finite time interval to improve the transient response of a system using the previous motion. For the purpose of emulating human learning, ILC uses knowledge obtained from the previous trial to adjust the control input for the current trial so that a better performance can be achieved. ILC is a memory based control technique since the input-output data should be stored after each iteration for updating the control input for the next iteration. Therefore, ILC requires less a priori knowledge about the controlled system in the controller design phase and also less computational effort than many other kinds of control. Besides, in terms of how to use tracking error signal of the previous iteration to form the control signal of the current iteration, ILC updating schemes can be classified as P-type, D-type, PD-type, and PID type. A typical ILC in the time domain is a simple open-loop control (off-line learning control) that only uses tracking error information in the previous iterations to form the control signal used in the current iteration and it cannot suppress the unanticipated, non-repeating disturbances. So, ILC is a technique of controlling systems operating in a repetitive mode with the additional requirement that a specified output trajectory $y_d(t)$ in an interval $[0,T]$ is followed to a high precision and through improving the performance from trial to trial in the sense that the tracking error is sequentially reduced. The basic strategy is to use an iteration of the form $u_{i+1}(t)=f(u_i(t),e_i(t)),\ e_i(t)=y_d(t)-y_i(t)$, where $f(.,.)$ defines the learning algorithm and remains to be specified, $y_i(t)$ is the output at the i-th operation resulting from an input $u_i(t)$, and $y_d(t)$ represents the desired output. The new control input $u_{i+1}(t)$ should make the system closer to the desired result in the next execution cycle.

In the real application, to overcome such drawbacks, an ILC scheme is usually performed together with a proper feedback controller for compensation [6], where we often design a learning operator for the closed-loop (on-line ILC) systems that have achieved a good performance. Since the theories and learning algorithms on ILC were firstly proposed, ILC has attracted considerable interests [3] due to its simplicity and effectiveness of the learning algorithm, and its ability to deal with the problems associated with nonlinear, time-delay, uncertainties, and, recently, singular systems. Besides, during the past years, singular systems have attracted attention of a lot of researchers from the mathematics and

1) University of Belgrade, Faculty of Mechanical Engineering, Department of Mechanics, Kraljice Marije 16, 11020 Belgrade 35, SERBIA
2) Messer Tehnogas AD, Banjički put 62, 11090 Belgrade, SERBIA
Correspondence to: Mihailo Lazarević; e-mail:mlazarevic@mas.bg.ac.rs

control communities due to the fact that singular systems can describe behavior of some physical systems better than regular systems such as: electrical network models [7], mechanical models [8, 9], etc. Naturally, many theoretical results for regular systems have been extended to singular cases. For example, the robot control systems can generally be described by some nonlinear ordinary differential equations. However, when the robots contact with the objects and the environment, they will usually be depicted by some end-point constraints. In that way, the constraints are generally described by nonlinear differential-algebraic equations which are modeled as singular systems, [9]. It is well known that the issues of concern for singular systems are much more complicated than those for regular systems, because for singular systems we need to consider not only stability, but also regularity and the absence of impulses at the same time [10]. Actually, elimination of algebraic constraints needs a suitable feedback control [11]. From the control point of view, it is also necessary to study the ILC for singular systems. Until now, there are few results reported on introducing ILC methods to studying of tracking control for singular systems [12, 13].

Recently, increasing attentions are paid to fractional differential equations and their applications in various science and engineering fields [14, 15]. Moreover, an increasing attention has been paid to the fractional calculus (FC) and its application in control and modeling of fractional-order singular systems [16, 17]. It is not difficult to conclude that other dynamic systems (robotic systems of fractional-order, etc.) [18] can be displayed in the singular form, especially in realization of various robotic tasks.

Recently, the application of ILC to the fractional-order systems has become a new topic [19-22]. Among different fractional order controllers, fractional order iterative learning controller (FOILC), the fractional order version of iterative learning control (ILC), is of interest in this paper. Also, in [23, 24] are presented new results for PD^{α} type of robust ILC for a given class of fractional order uncertain time delay system. Moreover, for the first time, in the paper [25] an iterative learning feedback control is considered for the *fractional-order singular systems* as well as in the paper [26] a robust iterative learning feedback control of the second-order for fractional-order singular systems is considered. Motivated by the mentioned investigations of ILC algorithms for ILC fractional order control in the tracking problems of these systems, (open)-closed-loop iterative learning control for given fractional-order singular systems described in the form of state space and output equations. The sufficient convergent conditions of the proposed ILC will be derived in time-domain and formulated by a theorem. A rigorous mathematical proof for the convergence of the iterative learning process is presented. Finally, the simulation results are presented to illustrate the performance of the proposed P-PD^{α} ILC scheme.

The remainder of this paper is arranged as follows: in the Section *Preliminaries and basics of fractional calculus*, some preliminaries as well as the fractional Caputo operators are presented. In Section *Open-loop fractional-order iterative learning control,* the first main result is derived where the convergence is guaranteed by mathematical proof rigorously, which includes the extensions of some of the basic result ILC of singular fractional-order systems with order α ∈ (0, 1) to uncertain fractional-order singular system. In the next section *Open-closed-loop fractional-order iterative learning control* the second main result is presented in the same manner where the open-closed-loop fractional-order ILC is introduced for the same singular fractional order system. In the section *Numerical simulations* suitable numerical examples are included to illustrate the performance of the proposed (P)-PD^{α} ILC schemes. Finally, the last section summarizes this work.

Preliminaries and basics of fractional calculus

The λ -norm, maximum norm, induced norm

For a later use in proving the convergence of the proposed learning control, the following norms are introduced [3] for the n-dimensional Euclidean space R^n: the sup-norm $\|x\|_{\infty}=\sup_{1\le k\le n}|x_k|$, $x=[x_1,x_2,...x_n]^T$, $|x_k|$-absolute value; the maximum norm $\|x\|_s=\max_{0\le t\le T}|x(t)|$, $x(t)=[x_1(t),x_2(t),...,x_n(t)]^T$; the matrix norm as $\|A\|_{\infty}=\max_{1\le k\le m}\left(\sum_{j=1}^{n}|g_k|\right)$, $A=[a_{kj}]_{mxn}$ and the λ -norm for a real function:

$$\begin{gathered} h(t),\ (t\in[0,T]),\ h:[0,T]\to\Re^n \\ \|h(t)\|_{\lambda}=\sup_{t\in[0,T]} e^{-\lambda t}\|h(t)\|,\ \lambda>0 \end{gathered} \tag{1}$$

A useful property associated with the λ-norm is the following inequality.

Property 1: λ norm has the next property

$$\begin{gathered} \sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \|f(.)\|e^{a(t-\tau)}d\tau = \\ \sup_{t\in[0,T]}\int_0^t e^{-\lambda t}\|f(.)\|e^{a(t-\tau)(a-\lambda)}d\tau \le \frac{1-e^{(a-\lambda)T}}{\lambda-a}\|f(.)\|_{\lambda} \end{gathered} \tag{2}$$

The induced norm of the matrix A is defined as:

$$\|A\|=\sup\left\{\frac{\|Ax\|}{x}:x\in X \text{ with } \|x\|\ne 0\right\} \text{ with,} \tag{3}$$

where $\|(.)\|$ denotes an arbitrary vector norm. In case $\|(.)\|_{\infty}$ it follows that

$$\|Ax\|_{\infty}\le\|A\|_{\infty}\|x\|_{\infty}, \tag{4}$$

where $\|A\|_{\infty}$ denotes the maximum value of the matrix A. For the previous norms, note that

$$\|h(t)\|_{\lambda}\le\|h(t)\|_{\infty}\le e^{\lambda T}\|h(t)\|_{\lambda}. \tag{5}$$

The λ - norm is thus equivalent to the ∞- norm. For simplicity, in applying the norm $\|(.)\|_{\infty}$ the index ∞ will be omitted. Before giving the main results, we first give the following Lemma 1, [27].

Lemma 1. Suppose a real positive series $\{a_n\}_1^{\infty}$ satisfies

$$\begin{gathered} a_n\le\rho_1 a_{n-1}+\rho_2 a_{n-2}+...+\rho_N a_{n-N}+\varepsilon \\ (n=N+1,\ N+2,\ ...,\ ) \end{gathered} \tag{6}$$

where $\rho_1 \geq 0 \; (i=1,2,...,N)$ $\varepsilon = 0$ and $\rho = \sum_{i=1}^{N} \rho_i < 1$. Then the following holds:

$$\lim_{n\to\infty} a_n \leq \varepsilon / (1-\rho). \tag{7}$$

Fractional calculus- Caputo operator

Fractional calculus (FC) is a generalization of classical calculus concerned with the operations of integration and the differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The three most frequently used definitions for the general fractional differential integral are: the Grunwald-Letnikov (GL) definition, the Riemann-Liouville (RL) and the Caputo definitions, [14, 15]. In this paper, Caputo fractional-order operator is used, where definition of the left Caputo fractional-order derivatives is given [14, 15] as follows:

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{8}$$

where $f^{(n)}(\tau) = d^n f(\tau)/d\tau^n$, $n-1 \leq \alpha < n \in \mathbb{Z}^+$, and $\Gamma(.)$ is the well-known Euler's gamma function. In the case $n=1$ we have $0 \leq \alpha < 1$ as well as

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{df(\tau)}{d\tau} d\tau. \tag{9}$$

In the following sections, D^{α} will denote ${}_{to}^{C}D_t^{\alpha}$ for brevity of notation.

Fractional-order autonomous linear singular system

Consider the following autonomous, singular, fractional-order system (SFOS) described by the state and output equations, respectively

$$ED^{\alpha}\, \mathbf{x}(t) = Ax(t), \quad n-1 < \alpha < n \ , \tag{10}$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \tag{11}$$

where admissible initial conditions for (10) are given by

$$x^{(k)}(0) = x_{0,k} \quad k = 0,1,2,...n-1. \tag{12}$$

Here, ${}_{0}^{C}D_t^{\alpha} = D^{\alpha}$ denotes the α th-order Caputo fractional derivative with respect to *t*, while E, A, and C are matrices with appropriate dimensions [28, 29]. In solving a singular problem, assuming regularity of the system, it is necessary to ensure the existence and uniqueness of the solution.

Definition 1.

a) The SFOS system (10) is said to be regular if $\det\left(s^{\alpha}E - A\right) \neq 0$,

b) The SFOS system (10) is said to be impulse free if (10) applies and

$$\deg\left(\det\left(s^{\alpha}E - A\right)\right) = rankE. \tag{13}$$

Lemma 1. The triplet (E, A, α) is called regular if and only if $\det\left(s^{\alpha}E - A\right) \neq 0$ for some $s \in \mathbb{C}$ [28, 29]. Also, if triplet (E, A, α) is regular, we call SFOS system (10) regular, and consequently SFOS system is solvable.

Lemma 2. If the function $f(t,x)$ is continuous, then the initial value problem

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x(t)), & 0 < \alpha < 1 \\ x(t_0) = x(0) \end{cases} \tag{14}$$

is the equivalent to the following nonlinear Volterra integral equation:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds \tag{15}$$

and its solutions are continuous, [30].

Closed-loop fractional-order iterative learning control

The fractional-order non-autonomous singular linear system

A non-integer (fractional) linear, singular system described in the form of pseudo state space and output equations is considered. The considered class of fractional-order $\alpha \in (0,1)$ non-autonomous singular linear system can be written as the state space equation and output equation

$$E\, D^{\alpha}\, \mathbf{x}(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1 \tag{16}$$

$$\mathbf{y}(t) = C\mathbf{x}(t). \tag{17}$$

Here, t is the time within the operation interval $J = [t_o, t_o + T], J \subset R$, while A, B, and C are matrices having appropriate dimensions. It is assumed that $\det E = 0$ and that SFOS system is regular.

Also, the initial conditions of fractional differential equations which were compared to the given fractional derivatives were considered by different authors [29, 31], assuming that there was no difficulty regarding the questions of existence, uniqueness, and continuity of solutions with respect to initial data. The following assumptions on the system (16), (17) are imposed.

A1. The desired trajectories $y_d(t)$, $x_d(t)$ are continuously differentiable in $[0, T]$.

A2. For the given desired output trajectory $y_d(t)$, there exists a control input $u_d(t)$ such that

$$ED^{\alpha}\, \mathbf{x}_d(t) = Ax_d(t) + Bu_d(t), \;\; 0 < \alpha < 1 \tag{18}$$

$$\mathbf{y}_d(t) = C\mathbf{x}_d(t). \tag{19}$$

A3. SFOS system is controllable and observable.

A4. Resetting the initial conditions holds for all iterations, i.e. $x_k(0) = x_d(0), \quad k = 0,1,2...,$

Convergence Analysis

Here, it is suggested the closed-loop fractional order PD^{α} learning algorithm which comprises control law a PD^{α} feedback law. Moreover, it was shown in [32] that the tracking speed was the fastest when the system and iterative learning scheme have the same order. In the feedback loop ILC, the PD^{α} controller provides stability of the system and keeps its state errors within uniform bounds. Besides, feedback control is introduced as follows, (see Fig.1):

$$u_{f\,i+1}(t) = \Pi\, {}_0D_t^{\alpha} e_{i+1}(t^-) + \Gamma e_{i+1}(t^-)\,, \qquad (20)$$

where $e_{i+1}(t^-) = e_{i+1}(t-\varepsilon),\ \varepsilon \to 0^+$ denotes a vector of the just realized tracking error signal at time. If the feedback delay can be neglected then: $e_{i+1}(t^-) = e_{i+1}(t)$. In that way, closed-loop fractional order PD^{α} learning algorithm takes the form of

$$u_{i+1}(t) = u_i(t) + \Gamma \cdot e_{i+1}(t) + \Pi \cdot {}_0D_t^{\alpha} e_{i+1}(t)\,, \qquad (21)$$

where Γ, Π are gain matrices of appropriate dimensions, $u(t)$ the value of the function at time. A sufficient condition for convergence of a proposed feedback ILC is given by the Theorem 1 and proved as follows.

Theorem 1: Suppose that the update law (21), is applied to the system (16), (17) and assumptions $A_i, i = 1,2,3,4$ satisfies. If matrix Π exists such that

$$\left\| \left[I - \Pi C \bar{B} \right] \right\| \le \rho < 1, \qquad (22)$$

where $\bar{B} = \left(E + B\Pi C\right)^{-1}$ and matrix Π is such that $\left(E + B\Pi C\right)$ is invertible, then, when $i \to \infty$ the bounds of the tracking errors $\|x_d(t) - x_i(t)\|$, $\|y_d(t) - y_i(t)\|$, $\|u_d(t) - u_i(t)\|$, converge asymptotically to zero.

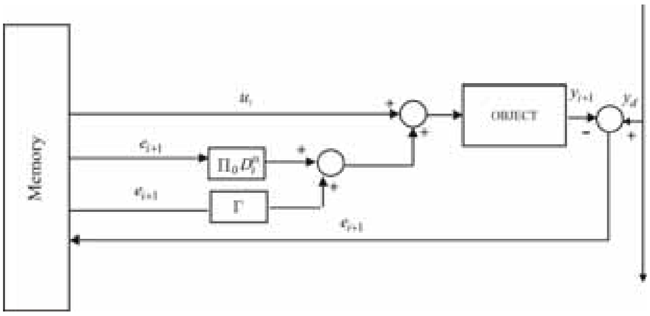

Figure 1. Block diagram of closed-loop PD^{α} iterative learning control for a LTI singular system

Proof. Let

$$\begin{aligned} \delta h_i &= h_d(t) - h_i(t), \quad h = x, x_d, u, u_d, f \\ D^{\alpha} \delta h_i\ (t) &= \delta h_i^{(\alpha)} = h_d^{(\alpha)}(t) - h_i^{(\alpha)}(t) \end{aligned} \qquad (23)$$

Tracking error can be obtained as follows:

$$e_i^{(\alpha)}(t) = \frac{d^{(\alpha)}}{dt^{(\alpha)}}\left(y_d(t) - y_i(t)\right) = C\delta x_i^{(\alpha)}(t)\,. \qquad (24)$$

Taking the proposed control law gives:

$$\delta u_{i+1} = u_d - u_{i+1} = \delta u_i - \Gamma e_{i+1} - \Pi e_{i+1}^{(\alpha)}\,, \qquad (25)$$

or, taking (24) it yields:

$$\delta u_{i+1} = u_d - u_{i+1} = \delta u_i - \Gamma C \delta x_{i+1} - \Pi C \delta x_{i+1}^{(\alpha)} \qquad (26)$$

Also, from (16), (17) one can find that

$$E\delta x_{i+1}^{(\alpha)} = A\delta x_{i+1} + B\delta u_{i+1} \qquad (27)$$

Substituting (26) into (27) it follows

$$E\delta x_{i+1}^{(\alpha)} = A\delta x_{i+1} + B\delta u_i - B\Gamma C\delta x_{i+1} - B\Pi C\delta x_{i+1}^{(\alpha)} \qquad (28)$$

After, rearranging (22) it becomes

$$\left(E + B\Pi C\right)\delta x_{i+1}^{(\alpha)} = \left(A - B\Gamma C\right)\delta x_{i+1} + B\delta u_i \qquad (29)$$

Using suitable gain matrix Π as well as taking into account previously introduced assumptions, matrix $\left(E + B\Pi C\right)$ is invertible, i.e. exists $\left(E + B\Pi C\right)^{-1}$. Multiplying on the left side expression (29) by $\left(E + B\Pi C\right)^{-1}$ we obtain (30) in the form

$$\begin{aligned} \delta x_{i+1}^{(\alpha)} &= \left(E + B\Pi C\right)^{-1}\left(A - B\Gamma C\right)\delta x_{i+1} + \\ &+ \left(E + B\Pi C\right)^{-1} B\delta u_i \end{aligned} \qquad (30)$$

If one adopts

$$\bar{A} = \left(E + B\Pi C\right)^{-1}\left(A - B\Gamma C\right),\ \bar{B} = \left(E + B\Pi C\right)^{-1} B \qquad (31)$$

then (30) becomes

$$\delta x_{i+1}^{(\alpha)} = \bar{A}\delta x_{i+1} + \bar{B}\delta u_i \qquad (32)$$

By replacing (32) into (26), we have

$$\delta u_{i+1} = \left[I - \Pi C\bar{B}\right]\delta u_i - \left[\Gamma C + \Pi C\bar{A}\right]\delta x_{i+1} \qquad (33)$$

Estimating the norms of (33) with $\|(.)\|$ and using the condition of Theorem 1 one gets

$$\begin{aligned} \|\delta u_{i+1}\| &\le \rho\|\delta u_i\| + \left\|\left[\Gamma C + \Pi C\bar{A}\right]\right\|\|\delta x_{i+1}\| = \\ &= \rho\|\delta u_i\| + \beta_0\|\delta x_{i+1}\| \end{aligned} \qquad (34)$$

Also, one can write the solutions of (32) in the form of the equivalent Volterra integral equations using assumption A4, as:

$$\delta x_{i+1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(A\delta x_{i+1}(s) + \bar{B}\delta u_i(s)\right)ds \qquad (35)$$

Applying the norm $\|(.)\|$ on the equation (35), if it is uniqueness solution, [29, 31] one obtains:

$$\begin{aligned} \|\delta x_{i+1}(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\left\|\bar{A}\right\|\|\delta x_{i+1}(s)\|ds + \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\left\|\bar{B}\right\|\|\delta u_i(s)\|ds \\ &\le \frac{a}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\|\delta x_{i+1}(s)\|ds + \frac{b}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\|\delta u_i(s)\|ds \end{aligned} \qquad (36)$$

where $a = \left\|\bar{A}\right\|,\ b = \left\|\bar{B}\right\|$. Moreover, applying λ norm to both sides of the previous (36), it follows

$$\|\delta x_{i+1}(t)\|_{\lambda} \le \sup_{0\le t\le T} e^{-\lambda t}\left\{ \begin{array}{l} \frac{a}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\|\delta x_{i+1}(s)\|ds + \\ +\frac{b}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}|\|\delta u_i(s)\|ds \end{array}\right\}, \quad (37)$$

$$\|\delta x_{i+1}(t)\|_{\lambda} \le \sup_{0\le t\le T}\left\{\int_0^t e^{-\lambda t}\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}\left[a\|\delta x_{i+1}(s)\|+b\|\delta u_i(s)\|\right]ds\right\}$$

$$\le \sup_{0\le t\le T}\int_0^t e^{-\lambda(t-s)}\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}\sup_{0\le t\le T} e^{-\lambda s}\left[a\|\delta x_{i+1}(s)\|+b\|\delta u_i(s)\|\right]ds$$

$$\le\left(a\|\delta x_{i+1}(t)\|_{\lambda}+b\|\delta u_i(t)\|_{\lambda}\right)\cdot\sup_{0\le t\le T}\int_0^t e^{-\lambda(t-s)}ds\ \sup_{0\le t\le T}\int_0^t\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}ds \quad (38)$$

or,

$$\|\delta x_{i+1}(t)\|_{\lambda} \le\left(a\|\delta x_{i+1}(t)\|_{\lambda}+b\|\delta u_i(t)\|_{\lambda}\right)\cdot\frac{\left(1-e^{-\lambda T}\right)}{\lambda}\frac{T^{\alpha}}{\Gamma(\alpha+1)} \quad (39)$$

Introducing $O\left(\lambda^{-1}\right)$, as

$$O\left(\lambda^{-1}\right)=\frac{\left(1-e^{-\lambda T}\right)}{\lambda}\frac{T^{\alpha}}{\Gamma(\alpha+1)}, \quad (40)$$

where (39) simplifies to

$$\|\delta x_{i+1}(t)\|_{\lambda} \le\left(a\|\delta x_{i+1}(t)\|_{\lambda}+b\|\delta u_i(t)\|_{\lambda}\right)O\left(\lambda^{-1}\right) \quad (41)$$

or, one may conclude

$$\|\delta x_{i+1}(t)\|_{\lambda} \le\frac{bO\left(\lambda^{-1}\right)}{\left(1-aO\left(\lambda^{-1}\right)\right)}\|\delta u_i(t)\|_{\lambda}\le O_{\gamma}\left(\lambda^{-1}\right)\|\delta u_i(t)\|_{\lambda}, \quad (42)$$

then, if a sufficiently large λ is used**,** one can obtain that:

$$\lambda\Gamma(\alpha+1)-a\left(1-e^{-\lambda T}\right)T^{\alpha}>0 \quad (43)$$

Taking the λ-norm of the expression (28) leads to:

$$\|\delta u_{i+1}\|_{\lambda}\le\rho\|\delta u_i\|_{\lambda}+\beta_0\|\delta x_{i+1}\|_{\lambda} \quad (44)$$

Finally, taking into account (36) we have

$$\|\delta u_{i+1}\|_{\lambda}\le\left(\rho+\beta_0 O_{\gamma}\left(\lambda^{-1}\right)\right)\|\delta u_i\|_{\lambda}=\rho'\|\delta u_i\|_{\lambda} \quad (45)$$

So that, there exists a sufficient large λ satisfying

$$\rho'=\left(\rho+\beta_0 O_{\gamma}\left(\lambda^{-1}\right)\right)<1 \quad (46)$$

Therefore, according to Lemma 1, [27] it can be concluded that:

$$\lim_{i\to\infty}\|\delta u_i\|_{\lambda}\to 0, \quad (47)$$

This completes the proof of Theorem 1. Moreover, due to uniqueness and existence of the theorem for fractional order singular system,[29] one can conclude that

$$\lim_{i\to\infty}x_i(t)=x_d(t),\quad \lim_{i\to\infty}y_i(t)=y_d(t). \quad (48)$$

Further, the case of the fractional order $\alpha\in(0,1)$ singular system non-autonomous singular linear system can be written as the state space equation and output equation is also discussed here:

$$ED^{\alpha}\mathbf{x}(t)=(A+\Delta A)x(t)+Bu(t),\quad 0<\alpha<1 \quad (49)$$

$$\mathbf{y}(t)=C\mathbf{x}(t), \quad (50)$$

Here, t is time in the operation interval $J=[t_0,t_0+T]$, $J\subset R$, as well as A,B and C are matrices with the appropriate dimensions; ΔA is unknown real norm-bounded matrice which represent parameter uncertainty in the system model.

Theorem 2. For the fractional order singular system (49), (50) with the PD$^{\alpha}$-type ILC scheme (21), and the assumptions A1-A4 where the convergence condition is given by (22), then when $i\to\infty$ the bounds of the tracking errors $\|x_d(t)-x_i(t)\|$, $\|y_d(t)-y_i(t)\|$, $\|u_d(t)-u_i(t)\|$, converge asymptotically to a residual ball centered at the origin.

Proof: The proof follows from the proof of Theorem 1. Namely, from (49), (50) one can easily find that

$$E\delta x_{i+1}^{(\alpha)}=(A+\Delta A)\delta x_{i+1}+B\delta u_{i+1}-\Delta Ax_d \quad (52)$$

Multiplying on the left side expression (52) by $(E+B\Pi C)^{-1}$ we obtain (53) in the form

$$\delta x_{i+1}^{(\alpha)}=\left(\bar{A}+\Delta\bar{A}\right)\delta x_{i+1}+\bar{B}\delta u_i-\Delta\bar{A}x_d \quad (53)$$

where

$$\bar{A}=(E+B\Pi C)^{-1}(A-B\Gamma C),\ \bar{B}=(E+B\Pi C)^{-1}B,$$

$$\Delta\bar{A}=(E+B\Pi C)^{-1}\Delta A. \quad (54)$$

By replacing (53) into (26), we have

$$\delta u_{i+1}=\left[I-\Pi C\bar{B}\right]\delta u_i-\left[\Gamma C+\Pi C\left(\bar{A}+\Delta\bar{A}\right)\right]\delta x_{i+1}+ +\Pi C\Delta\bar{A}x_d \quad (55)$$

Estimating the norms of (55) with $\|(.)\|$ and using the condition of Theorem 2 one gets

$$\begin{aligned}&\|\delta u_{i+1}\|\le\rho\|\delta u_i\|+\\&+\left\|\left[\Gamma C+\Pi C\left(\bar{A}+\Delta\bar{A}\right)\right]\right\|\beta_0\|\delta x_{i+1}\|\|\delta x_{i+1}\|+\\&+\|\Pi C\Delta\bar{A}\|\|x_d\|=\rho\|\delta u_i\|+\beta_0\|\delta x_{i+1}\|+\beta_1\|x_d\|\le\rho\|\delta u_i\|+\\&+\beta_0\|\delta x_{i+1}\|+\beta_1 c\end{aligned} \quad (56)$$

where it is fulfilled, $\|x_d(t)\|\le c,\ \forall t\in[0,T]$. Also, one can write the solutions of (53) in form of the equivalent Volterra integral equations using the assumption A4, as:

$$\delta x_{i+1}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\begin{matrix}\left[\bar{A}+\Delta\bar{A}\right]\delta x_{i+1}(s)+\\ \bar{B}\delta u_i(s)\end{matrix}\right)ds-\\ -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta\bar{A}x_d(s)\,ds \tag{57}$$

In a similar manner, applying the norm $\|(.)\|$ on the equation (57), if a uniqueness solution exists, [29, 31] where $a=\|\bar{A}\|$, $b=\|\bar{B}\|$, $a_{\Delta}=\|\Delta\bar{A}\|$, and applying λ norm, we have

$$\|\delta x_{i+1}(t)\|_{\lambda}\le\left((a+a_{\Delta})\|\delta x_{i+1}(t)\|_{\lambda}+b\|\delta u_i(t)\|_{\lambda}+a_{\Delta}c\right)\cdot\\ \cdot\frac{\left(1-e^{-\lambda T}\right)}{\lambda}\frac{T^{\alpha}}{\Gamma(\alpha+1)}. \tag{58}$$

or, one may conclude

$$\|\delta x_{i+1}(t)\|_{\lambda}\le\frac{bO\left(\lambda^{-1}\right)\|\delta u_i(t)\|_{\lambda}+a_{\Delta}O\left(\lambda^{-1}\right)}{\left(1-(a+a_{\Delta})O\left(\lambda^{-1}\right)\right)}\le\\ \le O_{\gamma}\left(\lambda^{-1}\right)\|\delta u_i(t)\|_{\lambda}+\varepsilon'\left(\lambda^{-1}\right) \tag{59}$$

Finally, taking the λ-norm of the expression (56) leads to:

$$\|\delta u_{i+1}\|_{\lambda}\le\rho\|\delta u_i\|_{\lambda}+\beta_0\|\delta x_{i+1}\|_{\lambda}+\beta_1 c \tag{60}$$

or, taking into account (59) we obtain

$$\|\delta u_{i+1}\|_{\lambda}\le\left(\rho+\beta_0 O_{\gamma}\left(\lambda^{-1}\right)\right)\|\delta u_i\|_{\lambda}+\\ +\beta_0\varepsilon'\left(\lambda^{-1}\right)+\beta_1 c=\rho'\|\delta u_i\|_{\lambda}+\varepsilon \tag{61}$$

So that, there exists a sufficient large λ satisfying

$$\rho'=\left(\rho+\beta_0 O_{\gamma}\left(\lambda^{-1}\right)\right)<1 \tag{62}$$

Therefore, taking into account Lemma1, [3] it yields:

$$\lim_{i\to\infty}\|\delta u_i\|_{\lambda}\le\frac{1}{1-\rho'}\varepsilon, \tag{63}$$

This completes the proof of Theorem 2.

Remark 1. In the case of no parameter uncertainty, i.e. $\Delta A=0$, one can obtain that when $i\to\infty$ bounds of the tracking errors $\|x_d(t)-x_i(t)\|$, $\|y_d(t)-y_i(t)\|$, and $\|u_d(t)-u_i(t)\|$ converge asymptotically to zero, as stated in Theorem 1, (i.e $\Delta A=0$, i.e $\varepsilon=0$).

Open-closed-loop fractional-order iterative learning control

Also, for the singular system defined by (10), open-closed-loop *P-PD*$^{\alpha}$-type iterative learning algorithm is proposed as follows:

$$u_{i+1}(t)=u_i(t)+\Gamma_1 e_i(t)+\Gamma_2\left({}_C D_{0,t}^{\alpha}e_{i+1}(t)+\Pi_2 e_{i+1}(t)\right), \tag{64}$$

where $u_i(t)$ and $y_i(t)$ are, respectively, the system input and output in the i_{th} iteration, $e_i(t)=y_d(t)-y_i(t)$ is the trajectory tracking error at *i*-th iteration, $u_{i+1}(t)$ is the system input of the $(i+1)_{th}$ trial, $y_d(t)=Cx_d(t)$ denotes desired output trajectory, and Γ_1,Γ_2,Π_2 are open-closed-loop learning matrices. In the closed loop, the PD^{α} controller $\Gamma_2\left({}_C D_{0,t}^{\alpha}e_{i+1}(t)+\Pi_2 e_{i+1}(t)\right)$ provides stability of the system and keeps its state errors within uniform bounds. A sufficient condition for convergence of the proposed open-closed- loop ILC is given by Theorem 3. The proof is as follows:

Theorem3: Suppose that the update law defined by (64) is applied to the non-autonomous singular linear system (16), (17) and assumptions $A_i, i=1,2,3,4$ are satisfied. If matrix Γ_2 exists such that

$$\left\|\left[I-\Gamma_2 C\bar{B}\right]\right\|\le\rho<1, \tag{65}$$

where $\bar{B}=\left(E+B\Gamma_2 C\right)^{-1}B$ and matrix Γ_2 is such that $\left(E+B\Gamma_2 C\right)$ is invertible, then, when $i\to\infty$, the bounds of the tracking errors $\|x_d(t)-x_i(t)\|$ $\|y_d(t)-y_i(t)\|$, $\|u_d(t)-u_i(t)\|$ converge asymptotically to a residual ball centered at the origin.

Proof.

The proof is similar to the proofs of the previous two theorems. Taking the proposed control law gives:

$$\delta u_{i+1}=u_d-u_{i+1}=\delta u_i-\Gamma_1 e_i-\Gamma_2\left(e_{i+1}^{(\alpha)}+\Pi_2 e_{i+1}\right), \tag{66}$$

or, based on equation (24), it follows:

$$\delta u_{i+1}=\delta u_i-\Gamma_1 C\delta x_{i+1}-\Gamma_2 C\delta x_{i+1}^{(\alpha)}-\Gamma_2\Pi_2 C\delta x_{i+1}. \tag{67}$$

as well as taking into account (27) one can find that

$$\left(E+B\Gamma_2 C\right)\delta x_{i+1}^{(\alpha)}=\\ =\left(A-B\Gamma_2\Pi_2 C\right)\delta x_{i+1}+B\delta u_i-B\Gamma_1 C\delta x_i \tag{68}$$

By using suitable gain matrix Γ_2, as well as by taking into account the previously introduced assumptions, the matrix $\left(E+B\Gamma_2 C\right)$ is invertible, i.e. there exists $\left(E+B\Gamma_2 C\right)^{-1}$. By multiplying the expression (68) by $\left(E+B\Gamma_2 C\right)^{-1}$, we obtain

$$\delta x_{i+1}^{(\alpha)}=\bar{A}\delta x_{i+1}+\bar{A}_1\delta x_i+\bar{B}\delta u_i, \tag{69}$$

where

$$\bar{A}=\left(E+B\Gamma_2 C\right)^{-1}\left(A-B\Gamma_2\Pi_2 C\right),$$

$$\bar{B}=\left(E+B\Gamma_2 C\right)^{-1}B,\ \ \bar{A}_1=\left(E+B\Gamma_2 C\right)^{-1}B\Gamma_1 C. \tag{70}$$

And after replacing (69) into (67), we have

$$\delta u_{i+1}=\left[I-\Gamma_2 C\bar{B}\right]\delta u_i-\left[\Gamma_2 C\bar{A}+\Gamma_2\Pi_2 C\right]\delta x_{i+1}\\ -\left[\Gamma_1 C+\Gamma_2 C\bar{A}_1\right]\delta x_i \tag{71}$$

Taking the norm of both sides of the equation (71) and using the condition of Theorem 3, this reduces to:

$$\|\delta u_{i+1}\|\le\rho\|\delta u_i\|+\left\|\left[\Gamma_2 C\bar{A}+\Gamma_2\Pi_2 C\right]\right\|\|\delta x_{i+1}\|+\\ +\left\|\left[\Gamma_1 C+\Gamma_2 C\bar{A}_1\right]\right\|\|\delta x_i\|=\rho\|\delta u_i\|+\beta_0\|\delta x_{i+1}\|+\beta_1\|\delta x_i\| \tag{72}$$

Again, we can obtain the solutions of (25) in form of the equivalent Volterra integral equations using the assumption A4, as:

$$\delta x_{i+1}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{pmatrix}\bar{A}\delta x_{i+1}(s)+\bar{A}_1\delta x_i(s)\\ +\bar{B}\delta u_i(s)\end{pmatrix}ds \quad (73)$$

Taking norms and using their properties, we have

$$\begin{aligned}&\|\delta x_{i+1}(t)\|\le\frac{a}{\Gamma(\alpha)}\int_0^t\left|(t-s)^{\alpha-1}\right|\|\delta x_{i+1}(s)\|ds+\\ &+\frac{a_1}{\Gamma(\alpha)}\int_0^t\left|(t-s)^{\alpha-1}\right|\|\delta x_i(s)\|ds+\frac{b}{\Gamma(\alpha)}\int_0^t\left|(t-s)^{\alpha-1}\right|\|\delta u_i(s)\|ds+\end{aligned}\quad (74)$$

where $a_{(..)}=\left\|\bar{A}_{(..)}\right\|,\ b=\left\|\bar{B}\right\|$. Furthermore, the next relation is fulfilled:

$$\begin{aligned}&t\in[0,T],\ \ \|\delta x_i(t)\|=\|\delta x_{i+1}(t)+x_{i+1}-x_i(t)\|\\ &\le\|\delta x_{i+1}(t)\|+\|x_{i+1}-x_i(t)\|\end{aligned}\quad (75)$$

Here, we may introduce $\eta_{i+1}=\sup_{t\in[0,T]}\|x_{i+1}-x_i(t)\|$, and $\|\delta x_i(t)\|\le\|\delta x_{i+1}(t)\|+\eta_{i+1}$ and applying λ norm to both sides leads to

$$\begin{aligned}&\|\delta x_{i+1}(t)\|_\lambda\le\sup_{0\le t\le T}\left\{\int_0^t e^{-\lambda t}\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}\begin{bmatrix}(a+a_1)\|\delta x_{i+1}(s)\|+\\ +b\|\delta u_i(s)\|\end{bmatrix}ds\right\}\\ &+\sup_{0\le t\le T}\left(e^{-\lambda t}\frac{a_1\eta_{i+1}t^\alpha}{\Gamma(\alpha+1)}\right)\end{aligned}\quad (76)$$

$$\begin{aligned}&\le\sup_{0\le t\le T}\int_0^t e^{-\lambda(t-s)}\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}\sup_{0\le t\le T}e^{-\lambda s}\begin{bmatrix}(a+a_1)\|\delta x_{i+1}(s)\|+\\ +b\|\delta u_i(s)\|\end{bmatrix}ds\\ &+\frac{a_1\eta_{i+1}}{\Gamma(\alpha+1)},\\ &\le\left((a+a_1)\|\delta x_{i+1}(t)\|_\lambda+b\|\delta u_i(t)\|_\lambda\right)\cdot\\ &\sup_{0\le t\le T}\int_0^t e^{-\lambda(t-s)}ds\ \sup_{0\le t\le T}\int_0^t\frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)}ds+\xi.\end{aligned}$$

where $\xi=a_1\eta_{i+1}/\Gamma(\alpha+1)$. Defining $O(\lambda^{-1})$, as

$$O(\lambda^{-1})=\frac{\left(1-e^{-\lambda T}\right)}{\lambda}\frac{T^\alpha}{\Gamma(\alpha+1)},\quad (77)$$

and substituting (77) into (76) produces

$$\begin{aligned}&\|\delta x_{i+1}(t)\|_\lambda\le\left((a+a_1)\|\delta x_{i+1}(t)\|_\lambda+b\|\delta u_i(t)\|_\lambda\right)\\ &O(\lambda^{-1})+\xi.\end{aligned}\quad (78)$$

It follows immediately from (78) that

$$\begin{aligned}&\|\delta x_{i+1}(t)\|_\lambda\le\frac{bO(\lambda^{-1})\|\delta u_i(t)\|_\lambda+\xi}{\left(1-(a+a_1)O(\lambda^{-1})\right)}\le\\ &\le O_\gamma(\lambda^{-1})\|\delta u_i(t)\|_\lambda+\xi'(\lambda^{-1})\end{aligned}\quad (79)$$

Now, it is possible to choose a sufficiently large λ such that

$$\lambda\Gamma(\alpha+1)-(a+a_1)\left(1-e^{-\lambda T}\right)T^\alpha>0.\quad (80)$$

Also,combining (72) and (75) and applying the λ-norm yields

$$\begin{aligned}&\|\delta u_{i+1}\|\le\rho\|\delta u_i\|+\beta_0\|\delta x_{i+1}\|+\beta_1\|\delta x_i\|\\ &=\rho\|\delta u_i\|+(\beta_0+\beta_1)\|\delta x_{i+1}\|+\beta_1\eta_{i+1}\\ &\|\delta u_{i+1}\|_\lambda\le\rho\|\delta u_i\|_\lambda+\beta'\|\delta x_{i+1}\|_\lambda+\beta''\end{aligned}\quad (81)$$

Finally, taking into account (79) we have

$$\begin{aligned}&\|\delta u_{i+1}\|_\lambda\le\left(\rho+\beta' O_\gamma(\lambda^{-1})\right)\|\delta u_i\|_\lambda+\beta'\xi'(\lambda^{-1})+\beta''\\ &=\rho'\|\delta u_i\|_\lambda+\varepsilon\end{aligned}\quad (82)$$

Therefore, there exists a sufficiently large λ satisfying

$$\rho'=\left(\rho+\beta' O_\gamma(\lambda^{-1})\right)<1.\quad (83)$$

Thus, from the fact that $\rho'<1$ and Lemma 1, [3] , it is immediate to achieve that $\lim_{i\to\infty}\|\delta u_i\|_\lambda\le\frac{1}{1-\rho'}\varepsilon$. This completes the proof of Theorem 3.

Numerical simulations

In this section, two numerical examples are presented to show the effectiveness of the proposed $(P)-PD^{\alpha}$ type iterative learning controller. First, consider the following fractional order linear singular system in state space form described by

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}D^{0.5}x_1(t)\\ D^{0.5}x_2(t)\end{bmatrix}=\begin{bmatrix}1&2\\1&-1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix}\quad (84)$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}\quad (85)$$

where $t\in[0,1],\ \alpha=0.5$. The desired trajectories are given by

$$y_{d1}(t)=1.5t\cdot(1-t),\ \ y_{d2}(t)=0.5t^2,$$

$$y_{d1,2}(0)=y_{i1,2}(0)=0\quad (86)$$

The learning gain matrices are chosen as follows

$$\Pi=\begin{bmatrix}0.95&1\\0&0.95\end{bmatrix},\ \Gamma=\begin{bmatrix}0.95&1\\0&0.95\end{bmatrix},\quad (87)$$

It is easy to show that the pair (E; A) is regular.

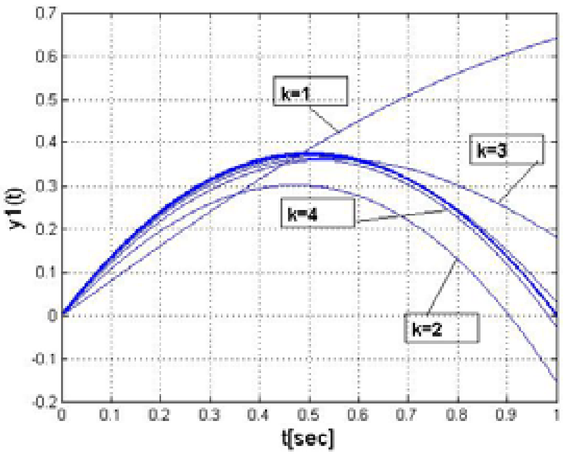

Figure 2. The tracking performance of the system output ($y_1(t)$**:** solid line, $y_{d1}(t)$**:** bold line)

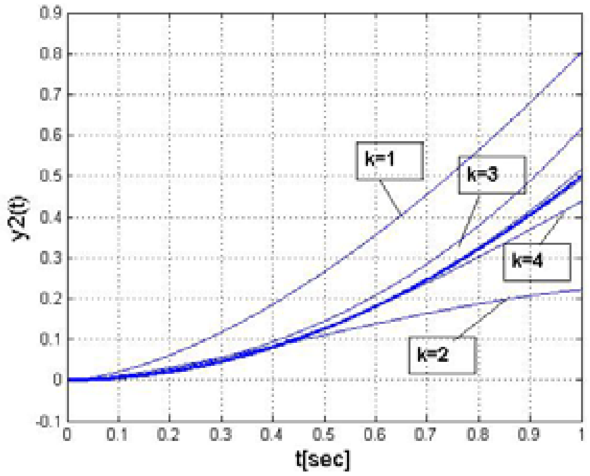

Figure 3. The tracking performance of the system output ($y_2(t)$**:** solid line, $y_{d2}(t)$ $y_{d2}(t)$ **:** bold line)

Simulation results in Figures 2-5 show the effectiveness of the developed ILC control scheme for the system (16), (17). The ILC rule (21) is used, (Figures 3, 4) show the tracking performance of the ILC system outputs on the interval $t \in [0,1]$. Also, we can find (see Figures 4, 5) that proposed requirement of tracking performance is achieved at the seventh iteration.

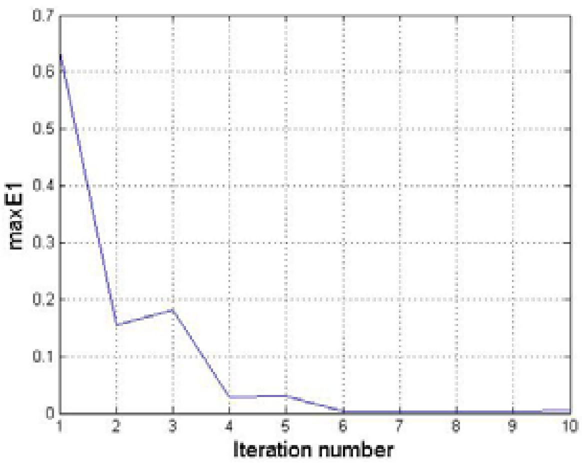

Figure 4. The sup-norm of tracking error $e_1(t)$ in each iteration

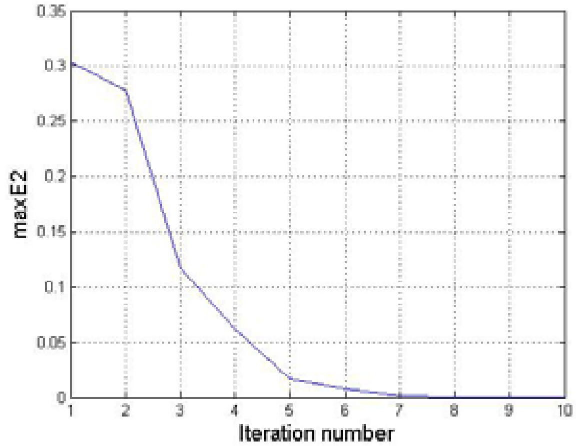

Figure 5. The sup-norm of tracking error $e_2(t)$ in each iteration

Now, we consider the same singular system where we apply open-closed-loop *P-PD*$^{\alpha}$-type iterative learning algorithm (64). In the simulation, we select the following gain matrices:

$$\Gamma_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} 0.95 & 0.4 \\ 0 & 0.95 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \quad (88)$$

To determine values of the gain matrices, it is necessary to satisfy the convergence condition of Theorem 2 and make a comprehensive consideration of the convergence speed. It is easy to show that the pair (E; A) is regular and $\left\| \left[I - \Gamma_2 C\overline{B} \right] \right\| = 0.7287 < 1$.

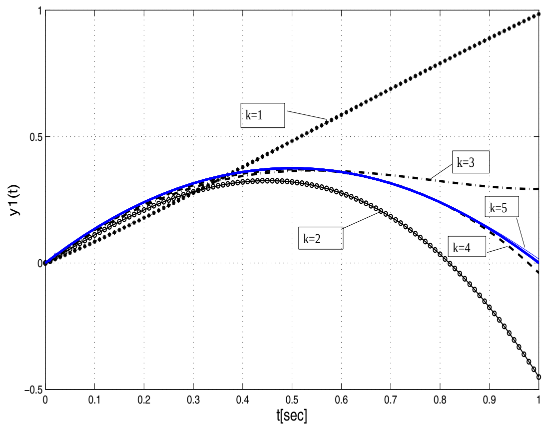

Figure 6. The tracking performance of the system output ($y_1(t)$ **:** solid line, $y_{d1}(t)$**:** bold line)

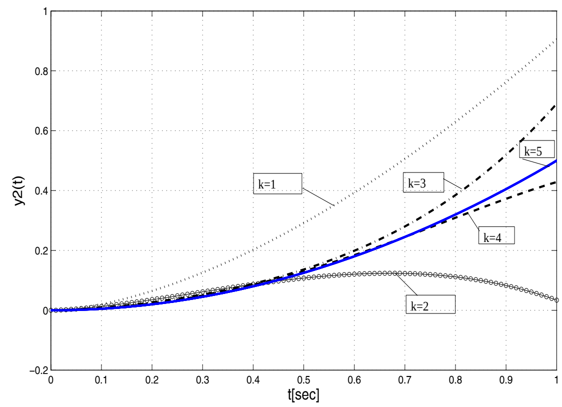

Figure 7. The tracking performance of the system output ($y_2(t)$ **:** solid line, $y_{d2}(t)$**:** bold line)

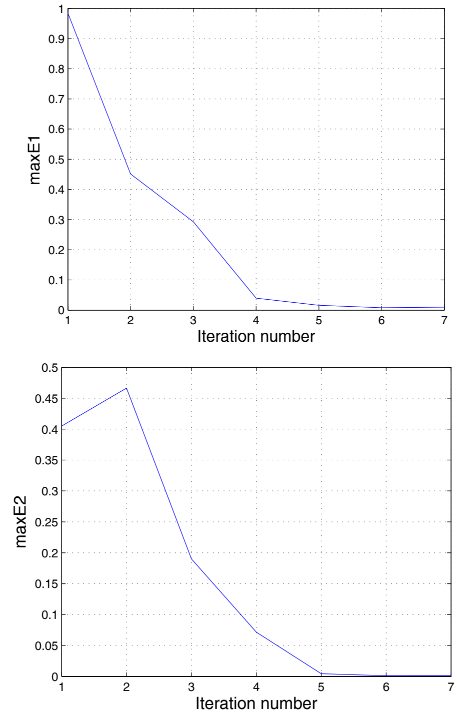

Figure 8. The sup-norms of tracking errors $e_1(t)$ and $e_2(t)$ in each iteration

Here the ILC rule (64) is used, where Figures 6, 7 show the tracking performance of the ILC system outputs over the interval $t \in [0,1]$. Also, we can find (see Fig.8) that the proposed requirement for the tracking performance is achieved at the fifth iteration.

Compared with the results shown in Figures 4 and 5, the ILC tracking errors presented in Fig.8 are bounded to a lower level. Beside using suggested open-closed ILC control as well as learning gains matrices, one may improve the speed convergence and transient behavior of the proposed ILC fractional order systems.

Conclusions

In this paper, a fractional order (P)-PDa type of ILC is proposed for a given class of fractional order singular systems and, using simulations, the effectiveness of the proposed ILC controller was investigated. Particularly, we considered two cases of ILC: closed-loop PD^{α} type of ILC as well as open-closed loop (P)-PD^{α} type of ILC. Sufficient conditions for the convergence in the time domain of a proposed ILC were given by the corresponding theorems and proved.

Finally, improved ILC performances by including (open)-closed ILC controller are illustrated by numerical simulations.

Acknowledgement

This work is supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the projects TR 35006, III 41006 (M.L) and TR 33047 (P.M) and partially supported by the Serbia-China bilateral project under the number 3-12.

Literature

[1] MOORE.K.L.: *Iterative learning control for deterministic systems, Advances in Industrial Control*, Springer-Verlag, London, 1993.

[2] BIEN,Z., XU,J.: *Iterative Learning Control: Analysis*, Design, Integration and Applications, Kluwer Academic Publishers, 1989.

[3] AHN,H.S., MOORE,K., CHEN,Y.: *Iterative learning control robustness and monotonic convergence for interval systems*, Springer-Verlag London Limited, London, 2007.

[4] UCHIYAMA,M.: *Formulation of high-speed motion pattern of a mechanical arm by trial* (in Japanese), Trans. SICE (Soc. Instrum. Control Eng.) 1978, 14, pp.706–712.

[5] ARIMOTO,S., KAWAMURA,S., MIYAZAKI,F.: *Bettering operation of robots by learning*, J. Rob. Syst., 1984, 2(1), pp.123-140.

[6] KUC,T.Y., NAM,K., LEE,J.S.: *An iterative learning control of robot manipulators*, IEEE Trans. on Robot. Auton., 1991, 7(6), pp.835-842.

[7] ROSENBROCK,H H.: *Structural properties of linear dynamical systems*, Int. J. Control, 1974, 20(2), pp.191-202.

[8] MÜLLER,P.C.: *Stability of Linear Mechanical Systems with Holonomic Constraints*, Appl. Mech. Rev., 1993, 46(11), pp.160-164.

[9] BUZUROVIC,I.: *Dynamic model of medical robot represented as descriptor system*, Int. J. Inf. Syst. Sci., 2007, 2(2), pp.316-333.

[10] FANG,C.H., CHANG,F.R., *Analysis of stability robustness for generalized state-space systems with structured perturbations*, Syst. Control Lett., 1993, 21(2), pp.109−114.

[11] DAI,L.: *Singular Control Systems,118 Lecture Notes in Control and Information Sciences*. New York: Springer-Verlag, 1989.

[12] XIE,S.L,. XIE,Z.D, LIU,Y.Q.: *Learning Control Algorithm for State Tracking of Singular Systems with Delay*, Syst. Eng. Electron., 1999, 21(5), pp.10-16.

[13] PIAO,F.X., ZHANG,Q.L., WANG,Z.F.: *Iterative Learning Control for a Class of Singular Systems, Acta Autom*. Sin., 2007, 33(6), pp.658-659.

[14] PODLUBNY,I.: *Fractional Differential Equations*. Academic Press, San Diego,1999.

[15] MONJE,C.A., CHEN,Y.Q., VINAGRE,B.M., XUE,D.Y., FELIU-BATLLE,V.: *Fractional-order Systems and Controls:Fundamentals and Applications*. London:Springer-Verlag, 2010.

[16] MITKOWSKI,S.W.: *Fractional-Order Models of the Ultracapacitors*. In Mitkowski W, Kacprzyk J and Baranowsk J (eds) Advances in the Theory and Applications of Non-integer Order Systems. Switzerland: Springer, 2013, pp.281-294.

[17] KACZOREK,T.: *Singular fractional continuous-time and discrete-time linear systems*. Acta Mech. Autom., 2013, (7)1, pp.26-33.

[18] CAJIĆ,M., LAZAREVIĆ,P.M.: *Fractional-order spring/spring-pot/actuator element in a multibody system:Application of an expansion formula*, Mech. Res. Commun., 2014, 62, pp.44–56.

[19] PRELI CHEN,Y.Q, MOORE,K.L.,: *On D type iterative learning control*, In Proc. 40th IEEE Conference on Decision and Control, Orlando, FL USA, 2001, pp.4451-4456.

[20] LAZAREVIĆ,P.M.: *PD^{α}-type iterative learning control for fractional LTI system-16th Int.Cong*. CHISA 2004, G2.9, Praha,Czech Republic, August 2004, pp.22-26.

[21] LAZAREVIĆ,P.M., MANDIĆ,P.: *Feedback-feedforward iterative learning control for fractional-order uncertain time delay system–PD alpha type*, ICFDA'14 Catania, 23-25. June 2014, IE86, 2014.

[22] LI,Y., CHEN,Y.Q., AHN,H.S.: *Fractional-order iterative learning control for fractional-order linear systems*, Asian J. Control, 2011, 13 (1), pp.54- 63.

[23] LAZAREVIĆ,P.M: *Iterative Learning Control of Integer and Noninteger Order: an Overview*, Scientific Technical Review, ISSN 1820-0206, 2014, Vol.64, No.1, pp.35-47.

[24] LAZAREVIĆ,P.M: *Some New Results on Iterative Learning Control of Noninteger Order*, Scientific Technical Review, ISSN 1820-0206, 2014, Vol.64, No.2, pp.33-41.

[25] LAZAREVIĆ,P.M, TZEKIS,P.: *Iterative learning feeback control for singular fractional order system-PDa type*, IcETRAN 2014, Vrnjačka Banja, Serbia, 2-5. June, 2014, ISBN 978-86-80509-70-9, pp.AUI2.3, 1-6.

[26] LAZAREVIĆ,P.M., TZEKIS,P.: *Robust second-order PD type iterative learning control for a class of uncertain fractional-order singular systems*, Journal of Vibration and Control, 2016, Vol.22, No.8, pp.2004–2018.

[27] CHEN,Y., GONG,Z., WEN,C.: *Analysis of a high-order iterative learning control algorithm for uncertain nonlinear systems with state delays*. Automatica,1998, 34, pp.345–353.

[28] XIAONAS., LEIPO.L., ZHEN,W.: *Stabilization of Singular Fractional-Order Systems·A Linear Matrix Inequality Approach*, Proc.of the IEEE International Conference on Automation and Logistics Zhengzhou, China, August 2012, pp.19-23.

[29] YU,Y., JIAO,Z., SUN,C.: *Suffcient and Necessary Condition of Admissibility for Fractional-order Singular System*, Acta automatica sinica, December 2013, Vol.39, No.12

[30] DIETHEL,M.K., FORD,NJ.: *Analysis of fractional differential equations, Journal of Mathematical Analysis and Applications*, 2002, Vol.265, No.2, pp.229–248.

[31] ORTIGUEIRA,M.,COITO,F.: *The initial conditions of Riemann-Liouville and Caputo derivatives*, in Proc. 6th EUROMACH Nonlinear Dynamics Conference, (ENOC 2008), 2008.

[32] LI,Y., CHEN,Y.Q., AHN,H.S.: *Fractional-order iterative learning control for fractional-order linear systems*, Asian Journal of Control, January 2011, Vol.13, No.1, pp.54- 63.

Received: 22.06.2016.
Accepted:30.08.2016.

Iterativno upravljanje učenjem necelog reda za singularni sistem necelog reda: (P)-PD^{α} tip

Iterativno upravljanje putem učenja (ILC) predstavlja jedno od važnih oblasti u teoriji upravljanja i pogodno je za upravljanje šire klase mehatroničkih sistema, a posebno su pogodni za upravljanje kretanjem robotskih sistema. Ovaj se rad bavi problemom primene frakcionog reda ILC upravljanja za singularne sisteme frakcionog reda. Posebno, ovde se proučavaju singularni sistemi necelog reda u prostoru pseduo-stanja. U povratnoj sprezi frakcionog reda PD^{α} tip ILC upravljanje za singularni sistem frakcionog reda je istraživano. Takođe, frakcionog reda P-PD^{α} tip ILC upravljanje u direktnoj-povratnoj sprezi je razmatrano. Dovoljni uslovi za konvergenciju u vremenskom domenu predloženih šema ILC upravljanja su data odgovarajućim teoremama i koja su dokazana. Konačno, numeričke simualcije na primer pokazuje izvodljivost i efikasnost predloženog pristupa.

Ključne reči: **teorija upravljanja, iterativno upravljanje, upravljanje učenjem, necelobrojni red, singularni sistem, konvergencija metode, robotski sistem.**

Итерационное управление обучением дробного порядка для особой сингулярной системы дробного порядка: (P)-PD$^{\alpha}$ тип

Итерационное управление с помощью процесса обучения (ILC) является одним из важных направлений в теории управления и является идеальным решением для управления более широким классом мехатронных систем и особенно хорошо подходит для управления движением робототехнических систем. Эта статья имеет дело с применением систем управления ILC дробного порядка для сингулярных систем дробного порядка. В частности, мы изучаем сингулярные системы дробного порядка в пространстве псевдосостояния. Здесь исследуется тип ILC управления с замкнутым циклом дробной степени порядка PD$^{\alpha}$ в сингулярной системе дробного порядка. Кроме того, обсуждается и тип ILC управления дробного порядка P-PD$^{\alpha}$ в прямой обратной связи. Достаточные условия сходимости во временной области предлагаемых схем ILC управления приведены в соответствующих теоремах и доказаны. Наконец, численные моделирования, например, показывают целесообразность и эффективность предлагаемого подхода.

Ключевые слова: **теория управления, итерационное управление, управление обучением, дробный порядок, сингулярная система, методы конвергенции, роботизированная система.**

Contrôle itératif par l'étude de l'ordre fractionnel pour le système singulier de l'ordre fractionnel: (P)-PD^{α} type

Le contrôle itératif par l'étude (ILC) représente un domaine important dans la théorie de contrôle et il est convenable pour le contrôle d'une large classe des systèmes mécatroniques en particulier pour le contrôle du mouvement chez les systèmes robotiques. Ce papier s'occupe du problème d'application de l'ordre fractionnel de contrôle ILC pour les systèmes singuliers de l'ordre fractionnel. On étudie ici spécialement les systèmes singuliers de l'ordre fractionnel dans l'espace de pseudo état. Dans les réactions de l'ordre fractionnel PDα le type de contrôle ILC pour le système singulier de l'ordre fractionnel a été examiné. On a considéré aussi le contrôle dans les réactions directes de l'ordre fractionnel du type P-PDα . Les conditions suffisantes pour la convergence dans le domaine temporel des schémas proposés du contrôle ILC sont données par les théorèmes correspondants et prouvées. Enfin les simulations numériques démontrent la faisabilité et l'efficacité de l'approche proposée.

Mots clés: **théorie de contrôle, contrôle itératif, contrôle par l'étude, ordre fractionnel, système singulier, convergence de méthode,système robotique.**