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On nonlinear quasi-contractions on TVS-cone metric spaces

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ABSTRACT

Recently, Du [W.-S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. (2009), doi:10.1016/j.na.2009.10.026] introduced the notion of TVS-cone metric space. In this paper we present fixed point theorem for nonlinear quasi-contractive mappings defined on TVS-cone metric space, which generalizes earlier results obtained by Ilić and Rakočević [D. Ilić, V. Rakočević, Quasi-contractions on a cone metric space, Appl. Math. Lett. 22 (2009) 728–731] and Kadelburg, Radenović and Rakočević [Z. Kadelburg, S. Radenović, V. Rakočević, Remarks on quasi-contractions on a cone metric space, Appl. Math. Lett. 22 (2009) 1674–1679].

1. Introduction

There has been a number of generalizations of metric space. One such generalization is the notion of a cone metric space (also known as K-metric space) introduced by several Russian authors (for historical notes see [1]) in the mid-20th century. Huang and Zhang [2] re-introduced such spaces, and also went further, defining convergent and Cauchy sequences in terms of interior points of the underlying cone. The concept of a cone metric space is more general than that of a metric space (see [2]—Example 1 and [3]—Examples 1.1 and 1.2). They have proved some fixed point theorems for this class of spaces. Further fixed point results for this class of spaces were obtained by Ilić and Rakočević [4], Rezapour and Hamlbarani [5], Abbas and Rhoades [6], Veazpour and Raja [3], Arshad et al. [7], Abbas and Rhoades [6], Kadelburg et al. [8], Włodarczyk, Plebaniak and Doliński [9], etc. Common fixed point results for this class of spaces were presented in papers by Vetro [10], Arshad et al. [11], Di Bari and Vetro [12,13], etc.

Applications of such results in the theory of integral equations and topological theory of set-valued dynamic systems can be found in the papers [7,3,9].

Ćirić [14] first introduced the notion of quasi-contraction and proved the fixed point theorem for this class of mappings. Ćirić's result was extended to cone metric spaces by Ilić and Rakočević [4] and Kadelburg, Radenović and Rakočević [8]. In the first of these papers authors considered cone metric spaces with normal cone. In the second paper, normality condition was omitted. However, in the proof of the existence of fixed point result, the authors use the strong assumption that the coefficient of quasi-contractivity does not exceed $\frac{1}{2}$. Ćirić's fixed point theorem was extended to nonlinear quasi-contractions on metric spaces by Ivanov [15], Danes [16] and Aranđelović et al. [17].

Recently, Du [18] introduced the notion of TVS-cone metric space.

In this paper we present fixed point theorem for nonlinear quasi-contractive mappings defined on TVS-cone metric space, which generalizes the earlier results presented in [4,8,17,15,16,14].

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2. Preliminary notes

Definition 1. Let E be a linear topological space. By Θ we denote the zero element of E. A subset P of E is called a cone if:

(1) P is closed, nonempty and $P \neq \{\Theta\}$;
(2) $a, b \in \mathbf{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(3) $P \cap (-P) = \{\Theta\}$.

Given a cone $P \subseteq E$, we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$, whereas $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

Let E be a linear topological space and let $P \subseteq E$ be a cone. We say that P is a solid cone if and only if $\text{int}P \neq \emptyset$.

In the following, we always suppose that E is a locally convex Hausdorff topological vector space, P is a solid cone in E such that $\leq$ is a partial ordering on E with respect to P. By I we denote the identity operator on E i.e. $I(x) = x$ for each $x \in E$.

Let $e \in \text{int}P$. The nonlinear scalarization function ξ_e defined by

$$\xi_e(y) = \inf\{t \in \mathbf{R} : y \in te - P\}, \quad \text{for all } y \in E,$$

has the following properties (see [18]):

(1) ξ_e is positively homogeneous and continuous on E;
(2) if $y_1 \in y_2 + P$, then $\xi_e(y_2) \leq \xi_e(y_1)$;
(3) ξ_e is subadditive on E.

Also, there are some more properties of the function ξ_e not mentioned by Du.

Lemma 1. *Let $e \in \text{int}P$, and let $\xi_e(x) = \inf\{t \in \mathbb{R} : te - x \in P\}$ be its scalarization function.*

(a) *Let $\Theta \ll x \ll \lambda e$, for some real $\lambda > 0$. Then $0 \leq \xi_e(x), -\xi_e(-x) < \lambda$.*
(b) *The function $E \ni x \mapsto \|x\| = \max\{|\xi_e(x)|, |\xi_e(-x)|\}$ is a semi-norm on a locally convex topological space considered over real field.*

Proof. (a) The left hand side of the required inequality is obvious, as well as it holds $\xi_e(x) \leq \xi_e(\lambda e) = \lambda$. Since $\lambda e - x \in \text{int}P$, there exists a neighborhood W of Θ such that $\lambda e - x + W \subseteq P$. Let $\delta > 0$ be such that $-\delta e \in W$. Then we have

$$P \ni \lambda e - x - \delta e = (\lambda - \delta)e - x, \tag{1}$$

and therefore $\xi_e(x) \leq \lambda - \delta < \lambda$.

Further, from (1), we conclude that $(\delta - \lambda)e + x \notin P$, and hence $\xi_e(-x) > -\lambda$.

(b) Obviously, $\|x\| \geq 0$, and $\|-x\| = \|x\|$. For $\alpha > 0$ we have $\xi_e(\alpha x) = \alpha\xi_e(x)$, and also $-\xi_e(-\alpha x) = -\alpha\xi_e(-x)$, whereas, for $\alpha < 0$ we have $\xi_e(\alpha x) = -\alpha\xi_e(-x)$, and also $-\xi_e(-\alpha x) = \alpha\xi_e(x)$ implying in both cases $\|\alpha x\| = |\alpha|\,\|x\|$. Subadditivity follows from subadditivity of the functions $x \mapsto \xi_e(x)$, $x \mapsto -\xi_e(-x)$, and max. □

Remark. One might try to generalize the previous lemma to complex semi-norms, considering the function $x \mapsto \sup_{0 \leq \varphi \leq 2\pi} |\xi_e(\exp(\mathrm{i}\varphi)x)|$, which is obviously a complex semi-norm. However, part (a) does not hold. Namely, $\mathbb{C}$ is a complex vector space (over itself) with the standard topology, and $P = \{z \in \mathbb{C} : \text{Im}z \geq 2|\text{Re}z|\}$ is a solid cone. Considering vectors $e = i$ and $x = ci$, for $0 < c < 1$ we find that $0 \ll x \ll e$, but $\xi_e(\exp(\mathrm{i}\pi/2)) > 1$ for c close enough to 1.

Definition 2. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:

(1) $\Theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \Theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a TVS-cone metric on X and (X, d) is called a TVS-cone metric space.

Remark. Cone metric spaces in the Huang–Zhang sense [2] are included in our definition since every Banach space is a locally convex Hausdorff topological vector space.

The following definitions were introduced in [2].

Definition 3. Let (X, d) be a solid TVS-cone metric space, $x \in X$ and (x_n) a sequence in X. Then

(1) (x_n) TVS-cone converges to x if for every $c \in \text{int}P$ there exists a positive integer N such that for all $n \geq N d(x_n, x) \ll c$. We denote this by $\lim x_n = x$ or $x_n \to x$;
(2) (x_n) is a TVS-cone Cauchy sequences if for every $c \in \text{int}P$ there exists a positive integer N such that for all $m, n \geq N d(x_m, x_n) \ll c$;
(3) (X, d) is a TVS-cone complete cone metric space if every Cauchy sequence is convergent.

Let (X, d) be a TVS-cone metric space. In [18] Du proved that for any $e \in \text{int}P$:

(1) Function $d_e : X \times X \to [0, +\infty)$ defined by $d_e = \xi_e \circ d$ is a metric.
(2) If (x_n) TVS-cone converges to x, then $\lim d_e(x_n, x) = 0$.
(3) If (x_n) is a TVS-cone Cauchy sequences then (x_n) is a Cauchy sequences (in the usual sense) in (X, d_e).
(4) If (X, d) is a TVS-cone complete metric space, then (X, d_e) is a complete metric space.

Lemma 2. *Let E be a TVS-cone space, let $P \subseteq E$ be a solid cone. A sequence $x_n \in E$ converges to $x \in E$ in TVS-cone metric d, if and only if $x_n \to x$ in all metrics d_e, where e runs through* int$P$.

Also, x_n is a TVS-cone Cauchy sequence if and only if x_n is a Cauchy sequence in all metrics d_e.

Proof. The "only if" part of the statement is contained in the previous properties (2) and (3). Let us prove the "if" part of the statement. Let us suppose that $x_n \to x$ in all metrics d_e, let $e \in \mathrm{int}(P)$, and let $\varepsilon < 1$. Then $d_e(x_n, x) = \xi_e(d(x_n, x)) \to 0$, and therefore there exists a positive integer n_0 such that for all $n \geq n_0$ we have

$$\inf\{r \mid d(x_n, x) \in re - P\} = \xi_e(d(x_n, x)) \leq \varepsilon/2,$$

which implies that there exists $r < \varepsilon$ such that $d(x_n, x) \in re - P$, i.e. $re - d(x_n, x) \in P$, i.e. $d(x_n, x) \leq re \ll e$.

The proof of the other statement dealing with Cauchy sequences can be written down in a similar way. □

Definition 4. By Φ we denote the set of all real functions $\varphi : [0, \infty) \to [0, \infty)$ which have the following properties:

(a) $\varphi(0) = 0$;
(b) $\varphi(x) < x$ for all $x > 0$;
(c) $\lim_{x\to\infty}(x - \varphi(x)) = \infty$.

Define $\Phi_1 = \{\varphi \in \Phi : \varphi$ is monotone nondecreasing and $\overline{\lim}_{t\to r+}\varphi(t) < r$ for any $r > 0\}$ and $\Phi_2 = \{\varphi \in \Phi : \overline{\lim}_{t\to r}\varphi(t) < r$ for any $r > 0\}$.

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. The element $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that the sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

In [17] the authors presented the following fixed point theorem which generalizes the earlier results obtained by Ivanov [15], Danes [16] and Ćirić [14].

Theorem 1. *Let (X, d) be a complete metric space and $f : X \to X$. If there exist $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_1 \cup \Phi_2$ such that*

$$\begin{aligned} d(f(x), f(y)) \leq \max\{&\varphi_1(d(x, y)), \varphi_2(d(x, f(x))), \varphi_3(d(y, f(y))), \\ &\varphi_4(d(x, f(y))), \varphi_5(d(f(x), y))\}, \end{aligned}$$

for any $x, y \in X$, then f has the unique fixed point $y \in X$ and for each $x \in X$ sequence of Picard iterates defined by f at x converge to y.

Remark. As is easy to see, $\Phi_1 \subseteq \Phi_2$. Indeed, if φ is nondecreasing, then $\overline{\lim}_{t\to r-}\varphi(t) \leq \varphi(r) < r$. Thus, the condition $\varphi_1, \ldots, \varphi_5 \in \Phi_1 \cup \Phi_2$ can be simplified to $\varphi_1, \ldots, \varphi_5 \in \Phi_2$.

3. Main results

By Ψ we denote the set of all functions $\psi : P \to P$ which have the following properties:

(a) $\psi(\Theta) = \Theta$.
(b) $(I - \psi)(\mathrm{int}(P)) \subseteq \mathrm{int}P$.
(c) $\lim_{t\to\infty}(\|tx\| - \|\psi(tx)\|) = \infty$, for any $x \in P \setminus \{\Theta\}$, where $\|.\|$ is an arbitrary real semi-norm on E.

For a function $\psi \in \Psi$ we say that $\psi \in \Psi_2$ if for any $x \in \mathrm{int}P$ and for each $(x_n) \subseteq \mathrm{int}P$ such that $x_n \to x$, there exists a positive integer n_0 such that $n > n_0$ implies $\psi(x_n) \leq (1 - \varepsilon)x$, where ε does not depend on the choice of the sequence (x_n).

Now we shall prove our main result, which is a generalization of the earlier results presented in [17,15,16].

Theorem 2. *Let (X, d) be a complete TVS-cone metric space and $f : X \to X$. Assume that there exists $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5 \in \Psi_2$ such that for any $x, y \in X$ there exists*

$$u \in \{\psi_1(d(x, y)), \psi_2(d(x, f(x))), \psi_3(d(y, f(y))), \psi_4(d(x, f(y))), \psi_5(d(f(x), y))\} \tag{2}$$

such that

$$d(f(x), f(y)) \leq u. \tag{3}$$

Then f has the unique fixed point $y \in X$ and for each $x \in X$ the sequence of Picard iterates defined by f at x converge to y, in the TVS-cone metric.

Proof. Let $e \in \mathrm{int}P$, $k \in \{1, \dots, 5\}$ and $\varphi_k : [0, \infty) \to [0, \infty)$ defined by

$$\varphi_k(t) = \max\{\xi_e(\psi_k(te)), -\xi_e(-\psi_k(te))\} =: \|\psi_k(te)\|.$$

Then

(a) $\varphi_k(0) = 0$ since $\xi_e(\Theta) = 0$ and $\psi_k(\Theta) = \Theta$.
(b) By Lemma 1, taking into account $\psi_k(te) \ll te$, we have $\varphi_k(t) = \|\psi_k(te)\| < t$.
(c) $\lim_{t\to\infty}(t - \varphi_k(t)) = \lim_{t\to\infty}(\|te\| - \|\psi_k(te)\|) = \infty$.
Therefore, $\varphi_k \in \Phi$.
(d) If $\psi_k \in \Psi_2$ then $\varphi_k \in \Phi_2$.

Let $t \in (0, +\infty)$ and $(t_n) \to t$. From $\psi_k \in \Psi_2$ and $t_n e \to te$ it follows that there exists a positive integer n_0 such that $n > n_0$ implies $\psi_k(t_n e) \le (1 - \varepsilon)t_n e \ll (1 - \varepsilon/2)t_n e$. By Lemma 1, it follows that $\varphi_k(t_n) = \|\psi_k(t_n)\| < (1 - \varepsilon/2)t_n$, for $n > n_0$. Hence $\overline{\lim}_{s\to t}\varphi(s) \le (1 - \varepsilon/2)t < t$ because t_n and t are arbitrary, which implies that $\varphi_k \in \Phi_2$.

Now, (X, d_e) is a complete metric space and $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_1 \cup \Phi_2$. From (2) and (3) it follows that

$$\xi_e(d(f(x), f(y))) \le \xi_e(u) \le \|u\|$$

for some

$$u \in \{\psi_1(d(x, y)), \psi_2(d(x, f(x))), \psi_3(d(y, f(y))), \psi_4(d(x, f(y))), \psi_5(d(f(x), y))\}$$

which implies

$$d_e(f(x), f(y)) \le \max\{\varphi_1(d_e(x, y)), \varphi_2(d_e(x, f(x))), \varphi_3(d_e(y, f(y))), \varphi_4(d_e(x, f(y))), \varphi_5(d_e(f(x), y))\},$$

for any $x, y \in X$. The statement follows from Theorem 1 and Lemma 2. □

Definition 5. If $A : E \to E$ is a one to one function such that $A(P) = P$, $(I - A)$ is one to one and $(I - A)(P) = P$ then we say that A is a contractive operator.

The next corollary includes the results presented in [4,8,14,2,5].

Corollary 1. *Let (X, d) be a TVS-cone metric space and $f : X \to X$. Assume that there exists contractive bounded linear operators A_1, A_2, A_3, A_4, A_5 such that for any $x, y \in X$ there exists*

$$u \in \{A_1(d(x, y)), A_2(d(x, f(x))), A_3(d(y, f(y))), A_4(d(x, f(y))), A_5(d(f(x), y))\}$$

such that $d(f(x), f(y)) \le u$. Then f has the unique fixed point $y \in X$ and for each $x \in X$ the sequence of Picard iterates defined by f at x converge to y in the TVS-cone metric.

Proof. Let $k \in \{1, \dots, 5\}$. Then $A_k(\Theta) = \Theta$ since A_k is linear. From $(I - A)(P) = P$ by Open mapping theorem (see [19]) it follows that $(I - A)(\mathrm{int}P) \subseteq \mathrm{int}P$. So $A(x) \ll x$ for any $x \in \mathrm{int}P$. Let $x \in \mathrm{int}P$, $(x_n) \subseteq \mathrm{int}P$ and $x_n \to x$. Then $\lim A(x_n) \ll x$ since A is continuous.

$$\lim_{t\to\infty}(\|tx\| - \|A_k(tx)\|) = (\|x - A_k(x)\|)\lim_{t\to\infty} t = \infty,$$

for any $x \in P \setminus \{\Theta\}''$, where $\|.\|$ is an arbitrary real semi-norm on E.

Hence $A \in \Psi_2$. The statement follows from Theorem 2. □

Example 1. Let $X = [0, 1]$, $E = C_{\mathbf{R}}[0, 1]$ equipped with the strongest locally convex topology and $P = \{g \in E : g(t) \ge 0, t \in [0, 1]\}$. Then E is a locally convex not normable space by exercise 7 from [20], and P is a non-normal solid cone, by Theorem 2.3 from [21]. Also, $d : X^2 \to P$, defined by

$$d(x, y) = |x - y|e^t,$$

is a TVS-cone metric on E. Define $f : X \to X$ by

$$f(x) = \begin{cases} \dfrac{x}{3} & \left(0 \le x < \dfrac{3}{4}\right) \\ \dfrac{x}{4} & \left(\dfrac{3}{4} \le x \le 1\right). \end{cases}$$

Then existence and uniqueness follows from Corollary 1. The results of a recent work [5,2,4,22] are not applicable in this case because they deal either with full contractions or with normal cones.

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