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Elements of Dynamic Parameters Modification and Sensitivity

Structural modifications can cause changes in the matrices of some elements. Depending on the type of structure, or a desired change, a group of elements can be modified. This paper analyzes the effects of small and large-scale modifications of some groups of elements on structural eigenvalues and frequencies. The process of analysis is done using a computer program, based on using of finite element methods and the implementation of structure energy distributions.

Keywords: *Kinetic and potential energy, sensitivity, eigenvalues, eigenvectors, dynamic modification.*

1. INTRODUCTION

Specific computations allow for a more profound knowledge of the structure's behaviour [1]. The elements of modification (distribution of membrane and bending stresses [2], deformation energy and kinetic and potential energy per structural component) contribute to very efficient identification of the structure's behaviour. They define the modifications that have to be performed on the structure in order to improve its behaviour during service life. The modification problem, in terms of mathematical form, implies minimizing the structure's function (weight, deformation energy, stress level, eigenvalues) as a function of id parameters of modification v_j (coordinates of the points, cross-sectional area, thickness) with constraints $g(v_j)$ (constraints of stress values, displacements, length, surface, volume, frequency). In a general case, the functions considered are nonlinearly implicit. The basis for optimization is represented by the analysis of structural function sensitivity [4-6].

2. KINETIC AND POTENTIAL ENERGY

In conservative systems the total mechanical energy is constant and differential equations of motion can be set applying the principle of the conservation of energy [3]. The energy of a mechanical system is divided into potential and kinetic energy. Kinetic energy Ek is contained in the mass by means of its velocity and potential energy Ep is contained in the form of stress energy of the deformation work done. The total energy for free undamped oscillations remains constant, its velocity of change equals zero, which is illustrated by the equations as follows:

$$E_k + E_p = const\ ,\ \frac{d}{dt}\left(E_k + E_p\right) = 0$$

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If eigenvalues of the system are the only subject of interest to consider, from the principle of the conservation of energy, the following relation can be written:

$$E_{k1} + E_{p1} = E_{k2} + E_{p2}$$

here indices 1 and 2 represent two instants of time. Let 1 be an instant of time when mass passes through the position of static equilibrium, where $E_{p1}=0$ (potential energy referential level) can be chosen to hold. Also, let 2 be the time with a corresponding largest displacement of mass. In this position, the mass velocity equals zero, therefore $E_{k2}=0$. It can be then written that:

$$E_{k1} + 0 = 0 + E_{p2}\ .$$

However, if the system is subject to harmonic motion, then E_{k1} and E_{p2} are maximum values and there holds:

$$E_{k,\max} = E_{p,\max}$$

The above described procedure can be used as an introduction to essential understanding of dynamic modification methodology developed in this paper.

2.1 Distribution of potential and kinetic energy on the main forms of oscillation

The matrix form of differential equations of motion of a mechanical system represented by the finite element model for the case when there are no external forces acting is as follows:

$$[M]\left\{\ddot{\underset{\sim}{Q}}(t)\right\} + [K]\left\{\underset{\sim}{Q}(t)\right\} = \{0\} \qquad (1)$$

Where $[K]$ and $[M]$ are the matrices of stiffness and masses, respectively

The eigensolution of the above differential equation for the i-th mode is:

$$[K]\left\{\underline{Q}_i\right\} - \lambda_i[M]\left\{\underline{Q}_i\right\} = \{0\} \qquad (2)$$

where λ_i is the *i-th* eigenvalue, and $\{Q_i\}$ is the *i-th* eigenvector of the structure.

If the above equation is multiplied on the left side by the transposed value of the i-th eigenvector and is divided by 2, one obtains the potential and kinetic energy balance equation for the structure on the main forms of oscillation:

$$\frac{1}{2}\{Q_i\}^T [K]\{Q_i\} = \frac{1}{2}\lambda_i \{Q_i\}^T [M]\{Q_i\} \tag{3}$$

So, potential energy for the structure on the r-th main form of oscillation, considering the above equation, can be written in the form:

$$E_{p,r} = \frac{1}{2}\{Q_r\}^T [K]\{Q_r\} \tag{4}$$

whereas, in that case, kinetic energy is:

$$E_{k,r} = \frac{1}{2}\lambda_r \{Q_r\}^T [M]\{Q_r\} \tag{5}$$

Considering Eq. (3) as well as an extract presented above from the theory, there follows the principle of the conservation of total energy on the main forms of oscillation:

$$E_{p,r} = E_{k,r} = E_r \tag{6}$$

If the structure is discretized to N finite elements, then kinetic and potential energy of the entire structure can be represented as an algebraic sum of energies of all elements as follows. Let be

$\{q_r^s\}_e$ - the corresponding *r*-th eigenvector of the *e*-th element with *s* degrees of freedom,

$(e_{p,r})_e = \frac{1}{2}\{q_r^s\}_e^T [k]_e \{q_r^s\}_e$ - the potential energy of the *e*-th element on the *r*-th main form of oscillation,

$(e_{k,r})_e = \frac{1}{2}\omega_r^2 \{q_r^s\}_e^T [m]_e \{q_r^s\}_e$ - the kinetic energy of the *e-th* element on the *r-th* main form of oscillation, then the structural total kinetic energy on the *r-th* main form of oscillation can be represented by the sum:

$$E_{k,r} = \sum_{e=1}^{N}(e_{k,r})_e = \frac{1}{2}\sum_{e=1}^{N}\omega_r^2 \{q_r^s\}_e^T [m]_e \{q_r^s\}_e \tag{7}$$

Analogously, the structure's potential energy on the *r-th* main form of oscillation can be also represented by the sum:

$$E_{p,r} = \sum_{e=1}^{N}(e_{p,r})_e = \frac{1}{2}\sum_{e=1}^{N}\{q_r^s\}_e^T [k]_e \{q_r^s\}_e \tag{8}$$

If structural *N* finite elements are divided into *P* characteristic subgroups (subgroups can be formed according to the type of finite elements lines, surfaces, volumes), or according to unit groups in complex machine systems, then the total kinetic energy and the total potential energy can be represented as a sum per subgroup:

$$E_{p,r} = \sum_{l=1}^{P}(E_{p,r})_l, \quad E_{k,r} = \sum_{l=1}^{P}(E_{k,r})_l \tag{9}$$

Where $(E_{k,r})_l$ and $(E_{p,r})_l$ are kinetic and potential energy of the *l-th* subgroup of elements, respectively. The distribution of kinetic and potential energy per structural subgroup on the *r-th* main form of oscillation can be expressed in per cent, respectively:

$$\eta_{k,r}[\%] = \frac{(E_{k,r})_l}{E_{k,r}} \cdot 100[\%],$$

$$\eta_{p,r}[\%] = \frac{(E_{p,r})_l}{E_{p,r}} \cdot 100[\%] \tag{10}$$

So, based on the distribution of kinetic and potential energy expressed in per cent, a group of elements suitable for dynamic analysis can be roughly selected, to be discussed in detail below. And before that, a simple example will demonstrate how to calculate kinetic and potential energy per element, for the first three forms of oscillation.

2.2 Structural modifications

Structural modification can cause changes in the matrices of some elements. Depending on the type of structure, or desired changes, one element or a group of elements can be modified.

If more than one group of elements is modified, then the 'perturbed' matrix of the system stiffness equals:

$$[\Delta K]_{system} = \sum_{e=1}^{L}[\Delta k]_e, [\Delta M]_{system} = \sum_{e=1}^{L}[\Delta m]_e \tag{11}$$

where *L* is the number of modified elements. Furthermore, each growth matrix for individual elements can be represented as the function of matrices of the original system via the *coefficient of modification* (or as a sum of terms especially referring to bending, axial strain or torsion, if it is needed):

$$[\Delta k]_e = \alpha_e \cdot [k]_e, [\Delta m]_e = \beta_e \cdot [m]_e \tag{12}$$

These relations can be linear or nonlinear. For example, the effect of plate thickness on axial stiffness is linear, while the effect on bending stiffness is of the third order. These relations for mass modification are commonly linear. If necessary, the range of structural modifications can be sometimes also expressed by corresponding inequalities.

Example of the analysis of energies distribution for a simple structure composed of three articulated connecting rods

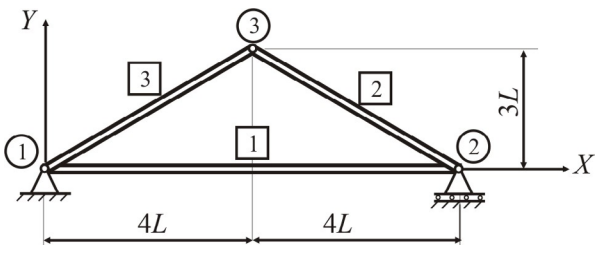


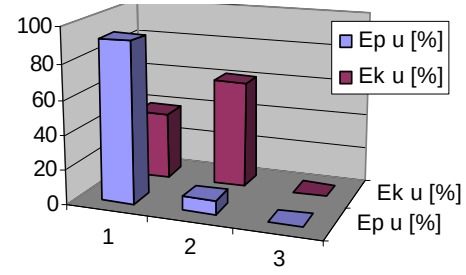
Figure 1.

Table 1.

	First form		Second form		Third form	
	E_p [J]	E_k [J]	E_p [J]	E_k [J]	E_p [J]	E_k [J]
Rod I	0.2982	0.1254	0	0	0.0935	0.4378
Rod II	0.0265	0.1992	0	0	1.0507	0.7062
Rod III	0	0	0.5123	0.5123	0	0

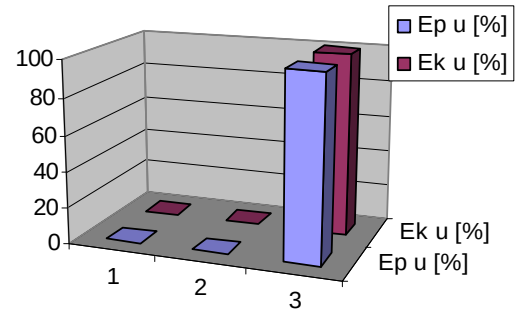
All terms for energy are multiplied by factor EA/L . The dimension for energy is [J]. Numbers in the table have the dimension [m^2]

Figure 1.



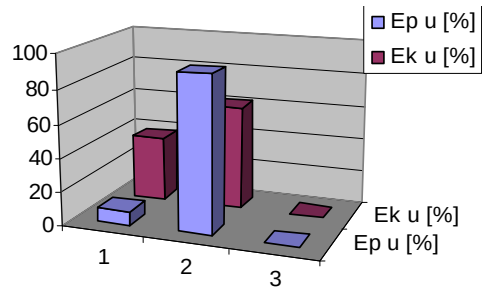
	1	2	3
Ep u [%]	91.8386	8.1614	0
Ek u [%]	38.6322	61.3678	0

Figure 2. Distribution of kinetic and potential energy per lattice rod on the 1st form of oscillation, in per cent.



	1	2	3
Ep u [%]	0	0	100
Ek u [%]	0	0	100

Figure 3. Distribution of kinetic and potential energy per lattice rod on the 2nd form of oscillation, in per cent.



	1	2	3
Ep u [%]	8.1716	91.8284	0
Ek u [%]	38.2692	61.7308	0

Figure 4. Distribution of kinetic and potential energy per lattice rod on the 3rd form of oscillation, in per cent.

2.3 Modified first-order equations for the system

If $[\Delta K]$ and $[\Delta M]$ are corresponding changes in the matrices of stiffness and masses, respectively, then the expression (2) can be also applied to a modified system, and so-called modified (In the literature, the term 'perturbation equation' is frequently used.) equation for the case of free oscillations reads $[K]'\left\{\underline{Q}_i\right\}' = \lambda_i'[M]'\left\{\underline{Q}_i\right\}'$, where it holds:

$$[K]=[K]'-[\Delta K],\ [M]=[M]'-[\Delta M],$$
$$\left\{\underline{Q}_i\right\}=\left\{\underline{Q}_i\right\}'-\left\{\Delta\underline{Q}_i\right\},\ \lambda_i=\lambda_i'-\Delta\lambda_i$$

and where $\Delta\lambda_i$ and $\left\{\Delta\underline{Q}\right\}_i$ are changes of eigenvalues and eigenvectors, respectively. Now, the equation of an original, unmodified system $[K]\left\{\underline{Q}_i\right\}=\lambda_i[M]\left\{\underline{Q}_i\right\}$ can be written as follows:

$$\begin{aligned}&\left([K]'-[\Delta K]\right)\left(\left\{\underline{Q}_i\right\}'-\left\{\Delta\underline{Q}_i\right\}\right)=\\&\left(\lambda_i'-\Delta\lambda_i\right)\left([M]'-[\Delta M]\right)\left(\left\{\underline{Q}_i\right\}'-\left\{\Delta\underline{Q}_i\right\}\right)\end{aligned}\tag{13}$$

The above equation is the third-order equation for its 'modified' terms, and if the potential and kinetic energy balance equation for the structure [3] is written in a 'perturbed' form, the fourth-degree equation is obtained:

$$\begin{aligned}&\left(\left\{\underline{Q}_i\right\}+\left\{\Delta\underline{Q}_i\right\}\right)^T\left([K]+[\Delta K]\right)\left(\left\{\underline{Q}_i\right\}+\left\{\Delta\underline{Q}_i\right\}\right)=\\&\left(\lambda_i+\Delta\lambda_i\right)\left(\left\{\underline{Q}_i\right\}+\left\{\Delta\underline{Q}_i\right\}\right)^T\cdot\\&\cdot\left([M]+[\Delta M]\right)\left(\left\{\underline{Q}_i\right\}+\left\{\Delta\underline{Q}_i\right\}\right)\end{aligned}\tag{14}$$

The above equations indicate that there are two approaches. First, these equations can be used to obtain modification of frequencies and modal forms as a result of modifications in the system stiffness and mass. This approach is referred to as 'advanced modification'. Second, these equations can be used for the inverse case, how to determine modifications in the system stiffness and mass when there are desired modifications of frequencies and modal forms. The approach is called 'inverse modification'. Assuming that structural modifications are small, it can be expected that modifications of the vectors of eigenvalues and eigenvectors will be also small. So, the higher-order terms in the following expression can be neglected [4] (Note that the 'order', as above used, refers to modified quantities and does not represent the terms of the modification parameters order. For example, [ΔK] can be of the third order when plate thickness is modification parameter, while [ΔM] for that case is of the first order. So, it is not quite clear when higher-order terms can be 'painlessly' neglected compared to first-order terms.):

$$\left([K]'-[\Delta K]\right)\left(\left\{\underline{Q}_i\right\}'-\left\{\Delta\underline{Q}_i\right\}\right)=$$

$$
\begin{aligned}
&(\lambda_i'-\Delta\lambda_i)([M]'-[\Delta M])\left(\left\{\underline{Q}_i\right\}'-\left\{\Delta\underline{Q}_i\right\}\right)\\
&[K]'\left\{\underline{Q}_i\right\}'-[K]'\left\{\Delta\underline{Q}_i\right\}-[\Delta K]\left\{\underline{Q}_i\right\}'+[\Delta K]\left\{\Delta\underline{Q}_i\right\}=\\
&\lambda_i'[M]'\left\{\underline{Q}_i\right\}'-\lambda_i'[M]'\left\{\Delta\underline{Q}_i\right\}-\\
&-\lambda_i'[\Delta M]\left\{\underline{Q}_i\right\}'-\Delta\lambda_i'[M]'\left\{\underline{Q}_i\right\}'+\ldots
\end{aligned}
\tag{14a}
$$

Considering that equations

$$
[K]'\left\{\underline{Q}_i\right\}'=\lambda_i'[M]'\left\{\underline{Q}_i\right\}',\ [K]'\left\{\Delta\underline{Q}_i\right\}=\lambda_i'[M]'\left\{\Delta\underline{Q}_i\right\}
$$

hold, when only first-order terms are retained, Eq (14) becomes a modified first-order equation:

$$
[\Delta K]\left\{\underline{Q}_i\right\}'\approx\lambda_i'[\Delta M]\left\{\underline{Q}_i\right\}'+\Delta\lambda_i[M]'\left\{\underline{Q}_i\right\}' \tag{15}
$$

If the above equation is multiplied on the left side by one half of the transposed value of the *i-th* eigenvector, there follows the expression:

$$
\begin{aligned}
&\frac{1}{2}\left\{\underline{Q}_i\right\}'^T[\Delta K]\left\{\underline{Q}_i\right\}'\approx\frac{1}{2}\lambda_i'\left\{\underline{Q}_i\right\}'^T[\Delta M]\left\{\underline{Q}_i\right\}'+\\
&+\frac{1}{2}\Delta\lambda_i\left\{\underline{Q}_i\right\}'^T[M]'\left\{\underline{Q}_i\right\}'
\end{aligned}
$$

wherefrom a change of the *i-th* eigenvalue can be expressed for the system modification, which is the goal of this procedure:

$$
\frac{\Delta\lambda_i}{\lambda_i'}=\frac{\frac{1}{2}\left\{\underline{Q}_i\right\}'^T[\Delta K]\left\{\underline{Q}_i\right\}'-\frac{1}{2}\lambda_i'\left\{\underline{Q}_i\right\}'^T[\Delta M]\left\{\underline{Q}_i\right\}'}{\frac{1}{2}\lambda_i'\left\{\underline{Q}_i\right\}'^T[M]'\left\{\underline{Q}_i\right\}'} \tag{16}
$$

The above expression can be considered *a basic expression for structural reanalysis* to improve dynamic characteristics. The expression in the numerator represents the difference in potential energy growth and kinetic energy growth between modified and unmodified states. Since the *i-th* eigenvalue growth is directly proportional to that difference, each term in the nominator is of vital interest for the analysis to be done in detail in considerations below. Another important question is raised with reference to the above expression. Namely, the notation ", ', " is used to denote the corresponding quantities related to the modified state. Frequently, in the modification process, due to the bulky nature of some problem it is impossible to readily arrive at those quantities. If modifications are small, which is a condition for the accuracy of derived expressions, the expression with quantities figuring in the unmodified system can be used quite reliably.

$$
\frac{\Delta\lambda}{\lambda}=\frac{\frac{1}{2}\left\{\underline{Q}\right\}[\Delta K]\left\{\underline{Q}\right\}-\frac{1}{2}\lambda\left\{\underline{Q}\right\}[\Delta M]\left\{\underline{Q}\right\}}{\frac{1}{2}\lambda\left\{\underline{Q}\right\}[M]\left\{\underline{Q}\right\}} \tag{17}
$$

If the *r-th* form of oscillation is observed, the growths of potential and kinetic energy are determined by the expressions:

$$
\begin{aligned}
&\Delta E_{p,r}=\frac{1}{2}\left\{\underline{Q}_r\right\}'^T[\Delta K]\left\{\underline{Q}_r\right\}'=\\
&\frac{1}{2}\left\{\underline{Q}_r\right\}'^T[K]'\left\{\underline{Q}_r\right\}'-\frac{1}{2}\left\{\underline{Q}_r\right\}'^T[K]\left\{\underline{Q}_r\right\}'\approx\\
&\approx\frac{1}{2}\left\{\underline{Q}_r\right\}^T[\Delta K]\left\{\underline{Q}_r\right\}\\
&\Delta E_{k,r}=\frac{1}{2}\lambda_r'\left\{\underline{Q}_r\right\}'^T[\Delta M]\left\{\underline{Q}_r\right\}'=\\
&\frac{1}{2}\lambda_r'\left\{\underline{Q}_r\right\}'^T[M]'\left\{\underline{Q}_r\right\}'-\frac{1}{2}\lambda_r'\left\{\underline{Q}_r\right\}'^T[M]\left\{\underline{Q}_r\right\}'\approx\\
&\approx\frac{1}{2}\lambda_r\left\{\underline{Q}_r\right\}^T[\Delta M]\left\{\underline{Q}_r\right\}
\end{aligned}
$$

The expression in the nominator of Eq (17) is kinetic energy on some form of oscillation, and considering the expression (3), it simultaneously represents potential energy due to energy balance on the main forms of oscillation

$$
\begin{aligned}
&E_{p,r}=\frac{1}{2}\left\{\underline{Q}_r\right\}^T[K]\left\{\underline{Q}_r\right\},\\
&E_{k,r}=\frac{1}{2}\lambda_r\left\{\underline{Q}_r\right\}^T[M]\left\{\underline{Q}_r\right\},\ E_{p,r}=E_{k,r}=E_r
\end{aligned}
\tag{18}
$$

If modification is performed on the *e-th* finite element, the matrices of masses and stiffness of that finite element become:

$$
\begin{aligned}
&\left[k\right]_e'=\left[k\right]_e+\left[\Delta k\right]_e=\left[k\right]_e+\alpha_e\left[k\right]_e,\\
&\left[m\right]_e'=\left[m\right]_e+\left[\Delta m\right]_e=\left[m\right]_e+\beta_e\left[m\right]_e,
\end{aligned}
\tag{19}
$$

where α_e and β_e are the quantities defining the modification of the *e-th* element and are called the *coefficients of modification.* For that case, in the matrices of growth of stiffness and masses matrices all terms equal zero, except for those corresponding to the *e-th* finite element, so that the nominator of Eq. (17) for the *r-th* form of oscillation becomes:

$$
\begin{aligned}
&\frac{1}{2}\left\{\underline{Q}_r\right\}^T\left[\Delta K\right]\left\{\underline{Q}_r\right\}-\frac{1}{2}\lambda_r\left\{\underline{Q}_r\right\}^T\left[\Delta M\right]\left\{\underline{Q}_r\right\}=\\
&\frac{1}{2}\alpha_e\left\{q_r^s\right\}_e^T\left[k\right]_e\left\{q_r^s\right\}_e-\frac{1}{2}\beta_e\lambda_r\left\{q_r^s\right\}_e^T\left[m\right]_e\left\{q_r^s\right\}_e=\\
&\frac{1}{2}\left(\alpha_e e_{p,r}-\beta_e e_{k,r}\right)
\end{aligned}
\tag{20}
$$

where:

$\left\{q_r^s\right\}_e$ - a corresponding *r-th* eigenvector of the *e-th* element with *s* degrees of freedom,

$e_{p,r}=\frac{1}{2}\left\{q_r^s\right\}_e^T\left[k\right]_e\left\{q_r^s\right\}_e$ - potential energy of the *e-th* element on the *r-th* main form of oscillation without structural modification,

$e_{k,r}=\frac{1}{2}\omega_r^2\left\{q_r^s\right\}_e^T\left[m\right]_e\left\{q_r^s\right\}_e$ - kinetic energy of the *e-th* element on the *r-th* main form of oscillation without structural modification.

After the analysis, the expression (17) can be written as follows:

$$\frac{\Delta\lambda_i}{\lambda_i}=\frac{\frac{1}{2}\left\{\underline{Q}_i\right\}^T[\Delta K]\left\{\underline{Q}_i\right\}-\frac{1}{2}\lambda_i\left\{\underline{Q}_i\right\}^T[\Delta M]\left\{\underline{Q}_i\right\}}{\frac{1}{2}\lambda_i\left\{\underline{Q}_i\right\}^T[M]\left\{\underline{Q}_i\right\}}=\frac{\alpha_e e_{p,r}-\beta_e e_{k,r}}{E_{k,r}} \quad (21)$$

The expression (20) indicates impact of some finite elements on the eigenvalue growth. If the distribution of energies per group of elements is expressed in per cent on each main form of oscillation, the information items needed for modification can be roughly obtained. The basic goal of dynamic modification is to increase eigenvalues and their mutual intervals.

2.3 Characteristics of modifications. Coefficients of modification

As above mentioned, and well-known, eigenvalue is equal to the square of the frequency ($\lambda = \omega 2$). Therefore, it can be claimed that eigenvalue is a 'simpler' variable than frequency. Modified eigenvalue can be expressed as follows:

$$\lambda+\Delta\lambda=\left(\omega+\Delta\omega\right)^2=\omega^2+2\omega\Delta\omega+\left(\Delta\omega\right)^2 \quad (22)$$

So,

$$\Delta\lambda=2\omega\Delta\omega+\left(\Delta\omega\right)^2 \quad (23)$$

If *relative modification relations* are incorporated in the following way $\psi=\frac{\Delta\lambda}{\lambda},\ \xi=\frac{\Delta\omega}{\omega}$ the above equation can be written in the form:

$$\psi\cdot\lambda=2\omega^2\cdot\xi+\omega^2\cdot\xi^2\Rightarrow\ \psi=2\cdot\xi+\xi^2$$

and for small modifications of frequencies there holds the approximate linearized form:

$$\psi\approx 2\cdot\xi$$

For example, if the increase of frequency is 10%, the following calculations hold:

$$\left.\begin{array}{ll}\omega'=\omega+\Delta\omega=1.1\omega & \xi=0.1\\ \lambda'=\lambda+\Delta\lambda=\left(\omega'\right)^2=1.21\lambda & \psi=0.21\end{array}\right\}\ \psi_{priblizno}=2\xi=0.2$$

However, if the increase of frequency is 30%, then it holds:

$$\left.\begin{array}{ll}\omega'=\omega+\Delta\omega=1.3\omega & \xi=0.3\\ \lambda'=\lambda+\Delta\lambda=\left(\omega'\right)^2=1.69\lambda & \psi=0.69\end{array}\right\}\ \psi_{priblizno}=2\xi=0.6$$

This difference between linearized and accurate values can cause troubles, and because of that nonlinear forms must be retained for larger modifications of frequency.

Some characteristic examples are used below to demonstrate the derivation of the *coefficients of modification.* The leading spindle, a line carrier, can be discretized in the shape of KE-beam with a ring cross-section, outside diameter D and inside diameter d. If structural modification is performed via diameter modification, then *relative modification relations* are given as follows:

$$\psi=\frac{\Delta D}{D},\ \ \xi=\frac{\Delta d}{d} \quad (24)$$

The matrices of stiffness and masses for finite element in the beam shape, as above mentioned, look like this:

$$\underset{4\times4}{\left[k_{xy}^{(e)}\right]}=\frac{EI_z}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix},$$

$$\underset{4\times4}{[m^{(e)}]}=\frac{\rho Al}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}.$$

Where $I_z=\iint_A y^2\cdot dA$ is axial moment of inertia of the cross-sectional area around the *z*-axis. For the case of a beam with a ring cross-section, and the ring cross—sectional area $A=\frac{D^2\pi}{4}\left[1-\left(\frac{d}{D}\right)^2\right]$.

So, the matrix of stiffness is the fourth-order function of the cross-sectional diameter, and the matrix of masses is the second-order function of the cross-sectional diameter, therefore, after applying the expression (19) the corresponding *coefficients of modification* for the case $\psi = \xi$ are:

$$\alpha_e=\left(1+\psi\right)^4-1,\ \beta_e=\left(1+\psi\right)^2-1$$

For the case of a beam element with a rectangular cross-section, whose dimensions are $b\times h$, axial moment of inertia for one of the main axes is $I_z=\frac{b^3h}{12}$. If structural modification is performed via modification of dimensions *b* and/or *h*, then *relative modification relations* are $\psi=\frac{\Delta b}{b}$ and $\xi=\frac{\Delta h}{h}$ and corresponding *coefficients of modification* α_e and β_e, for this case would be:

$$\alpha_e=\left(1+\psi\right)^3\left(1+\xi\right)-1,\ \beta_e=\left(1+\psi\right)\left(1+\xi\right)-1$$

Table 2.

I profile						
h [cm]	A [cm^2]	A_i/A_1	I_y [cm^4]	I_{yi}/I_{y1}	I_x [cm^4]	I_{xi}/I_{x1}
8.0	7.58	1.00	6.3	1.00	77.8	1.00
10.0	10.6	1.40	12.2	1.94	171	2.20
12.0	14.2	1.87	21.5	3.41	328	4.22
16	22.8	3.01	54.7	8.68	935	12.02
18	27.9	3.68	81.3	12.90	1450	18.64
20	33.5	4.42	117	18.57	2140	27.51
22	39.6	5.22	162	25.71	3060	39.33
24	46.1	6.08	221	35.08	4250	54.63
26	53.4	7.04	288	45.71	5740	73.78
28	61.1	8.06	364	57.78	7590	97.56
30	69.1	9.12	451	71.59	9800	125.96

In Tables 2-5, relative modification relations are indirectly represented by the axial moments of inertia - cross-sectional areas ratio for some characteristic profiles, as indicated by corresponding diagrams.

Table 3.

U profile						
h [cm]	A [cm^2]	A_i/A_1	I_y [cm^4]	I_{xi}/I_{x1}	I_x [cm^4]	I_{xi}/I_{x1}
6.5	9.03	1.00	14.1	1.00	57.5	1.00
8.0	11	1.22	19.4	1.38	106	1.84
10.0	13.5	1.50	29.3	2.08	206	3.58
12.0	17	1.88	43.2	3.06	364	6.33
14	20.4	2.26	62.7	4.45	605	10.52
16	24	2.66	85.3	6.05	925	16.09
18	28	3.10	114	8.09	1350	23.48
20	32	3.54	148	10.50	1910	33.22
22	37.4	4.14	197	13.97	2690	46.78
24	42.3	4.68	248	17.59	3600	62.61
26	48.3	5.35	317	22.48	4820	83.83
28	53.3	5.90	399	28.30	6280	109.22
30	58.8	6.51	495	35.11	8030	139.65

Table 4.

Boxy cross-section, thickness δ = 2 mm						
a,b [cm]	A [cm^2]	A_i/A_1	I_y [cm^4]	I_{xi}/I_{x1}	I_x [cm^4]	I_{xi}/I_{x1}
18	1.177	1.0000	0.404	1.0000	0.404	1.0000
20	1.337	1.1359	0.603	1.4926	0.603	1.4926
22	1.497	1.2719	0.857	2.1213	0.857	2.1213
25	1.737	1.4758	1.357	3.3589	1.357	3.3589
30	2.137	1.8156	2.558	6.3317	2.558	6.3317
35	2.537	2.1555	4.306	10.6584	4.306	10.6584
40	2.937	2.4953	6.701	16.5866	6.701	16.5866
50	3.737	3.1750	13.833	34.2401	13.833	34.2401
60	4.537	3.8547	24.754	61.2723	24.754	61.2723

Table 5.

Boxy cross-section, thickness δ = 3 mm						
a,b [cm]	A [cm^2]	A_i/A_1	I_y [cm^4]	I_{xi}/I_{x1}	I_x [cm^4]	I_{xi}/I_{x1}
40	4.208	1.0000	8.618	1.0000	8.618	1.0000
50	5.408	1.2852	18.510	2.1478	18.510	2.1478
60	6.608	1.5703	33.925	3.9365	33.925	3.9365
70	7.808	1.8555	56.065	6.5056	56.065	6.5056
80	9.008	2.1407	86.129	9.9941	86.129	9.9941
90	10.208	2.4259	125.317	14.5413	125.317	14.5413
100	11.408	2.7110	174.829	20.2865	174.829	20.2865
110	12.608	2.9962	235.865	27.3689	235.865	27.3689
130	15.008	3.5665	397.310	46.1023	397.310	46.1023

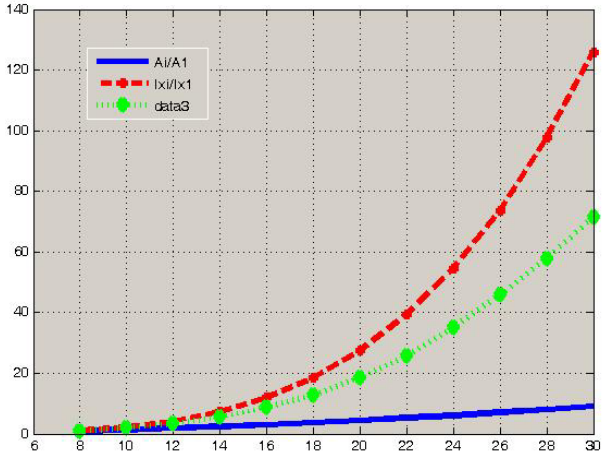


Figure 5. I profile.

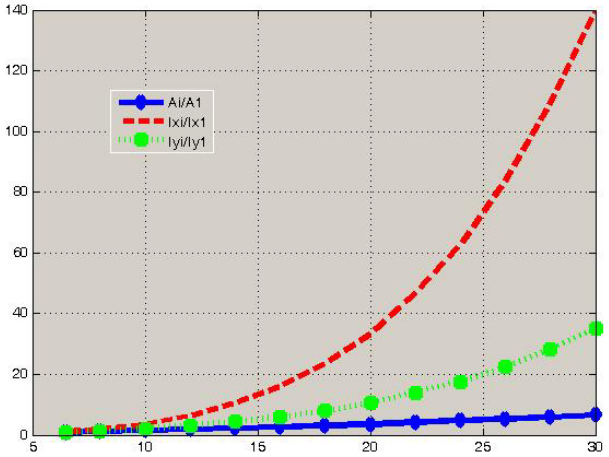


Figure 6. U profile.

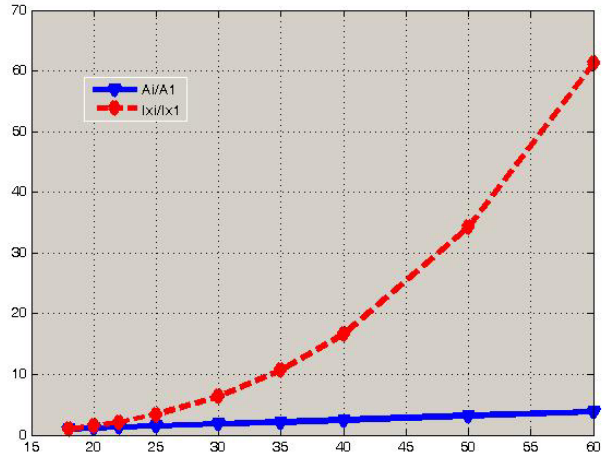


Figure 7. Profile of boxy cross-section, thickness δ=2mm.

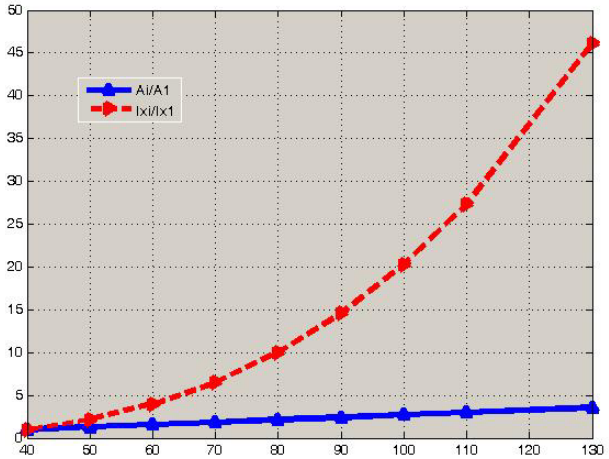


Figure 8. Profile of boxy cross-section, thickness δ=3mm.

3. CONCLUSION

The above diagrams allow for the following conclusions. Red line denotes relative relations between axial moments of inertia for one of the main central axes of the corresponding cross-section for an arbitrary cross-section and the corresponding referential one. Green line denotes those relations for another axis. Considering that beam bending stiffness is proportional to a corresponding axial moment of inertia, proper choice of the profile type can considerably increase

structural eigenfrequencies as early as in the stage of the original structure design. Namely, it is noticeable that U profile is more acceptable for one direction, while I profile is better for another direction. Boxy profiles are more acceptable because they have the same properties for both directions. This procedure can be applied to more complex structures. Establishing the most sensitive locations and making corresponding modifications of a set of elements, dynamic behavior of the entire structure can be improved.

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ЕЛЕМЕНТИ МОДИФИКАЦИЈЕ И ОСЕТЉИВОСТИ ДИНАМИЧКИХ ПАРАМЕТАРА

Наташа Тришовић, Ташко Манески, Зорана Голубовић, Стефан Сегла

Промене на конструкцијама могу изазвати промене у структурним матрицама неких елемената. У зависности од врсте конструкције, као и од жељених промена у њој, могуће је мењати већу или мању групу елемената. У овом раду се анализирају ефекти малих и великих модификација група елемената на сопствене вредности и фреквенције. Анализа се врши коришћењем компјутерских програма заснованим на методи коначних елемената и имплементацији дистрибуције потенцијалне и кинетичке енергије на главним облицима осциловања у посматраним елементима конструкције.