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Error bounds of Micchelli–Rivlin quadrature formula for analytic functions☆

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Abstract

We consider the well known Micchelli–Rivlin quadrature formula, of highest algebraic degree of precision, for the Fourier–Chebyshev coefficients. For analytic functions the remainder term of this quadrature formula can be represented as a contour integral with a complex kernel. We study the kernel, on elliptic contours with foci at the points ∓ 1 and a sum of semi-axes $\rho > 1$, for the quoted quadrature formula. Starting from the explicit expression of the kernel, we determine the locations on the ellipses where maximum modulus of the kernel is attained. So we derive effective L^{∞}-error bounds for this quadrature formula. Complex-variable methods are used to obtain expansions of the error in the Micchelli–Rivlin quadrature formula over the interval $[-1, 1]$. Finally, effective L^{1}-error bounds are also derived for this quadrature formula.

Keywords: Micchelli–Rivlin quadrature formula; Chebyshev weight function of the first kind; Error bound; Remainder term for analytic functions; Contour integral representation

1. Introduction

Micchelli and Rivlin [11] introduced a quadrature formula of highest algebraic degree of precision for the Fourier–Chebyshev coefficients $a_k(f)$, which is based on the divided differences of

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f' at the zeros of the Chebyshev polynomial T_n. Our aim in this paper is to analyze the remainder term and to obtain error bounds for this quadrature formula, when f is an analytic function.

Let $\{P_k\}_{k=0}^{\infty}$ be a system of orthonormal polynomials on $[a, b]$ with respect to a weight function ω, integrable, non-negative function on $[a, b]$ that vanishes only at isolated points. The approximation of f by partial sums $S_n(f)$ of its series expansions

$$f(x) = \sum_{k=0}^{\infty} a_k(f) P_k(x)$$

with respect to a given system of orthonormal polynomials $\{P_k\}_{k=0}^{\infty}$ is a classical way of recovery of functions. The numerical calculation of the coefficients $a_k(f)$, present in $S_n(f)$, is a main task in such a procedure. The computation of $a_k(f)$,

$$a_k(f) = \int_a^b P_k(t) f(t) \omega(t)\, dt,$$

requires the use of a quadrature formula. An application of the Gauss quadrature formula based on n values of the integrand $P_k(t) f(t)$ (with $k < 2n - 1$) will give the exact result for all polynomials of degree $2n - k - 1$. Is it possible to construct a formula based on n evaluations of f or its derivatives which gives the exact value of the coefficients $a_k(f)$ for polynomials f of higher degree? What is the highest degree of precision that can be attained by a formula based on n evaluations? Studying this question for the coefficients $a_k(f)$ of f with respect to the system of Chebyshev polynomials of the first kind $\{T_k\}_{k=0}^{\infty}$, orthogonal on $[-1, 1]$ with weight $\omega(t) = 1/\sqrt{1 - t^2}$,

$$T_k(t) = \cos(k \arccos t) = \frac{1}{2^{k-1}} (t - \xi_1) \cdots (t - \xi_k), \quad t \in (-1, 1).$$

Micchelli and Rivlin discovered in [11] the remarkable fact that the quadrature

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - t^2}} T_n(t) f(t)\, dt \approx \frac{\pi}{n 2^n} f'[\xi_1, \ldots, \xi_n] \tag{1.1}$$

is exact for all algebraic polynomials of degree $\leq 3n - 1$. Here, $g[x_1, \ldots, x_m]$ denotes the divided difference of g at the points $x_1, \ldots, x_m$, and thus formula (1.1) uses n function values of the derivative f', that is $f'(\xi_1), \ldots, f'(\xi_n)$. It is clear that there is no formula of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - t^2}} T_n(t) f(t)\, dt \approx \sum_{k=1}^{n} a_k f(x_k) + \sum_{k=1}^{n} b_k f'(x_k) \tag{1.2}$$

which is exact for all polynomials of degree $3n$. The polynomial $f(t) = T_n(t)(t - x_1)^2 \cdots (t - x_n)^2$ is a standard counterexample. Thus the Micchelli–Rivlin formula is of highest degree of precision among all formulas of the type (1.2). The question of uniqueness of this quadrature formula is reduced to the following problem which is also of independent interest: Prove that if Q is a polynomial of degree n with n zeros in $[-1, 1]$ and such that $|Q(\eta_j)| = 1$ at the extremal points $\eta_j = \cos(j\pi/n)$, $j = 0, 1, \ldots, n$, of the Chebyshev polynomial T_n, then $Q \equiv \pm T_n$. This property was proved by DeVore [3] and thus the uniqueness of Micchelli–Rivlin quadrature formula was settled (see [12]). For more details on this subject see also [1,2,14].

The paper is organized as follows. In Section 2 the remainder term of the Micchelli–Rivlin quadrature formula (1.1) for analytic functions is obtained. In Section 3, we shall derive effective L^{∞}-error bounds, i.e. (2.5) below, for the quadrature (1.1). In Section 4, complex-variable methods are used to obtain expansions of the error in the Micchelli–Rivlin quadrature formula over the interval $[-1, 1]$. Finally, effective L^1-error bounds, i.e. (2.8) below, for the quadrature (1.1) are also derived in Section 4. The results obtained here are an analogue of some results of Gautschi et al. [6] (see also [15,10]) and Hunter [9] concerning standard Gaussian quadratures.

2. The remainder term of Micchelli–Rivlin quadrature formulas for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathcal{D}$ its interior. Suppose that f is an analytic function in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$. If we know values of the function f and of the first derivative f' of f in the nodes $x_1, x_2, \ldots, x_n$ of the interval $[-1, 1]$, then the residue of Hermite interpolation of the function f can be written in the form (see Gončarov [7])

$$r_n(f; t) = f(t) - \sum_{\nu=1}^{n} \sum_{i=0}^{1} \ell_{i,\nu}(t) f^{(i)}(x_\nu) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\Omega_n(t)}{(z-t)\Omega_n(z)}\, dz, \tag{2.1}$$

where $\ell_{i,\nu}$ are the fundamental functions of Hermite interpolation and $\Omega_n(z) = \prod_{\nu=1}^{n}(z - x_\nu)^2$.

If we choose x_ν to be the zeros of the Chebyshev polynomial of the first kind, i.e., $x_\nu = \xi_\nu$, by multiplying (2.1) with $\omega(t)T_n(t)$, where $\omega(t) = 1/\sqrt{1-t^2}$, and integrating in t over $(-1, 1)$, we get a contour integral representation of the remainder term in (1.1), i.e., (1.2),

$$\begin{aligned} R_n(fT_n) = \int_{-1}^{1} r_n(f; t) T_n(t)\omega(t)\, dt &= \int_{-1}^{1} f(t)T_n(t)\omega(t)\, dt \\ &\quad - \sum_{\nu=1}^{n} \sum_{i=0}^{1} A_{i,\nu}(t) f^{(i)}(\xi_\nu), \end{aligned}$$

where $A_{i,\nu} = \int_{-1}^{1} \ell_{i,\nu}(t)T_n(t)\omega(t)dt$.

Finally, we get the representation

$$R_n(fT_n) = \frac{1}{2\pi i} \oint_{\Gamma} K_n(z) f(z) dz, \tag{2.2}$$

where the *kernel* is given by

$$K_n(z) = \frac{\rho_n(z)}{T_n^2(z)}, \tag{2.3}$$

and

$$\rho_n(z) = \int_{-1}^{1} \frac{\omega(t)}{z-t} T_n^3(t) dt. \tag{2.4}$$

Now we can obtain different kinds of estimates. The integral representation (2.2) leads to the error estimate

$$|R_n(fT_n)| \le \frac{\ell(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_n(z)| \right) \left(\max_{z\in\Gamma} |f(z)| \right), \tag{2.5}$$

where $\ell(\Gamma)$ is the length of the contour Γ.

Following [13, p. 119] (for $s = 0$) we get the L^r-error bound

$$\|R_n(fT_n)\| \le \frac{1}{2\pi}\|K_n\|_r\|f\|_{r'}, \quad 1 \le r \le +\infty,\ \frac{1}{r} + \frac{1}{r'} = 1, \tag{2.6}$$

where

$$\|f\|_r = \begin{cases} \left(\oint_\Gamma |f(z)|^r |dz|\right)^{1/r}, & 1 \le r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

In particular, we are interested in the L^∞- and L^1-error bounds, i.e.,

$$|R(fT_n)| \le \frac{1}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\oint_\Gamma |f(z)|\,|dz|\right), \tag{2.7}$$

and

$$|R_n(fT_n)| \le \frac{1}{2\pi}\left(\oint_\Gamma |K_n(z)|\,|dz|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.8}$$

respectively.

In this paper we take $\Gamma = \mathcal{E}_\rho$, where the ellipse $\mathcal{E}_\rho$ is given by

$$\mathcal{E}_\rho = \left\{z \in \mathbb{C}\,|\,z = \frac{1}{2}(u + u^{-1}), 0 \le \theta \le 2\pi\right\}, \quad u = \rho\, e^{i\theta}. \tag{2.9}$$

3. L^∞-error bounds based on the analysis of maximum of the modulus of the kernel of Micchelli–Rivlin quadrature formula

We have from (2.4), by substitution $t = \cos\theta$,

$$\rho_n(z) = \int_0^\pi \frac{[\cos n\theta]^3}{z - \cos\theta}d\theta = \frac{1}{4}\int_0^\pi \frac{1}{z-\cos\theta}\,(\cos 3n\theta + 3\cos n\theta)\,d\theta,$$

where we used the standard transformation for trigonometric function of triple argument. Now the kernel has the form

$$K_n(z) = \frac{\frac{1}{4}\int_0^\pi \frac{1}{z-\cos\theta}\,(\cos 3n\theta + 3\cos n\theta)\,d\theta}{[T_n(z)]^2},$$

i.e.

$$K_n(z) = \frac{\frac{1}{4}\frac{\pi}{\sqrt{z^2-1}}\left((z-\sqrt{z^2-1})^{3n} + 3(z-\sqrt{z^2-1})^n\right)}{[T_n(z)]^2},$$

where we used (see [6], for example)

$$\int_0^\pi \frac{\cos m\theta}{z-\cos\theta}d\theta = \frac{\pi}{\sqrt{z^2-1}}(z-\sqrt{z^2-1})^m, \quad m \in \mathbb{N}_0.$$

Substituting $z = (u + u^{-1})/2$, $u = z + \sqrt{z^2-1}$, using

$$T_n(z) = \left(u^n + u^{-n}\right)/2, \tag{3.1}$$

we get

$$K_n(z) = \frac{2\pi}{(u-u^{-1})u^n[u^n+u^{-n}]^2}\left(\frac{1}{u^{2n}}+3\right).$$

If we use the usual notation (see [6]) $a_j = a_j(\rho) = \frac{1}{2}(\rho^j+\rho^{-j})$, $j \in \mathbb{N}$ $(\rho > 1)$, when $u = \rho e^{i\theta}$, we have

$$|u^{-2n}+3|^2 = \rho^{-4n}+9+6\rho^{-2n}\cos 2n\theta,$$
$$|u-u^{-1}|^2 = 2(a_2-\cos 2\theta),$$
$$|u^n+u^{-n}|^2 = 2(a_{2n}+\cos 2n\theta),$$

and

$$|K_n(z)|^2 = \frac{\pi^2}{2\rho^{2n}}\cdot\frac{\rho^{-4n}+9+6\rho^{-2n}\cos 2n\theta}{(a_2-\cos 2\theta)(a_{2n}+\cos 2n\theta)^2}. \tag{3.2}$$

Now we can formulate the main statement.

Theorem 3.1. *For each fixed $\rho > 1$ there exists $n_0 = n_0(\rho)$ such that*

$$\max_{z\in\mathcal{E}_\rho}|K_n(z)| = \left|K_n\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|,$$

for each $n > n_0$.

Proof. The inequality

$$\rho^{-4n}+9+6\rho^{-2n}\cos 2n\theta \le \rho^{-4n}+9+6\rho^{-2n}$$

is obvious, so it is enough to prove that

$$\frac{1}{(a_2-\cos 2\theta)(a_{2n}+\cos 2n\theta)^2} \le \frac{1}{(a_2-1)(a_{2n}+1)^2},$$

i.e., $(a_2-\cos 2\theta)(a_{2n}+\cos 2n\theta)^2 \ge (a_2-1)(a_{2n}+1)^2$, for each $n \in \mathbb{N}_0$ greater than some $n_0 = n_0(\rho)$. First, let us transform the difference of the squares on the left and the right side:

$$\begin{aligned}(a_{2n}+1)^2-(a_{2n}+\cos 2n\theta)^2 &= (1-\cos 2n\theta)(2a_{2n}+1+\cos 2n\theta)\\ &= 2\sin^2 n\theta\cdot(2(a_{2n}+1)-2\sin^2 n\theta) = 2\sin^2 n\theta\cdot\Delta,\end{aligned}$$

where $\Delta = 2(a_{2n}+1)-2\sin^2 n\theta$.

The inequality we need now can be written in the following way

$$(a_2-1+2\sin^2\theta)((a_{2n}+1)^2-2\sin^2 n\theta\cdot\Delta) \ge (a_2-1)(a_{2n}+1)^2,$$

which reduces to $2\sin^2\theta(a_{2n}+1)^2-2\sin^2 n\theta(a_2-1+2\sin^2\theta)\Delta \ge 0$, i.e.

$$(a_{2n}+1)^2-\frac{\sin^2 n\theta}{\sin^2\theta}(a_2-1+2\sin^2\theta)\Delta \ge 0.$$

Since

$$\Delta \ge 2(a_{2n}+1)-2\sin^2 n\theta \ge 2(a_{2n}+1)-2 = 2a_{2n} > 0,$$

Table 3.1
The smallest possible value of n_0 for which both roots of $g(y)$ become greater than or equal to 1.

ρ	1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.1	1.15	1.2	1.25	1.3	1.4
n_0	160	81	54	41	33	28	24	21	19	17	12	9	8	7	5

using the well-known inequality $|\sin n\theta/\sin\theta| \le n$, we get

$$\begin{aligned}
&(a_{2n}+1)^2 - \frac{\sin^2 n\theta}{\sin^2\theta}(a_2 - 1 + 2\sin^2\theta)\Delta \\
&\quad = (a_{2n}+1)^2 - \frac{\sin^2 n\theta}{\sin^2\theta}(a_2-1)\Delta - 2\sin^2 n\theta\cdot\Delta \\
&\quad \ge (a_{2n}+1)^2 - n^2(a_2-1)\Delta - 2\sin^2 n\theta\cdot\Delta \\
&\quad = (a_{2n}+1)^2 - (2(a_{2n}+1) - 2\sin^2 n\theta)(n^2(a_2-1) + 2\sin^2 n\theta) \\
&\quad = (a_{2n}+1)^2 - 2((a_{2n}+1) - y)(n^2(a_2-1)+2y),
\end{aligned}$$

where $y = \sin^2 n\theta (\in [0,1])$. Hence all we need to show is positivity of the quadratic function

$$g(y) = (a_{2n}+1)^2 - 2((a_{2n}+1)-y)(n^2(a_2-1)+2y) = 4y^2 + By + C$$

on the interval $[0, 1]$, where $B = 2n^2(a_2-1) - 4(a_{2n}+1)$ and $C = (a_{2n}+1)^2 - 2n^2(a_{2n}+1)(a_2-1)$.

The discriminant is

$$D = B^2 - 4AC = 16n^2(a_{2n}+1)(a_2-1) + 4n^4(a_2-1)^2 > 0,$$

so the roots of g,

$$y_1 = \frac{-B-\sqrt{D}}{8}, \qquad y_2 = \frac{-B+\sqrt{D}}{8},$$

are real.

Since the leading coefficient of $g(y)$ is positive, g will be non-negative on $[0, 1]$ if and only if $y_1 > 1$ or $y_2 < 0$. As we can see, the first condition will be satisfied when n is enough large (ρ is fixed). Namely, it obtains the form $-B-8 > \sqrt{D}$, i.e.,

$$2(a_{2n}+1) - n^2(a_2-1) - 4 > \sqrt{4n^2(a_{2n}+1)(a_2-1) + n^4(a_2-1)^2}. \tag{3.3}$$

Let us note that the left-hand side of the last inequality will be positive for enough large n, because exponential function of n increases much faster than quadratic function of n. So we can square the both sides of the last inequality. Then the member which increases the fastest in the left-hand side is ρ^{4n}, and in the right-hand side it is $n^2\rho^{2n+2}$, so the left-hand side will indeed become larger for n ($n > 1$) enough large (ρ is fixed). The proof follows. □

From the practical point of view, we are interested in the value $n_0 = n_0(\rho)$, so that for each $n > n_0$ the function, obtained from (3.3),

$$F(n) \equiv F_\rho(n) = 2(a_{2n}+1) - n^2(a_2-1) - 4 - \sqrt{4n^2(a_{2n}+1)(a_2-1) + n^4(a_2-1)^2}$$

is positive. Some of the obtained values n_0 are displayed in Table 3.1.

From the previous proof it is clear that it would be correct if we would fix $n(\ge 2)$ and let ρ change. That means that the following statement holds.

Table 3.2
The smallest possible value of ρ_0 for which both roots of $g(y)$ become greater than or equal to 1.

n	2	5	10	20	30	50	100	200
ρ_0	2.5154	1.3841	1.1739	1.083	1.055	1.033	1.017	1.008

Theorem 3.2. *For each fixed $n > 1$ there exists $\rho_0 = \rho_0(n)$ such that*

$$\max_{z\in\mathcal{E}_\rho} |K_n(z)| = \left| K_n\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|,$$

for each $\rho > \rho_0$.

Some of the obtained values ρ_0 are displayed in Table 3.2.

For $n = 1$ a little-bit different statement holds.

Theorem 3.3. *For each $\rho > 1$, there holds*

$$\max_{z\in\mathcal{E}_\rho} |K_1(z)| = \left| K_1\left(\frac{i}{2}(\rho+\rho^{-1})\right)\right|.$$

Proof. We have to prove

$$\frac{\rho^{-4}+9+6\rho^{-2}\cos 2\theta}{(a_2-\cos 2\theta)(a_{2n}+\cos 2\theta)^2} \le \frac{\rho^{-4}+9-6\rho^{-2}}{(a_2+1)(a_2-1)^2},$$

which after substitution $x = \cos 2\theta$, since all expressions are evidently positive, becomes

$$(a_2-x)(a_2+x)^2(\rho^{-4}+9-6\rho^{-2}) - (a_2+1)(a_2-1)^2(\rho^{-4}+9+6\rho^{-2}x) \ge 0,$$

for $x \in [-1, 1]$. The last expression can be transformed to $(x+1)h_\rho(x)/(2\rho^8)$, where

$$\begin{aligned} h_\rho(x) &= -2\rho^4(1-3\rho^2)^2x^2 - \rho^2(3\rho^4-4\rho^2+1)^2x + 3\rho^{12} + 9\rho^{10} \\ &\quad - 13\rho^8 + 10\rho^6 - \rho^4 + \rho^2 - 1. \end{aligned}$$

Since $h_\rho(-1) = (\rho^2-1)^2(3\rho^8+24\rho^6-10\rho^4-1) > 0$ and $h_\rho(1) = (\rho^4-1)^2(3\rho^4-1) > 0$ for $\rho > 1$, we conclude that the quadratic function $h_\rho(x)$ is concave and positive on $[-1, 1]$. Therefore, $(x+1)h_\rho(x)/(2\rho^8) \ge 0$ for each $x \in [-1, 1]$, and the proof is completed. □

4. Error bounds based on an expansion of the remainder term and L^1-error bounds

If f is an analytic function in the interior of $\mathcal{E}_\rho$, it has the expansion

$$f(z) = {\sum_{k=0}^{\infty}}' \alpha_k T_k(z), \tag{4.1}$$

where α_k are given by

$$\alpha_k = \frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{-1/2} f(t)T_k(t)dt.$$

The series (4.1) converges for each z in the interior of $\mathcal{E}_\rho$. The prim in the corresponding sum denotes that the first term is taken with the factor $1/2$.

Lemma 4.1. *If $z \notin [-1, 1]$, then holds the following expansion*

$$\frac{1}{[T_n(z)]^2} = \sum_{k=0}^{+\infty} \beta_{n,k} u^{-2n-k}, \tag{4.2}$$

where

$$\beta_{n,k} = \begin{cases} 4(-1)^j (j+1), & k = 2jn, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.3}$$

Proof. We know that if $x \in \mathbb{C}$, $|x| < 1$, then

$$\frac{1}{(1-x)^{\nu+1}} = \sum_{k=\nu}^{+\infty} \binom{k}{\nu} x^{k-\nu} \quad (\nu = 0, 1, 2, \ldots). \tag{4.4}$$

Using this fact and (3.1), with $u = \rho e^{i\theta}$, $\rho > 1$, $z = (u + u^{-1})/2$, we get

$$\frac{1}{[T_n(z)]^2} = \left[\frac{1}{2}(u^n + u^{-n})\right]^{-2} = 4u^{-2n} \left(\frac{1}{1 + u^{-2n}}\right)^2 = 4 \sum_{j=0}^{+\infty} (-1)^j (j+1) u^{-2n-2nj},$$

which completes the proof. □

Lemma 4.2. *If $z \notin [-1, 1]$, ρ_n can be expanded as*

$$\rho_n(z) = \sum_{k=0}^{+\infty} \gamma_{n,k} u^{-n-k-1}, \tag{4.5}$$

where

$$\gamma_{n,k} = \begin{cases} \dfrac{3\pi}{2}, & k = 0, 2, \ldots, 2n-2, \\ 2\pi, & k = 2n, 2n+2, \ldots, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.6}$$

Proof. It is well-known that when ω is a weight function, then $D_n(t) = [T_n(t)]^2 \omega(t)$ is also a weight function (see Engels [5, pp. 214–226]). We have

$$\rho_n(z) = \int_{-1}^{1} D_n(t) \frac{T_n(t)}{z - t} dt = \sum_{k=0}^{+\infty} \gamma_{n,k} u^{-n-k-1},$$

where

$$\gamma_{n,k} = 2 \int_{-1}^{1} \omega(t) [T_n(t)]^3 U_{n+k}(t)\, dt \quad (k = 0, 1, \ldots). \tag{4.7}$$

The last expression is equal to

$$\gamma_{n,k} = 2 \int_{0}^{\pi} [\cos(n\theta)]^3 \frac{\sin(n+k+1)\theta}{\sin\theta} d\theta,$$

which can be calculated using the formulas 1.320.5 and 1.320.7 in [8] and combining them with

$$\frac{\sin(m+1)x}{\sin x} = 2 \sum_{k=0}^{[m/2]}{}'' \cos(m-2k)x,$$

where the “the double prim” denotes that the last summand has to be halved if m is even. In that way we obtain exactly we need. □

Now, substituting (4.2) and (4.5) in (2.3), we obtain

$$K_n(z) = \sum_{k=0}^{+\infty} \omega_{n,k} u^{-3n-k-1}, \tag{4.8}$$

where

$$\omega_{n,k} = \sum_{j=0}^{k} \beta_{n,j} \gamma_{n,k-j}. \tag{4.9}$$

Theorem 4.3. *The remainder term $R_n(fT_n)$ can be represented in the form*

$$R_n(fT_n) = \sum_{k=0}^{+\infty} \alpha_{3n+k} \epsilon_{n,k}, \tag{4.10}$$

where the coefficients $\epsilon_{n,k}$ are independent on f. Furthermore, if f is an even function then $\epsilon_{n,2j+1} = 0$ $(j = 0, 1, \ldots)$.

Proof. By substitution (4.1) and (4.8) in (2.2), we obtain

$$\begin{aligned} R_n(fT_n) &= \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} \left({\sum_{k=0}^{\infty}}' \alpha_k T_k(z) \sum_{k=0}^{+\infty} \omega_{n,k} u^{-3n-k-1} \right) dz \\ &= \sum_{k=0}^{+\infty} \left(\frac{1}{2\pi i} {\sum_{j=0}^{+\infty}}' \alpha_j \int_{\mathcal{E}_\rho} T_j(z) u^{-3n-k-1} dz \right) \omega_{n,k}. \end{aligned}$$

Applying Lemma 5 from [9], this reduces to (4.10) with

$$\epsilon_{n,0} = \frac{1}{4}\omega_{n,0}, \qquad \epsilon_{n,1} = \frac{1}{4}\omega_{n,1}, \qquad \epsilon_{n,k} = \frac{1}{4}(\omega_{n,k} - \omega_{n,k-2}), \quad k = 2, 3, \ldots. \tag{4.11}$$

When k is odd, since $\omega(t) = \omega(-t)$ it follows from (4.9) and Lemmas 4.1 and 4.2 that $\omega_{n,k} = 0$, and hence $\epsilon_{n,k} = 0$. □

4.1. *Error bounds based on the expansion*

In general, the Chebyshev–Fourier coefficients α_k in (4.1) are unknown. However, Elliot [4] described a number of ways of estimating or bounding them. in particular, under our assumptions

$$|\alpha_k| \le \frac{2}{\rho^k} \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right). \tag{4.12}$$

By using (4.3), (4.6), (4.9), if and only if $k = 2jn$, $j \in \mathbb{N}_0$, we have

$$\begin{aligned} \omega_{n,2jn} &= \beta_0 \gamma_{2jn} + \beta_{2n} \gamma_{(2j-2)n} + \cdots + \beta_{(2j-2)n} \gamma_{2n} + \beta_{2jn} \gamma_0, \\ \omega_{n,2jn-2} &= \beta_0 \gamma_{2jn-2} + \beta_{2n} \gamma_{(2j-4)n} + \cdots + \beta_{(2j-2)n} \gamma_{2n-2}, \end{aligned}$$

which implies

$$\begin{aligned}\omega_{n,2jn} - \omega_{n,2jn-2} &= \beta_{(2j-2)n}(\gamma_{2n} - \gamma_{2n-2}) + \beta_{2jn}\gamma_0 \\ &= 4(-1)^{j-1} j \frac{\pi}{2} + 4(-1)^j (j+1) \frac{3\pi}{2},\end{aligned}$$

i.e.

$$\epsilon_{n,2jn} = (-1)^j \frac{\pi}{2}(-j + 3(j+1)) = (-1)^j \frac{\pi}{2}(2j+3).$$

Otherwise, $\epsilon_{n,k} = 0$ for $k \neq 2jn$. Using the obtained results, we get

$$|R_n(fT_n)| = \left|\sum_{k=0}^{+\infty} \alpha_{3n+k}\epsilon_{n,k}\right| = \left|\sum_{j=0}^{+\infty} \alpha_{3n+2jn}\epsilon_{n,2jn}\right| \le \frac{\pi}{\rho^{3n}} \left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right) \sum_{j=0}^{+\infty} \frac{2j+3}{\rho^{2jn}}.$$

Since for $|x| < 1$

$$\sum_{j=0}^{+\infty}(2j+3)x^j = \frac{2x}{(1-x)^2} + \frac{3}{1-x} = \frac{3-x}{(1-x)^2},$$

with $x = \rho^{-2n}$, the previous inequality reduces to

$$|R_n(fT_n)| \le \frac{\pi}{\rho^{3n}} \left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right) \frac{3-\rho^{-2n}}{(1-\rho^{-2n})^2} = \pi \left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right) \frac{3\rho^{2n}-1}{\rho^n(\rho^{2n}-1)^2}. \tag{4.13}$$

4.2. L^1-error bounds

According to (2.8) we study now the quantity $L_n(\mathcal{E}_\rho) = \frac{1}{2\pi}\oint_{\mathcal{E}_\rho} |K_n(z)|\ |dz|$, where $|K_n(z)|$ may be obtained from (3.2). Since $z = (u+u^{-1})/2$, $u = \rho e^{i\theta}$, and $|dz| = (1/\sqrt{2})\cdot\sqrt{a_2 - \cos 2\theta}\ d\theta$ (see [9]), the quantity $L_n(\mathcal{E}_\rho)$ reduces to

$$\begin{aligned}L_n(\mathcal{E}_\rho) &= \frac{1}{2\pi\sqrt{2}} \int_0^{2\pi} |K_n(z)| \sqrt{a_2 - \cos 2\theta}\, d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{\sqrt{\rho^{-4n} + 9 + 6\rho^{-2n}\cos 2n\theta}}{\rho^n(a_{2n} + \cos 2n\theta)} d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{\sqrt{\rho^{-2n} + 9\rho^{2n} + 6\cos 2n\theta}}{\rho^{2n}(a_{2n} + \cos 2n\theta)}\, d\theta.\end{aligned}$$

Applying Cauchy inequality to the last expression, we obtain

$$L_n(\mathcal{E}_\rho) \le \frac{\sqrt{\pi}}{2\rho^{2n}} \sqrt{\int_0^{\pi} \frac{\rho^{-2n} + 9\rho^{2n} + 6\cos 2n\theta}{(a_{2n} + \cos 2n\theta)^2}\, d\theta}.$$

Introducing $x = \rho^{4n}$, and using [8, Eq.3.616.7], we obtain

$$\begin{aligned}&(\rho^{-2n} + 9\rho^{2n}) \int_0^{\pi} \frac{d\theta}{(a_{2n} + \cos 2n\theta)^2} + 6\int_0^{\pi} \frac{\cos 2n\theta d\theta}{(a_{2n} + \cos 2n\theta)^2} \\ &\quad = (\rho^{-2n} + 9\rho^{2n}) \frac{4\pi x(x+1)}{(x-1)^3} - \frac{48\pi x^{3/2}}{(x-1)^3}.\end{aligned}$$

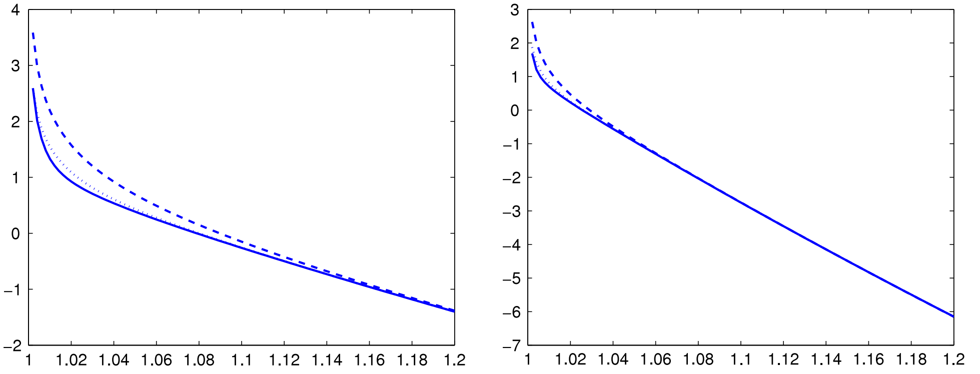

Fig. 4.1. $\log_{10}$ of the values $L_n(\mathcal{E}_\rho)$ (solid line) and its bounds given by (4.13) (dashed line) and (4.14) (dot-dashed line) for $n = 10$ (left), and $n = 30$ (right).

Therefore,

$$L_n(\mathcal{E}_\rho) \le \frac{\sqrt{\pi}}{2\rho^{2n}} \sqrt{(\rho^{2n} + 9\rho^{6n}) \frac{4\pi(\rho^{4n}+1)}{(\rho^{4n}-1)^3} - \frac{48\pi\rho^{6n}}{(\rho^{4n}-1)^3}}$$
$$= \frac{\pi}{\rho^n} \sqrt{\frac{9\rho^{8n} - 2\rho^{4n} + 1}{(\rho^{4n}-1)^3}}. \tag{4.14}$$

We drew graphs of the original $L_n(\mathcal{E}_\rho)$ and its bounds, namely their logarithms over the base 10, as functions of ρ, for different values n. As we can see, when n increases, corresponding graphs become more and more close each other (see Fig. 4.1).

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