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On the brachistochrone of a variable mass particle in general force fields

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ABSTRACT

The problem of the brachistochronic motion of a variable mass particle is considered. The particle moves through a resistant medium in the field of arbitrary active forces. Beginning from these general assumptions, and applying Pontryagin's minimum principle along with singular optimal control theory, a corresponding two-point boundary value problem is obtained and solved. The solution proposed involves an appropriate numerical procedure based upon the shooting method. In this numerical procedure, the evaluation of ranges for unknown values of costate variables is avoided by the choice of a corresponding Cartesian coordinate of the particle as an independent variable. A numerical example assuming the resistance force proportional to the square of the particle speed is presented. A review of existing results for related problems is provided, and it can be shown that these problems may be regarded as special cases of the brachistochrone problem formulated and solved in this paper under very general assumptions by means of optimal control theory.

1. Introduction

In 1696 Johann Bernoulli formulated the brachistochrone problem: find a smooth curve down which a particle slides from rest at a point *A* to a point *B* in a vertical plane influenced by its own gravity in the least time. After the brachistochrone problem had been independently solved by the Bernoulli brothers (Johann and Jacob), Newton, Leibnitz, Huygens and L'Hospital, various generalizations of the classical brachistochrone problem have been made. The brachistochronic motion of a particle was considered in various fields of active forces as well as under the action of various types of resistance forces (viscous friction forces, Coulomb friction force etc.). It is characteristic that the variational calculus method [1,2] was mainly used for various generalizations of the classical brachistochrone problem. However, from [3] to [4] and to [5], Pontryagin's minimum principle [6,7] and the singular optimal control theory [8] were included in solving this problem. Thus, in [3] the solution to the brachistochrone problem was realized within the theory of optimal control, while for the case of the brachistochrone in a resistant medium and the field of central forces the corresponding relations were developed only, representing the necessary optimality conditions. A numerical solution to the classical brachistochrone problem is given in [4]. An analytical solution to the problem of brachistochrone with Coulomb friction is presented in [5]. In other papers, generalizations of the problem were made using the variational calculus method: in the cases with Coulomb friction [9–13], where analytical solutions were obtained in [9–12] under various boundary conditions and various parameterizations of the brachistochrone curve, while in [13] the solution was obtained numerically; in the case with viscous friction, where as an independent variable was the particle speed in [14] and the slope angle of the brachistochrone curve in [15]; in the case with various fields of central forces in [16–19]. Among the mentioned types of generalizations of the brachistochrone problem, the paper [20] is interesting for its use of the Fermat principle from geometric optics. This principle was utilized

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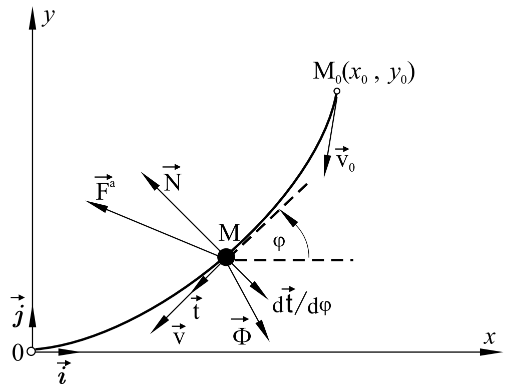

Fig. 1. Variable mass particle acted upon by arbitrary forces.

by Bernoulli as evident from the monograph [21]. The papers [22–24] considered the brachistochronic motion of a particle on a surface: in the cases of a smooth surface [23], Coulomb friction [22] and viscous friction [24]. In [22] it was shown that results from [9,11,15] represent special cases of the brachistochronic motion of a particle on a surface. Also, a certain number of references can be singled out [4,25–28] where the solution of the classical brachistochrone problem (cycloid) was used with the aim of testing various numerical methods in solving nonlinear engineering optimization problems. Thus, in [4] the classical Chebyshev method was utilized, in [27] the single-term Walsh series, in [25,26,28] the evolutionary computational technique based on the Darwinian principles of natural selection.

This paper considers the brachistochronic motion of a variable mass particle. This type of generalization of the brachistochrone problem has been considered only in the papers [29,30]. In [29] considerations involved the motion of a variable mass particle in the homogeneous gravitational field in a resistant medium, where it was taken that the relative velocity of the expelled (or gained) mass has the direction of the tangent to the particle path and a constant magnitude. At the initial particle position, the mass and speed of the particle were specified. A solution in parametric form was obtained with the slope angle of a tangent to the brachistochrone curve being taken as the parameter. This solution requires further numerical computations. In [30] the assumption of homogeneous gravitational field was retained and Coulomb friction force appears as the resistance force. The law of the rate of mass variation with respect to time as an exponential function of time was specified as well as the relative speed of the expelled mass as a function of the particle speed. The form of the particle trajectory was determined as well as the law of change in angle between the relative velocity of the expelled mass and the particle velocity so that the particle starting from a specified initial position reaches a specified final position with the given velocity in the shortest descent time. Like in [29], the task has not been completely analytically solved. In our paper it was assumed that a variable mass particle moves under the action of arbitrary active forces. The form of a brachistochrone curve is determined, which ensures that the particle starting from a specified initial position with a specified mass and initial speed comes to a specified final position in the shortest descent time. The formulated problem is solved by the application of Pontryagin's minimum principle [6,7] and the singular optimal control theory [8]. After a corresponding two-point boundary value problem has been formulated, a procedure for its numerical solving is presented. This procedure is based upon the shooting method [31] along with a combination of symbolic and numerical evaluations. The horizontal Cartesian coordinate of the particle has been chosen as an independent variable. This considerably simplifies the solution effort because the physical variables (mass, velocity, slope angle) are chosen in applying the shooting method. For these variables it is possible to assess the region in which their values are found, in contrast to the costate variables where it is difficult to do this. The results are illustrated by an example.

2. Formulation of the problem

Consider the motion of a particle M of variable mass m in the plane xOy of the Cartesian rectangular coordinate system along a smooth curve which is treated as a bilateral constraint (the particle must slide along the curve like a bead on a wire) under the action of arbitrary active forces (see Fig. 1). The axis Oy is directed vertically upward. The task consists of determining the form of the curve (brachistochrone) $y = y(x)$ which ensures that the particle, starting from the specified position $M_0(x_0 > 0, y_0 > 0)$ with the initial speed v_0 and mass m_0, reaches the origin O of the coordinate system in the shortest descent time t_f.

It is assumed below that the resultant of active forces, in a general case, is a function of the y coordinate, the mass m, the slope angle φ of the tangent to the particle path, and the particle speed v, i.e.

$$\vec{F}^a = \vec{F}^a(y, v, \varphi, m). \tag{1}$$

Also, it is assumed that there exists a unique solution $y = y(x)$ to the posed problem with the monotonically decreasing coordinate x, that is, $\dot{x} < 0$. From these assumptions it follows that the angle φ will vary within the interval

$$-\frac{\pi}{2} < \varphi < \frac{\pi}{2}. \tag{2}$$

The kinematics equations of the particle motion read

$$\dot{x} = -v\cos\varphi, \qquad \dot{y} = -v\sin\varphi \tag{3}$$

where an overdot denotes the derivative with respect to time t.

The differential equation describing the particle motion is

$$m\vec{a} = \vec{F}^{a} + \vec{N} + \vec{\Phi}, \tag{4}$$

where $\vec{N}$ is the constraint reaction force, while

$$\vec{\Phi} = \vec{\Phi}(v, \varphi, m) = \dot{m}\vec{v}_r \tag{5}$$

is the Meshchersky reactive force [32], where the law of the time-rate of mass variation is assumed to be known and of the form

$$\dot{m} = \dot{m}(m, v) \tag{6}$$

as well as the relative velocity of the expelled (or gained) mass

$$\vec{v}_r = \vec{v}_r(v, \varphi). \tag{7}$$

Similar to [11,12], introducing the following unit vectors

$$\vec{t} = (-\cos\varphi)\vec{i} + (-\sin\varphi)\vec{j}, \qquad \frac{\mathrm{d}\vec{t}}{\mathrm{d}\varphi} = (\sin\varphi)\vec{i} + (-\cos\varphi)\vec{j}, \tag{8}$$

the acceleration of the particle can be written in the form

$$\vec{a} = \dot{v}\vec{t} + v\dot{\varphi}\frac{\mathrm{d}\vec{t}}{\mathrm{d}\varphi}, \tag{9}$$

while resolution of the vector differential equation (4) into components along the directions of $\vec{t}$ and $\mathrm{d}\vec{t}/\mathrm{d}\varphi$ yields

$$\begin{aligned} m\dot{v} &= F_t^a(y, v, \varphi, m) + \Phi_t(v, \varphi, m), \\ mv\dot{\varphi} &= F_n^a(y, v, \varphi, m) + \Phi_n(v, \varphi, m) + N_n, \end{aligned} \tag{10}$$

where $\Phi_t = \vec{\Phi} \cdot \vec{t}$, $\Phi_n = \vec{\Phi} \cdot (\mathrm{d}\vec{t}/\mathrm{d}\varphi)$, $N_t = \vec{N} \cdot \vec{t} \equiv 0$, $N_n = \vec{N} \cdot (\mathrm{d}\vec{t}/\mathrm{d}\varphi)$, $F_t = \vec{F} \cdot \vec{t}$, $F_n = \vec{F} \cdot (\mathrm{d}\vec{t}/\mathrm{d}\varphi)$. For more details on the advantages of using the vector $\mathrm{d}\vec{t}/\mathrm{d}\varphi$ see [12]. Introducing the variable

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\varphi \tag{11}$$

a corresponding task of the optimal control can be formulated. In regard to this, using Eqs. (3), (6), (10) and (11) and taking the quantities y, p, v, and m as state variables, which are regarded as implicit functions of the variable x, the following state evolution equations can be formed

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = u, \\ &\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{F_t^a + \Phi_t}{m}\Psi(p, v) \triangleq \Psi_v(y, p, v, m), \qquad \frac{\mathrm{d}m}{\mathrm{d}x} = -\dot{m}\Psi(p, v) \triangleq \Psi_m(p, v, m), \end{aligned} \tag{12}$$

with the initial and final conditions

$$y(x_0) = y_0, \qquad v(x_0) = v_0, \qquad m(x_0) = m_0, \qquad y(0) = 0, \tag{13}$$

and where the optimal control u represents the second derivative of the coordinate y with respect to the variable x, which is directly related to the curvature of the particle path. In (12), the function $\Psi(p, v)$ is defined as:

$$\Psi(p, v) \equiv -\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\sqrt{1+p^2}}{v}. \tag{14}$$

Now, an optimal control should be defined

$$u = u(x) \tag{15}$$

and its corresponding optimal trajectory in the state space

$$y = y(x), \qquad p = p(x), \qquad v = v(x), \qquad m = m(x) \tag{16}$$

so that the functional, taking into account (14),

$$J=\int_0^{t_f}\mathrm{d}t=\int_{x_0}^{0}\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x=\int_0^{x_0}\Psi(p,v)\mathrm{d}x \tag{17}$$

has a minimum value, where Eq. (12) and the conditions (13) are satisfied.

If it were needed, the reaction of constraint $N_n(x)$ can be determined from Eqs. (3) and (10)–(12) as

$$N_n(x)=-\frac{muv^2}{(1+p^2)^{3/2}}-F_n^a-\Phi_n. \tag{18}$$

It should be pointed out that here a general case of active forces is considered. They involve both potential (e.g. gravitational) and non-potential forces (e.g. viscous friction forces proportional to arbitrary power of speed).

3. Solution of the problem within the framework of optimal control theory

To solve the tasks set applying the optimal control theory, the Hamiltonian [6,7] is formed

$$H=\Psi+\lambda_y p+\lambda_p u+\lambda_v\Psi_v+\lambda_m\Psi_m, \tag{19}$$

where $(\lambda_y,\ \lambda_p,\ \lambda_v,\ \lambda_m)$ are coordinates of the costate vector. The corresponding costate evolution equations read

$$\begin{aligned}
\frac{\mathrm{d}\lambda_y}{\mathrm{d}x}&=-\frac{\partial H}{\partial y}=-\lambda_v\frac{\partial\Psi_v}{\partial y},\\
\frac{\mathrm{d}\lambda_p}{\mathrm{d}x}&=-\frac{\partial H}{\partial p}=-\frac{\partial\Psi}{\partial p}-\lambda_y-\lambda_v\frac{\partial\Psi_v}{\partial p}-\lambda_m\frac{\partial\Psi_m}{\partial p},\\
\frac{\mathrm{d}\lambda_v}{\mathrm{d}x}&=-\frac{\partial H}{\partial v}=-\frac{\partial\Psi}{\partial v}-\lambda_v\frac{\partial\Psi_v}{\partial v}-\lambda_m\frac{\partial\Psi_m}{\partial v}\triangleq\Omega_v(y,p,v,m,\lambda_v,\lambda_m),\\
\frac{\mathrm{d}\lambda_m}{\mathrm{d}x}&=-\frac{\partial H}{\partial m}=-\lambda_v\frac{\partial\Psi_v}{\partial m}-\lambda_m\frac{\partial\Psi_m}{\partial m}\triangleq\Omega_m(y,p,v,m,\lambda_v,\lambda_m)
\end{aligned} \tag{20}$$

with the boundary conditions (transversality conditions)

$$\lambda_p(x_0)=0,\qquad \lambda_p(0)=0,\qquad \lambda_v(0)=0,\qquad \lambda_m(0)=0. \tag{21}$$

If a Hamiltonian H linearly depends upon a scalar control variable u (a typical case in which singular control arises in applications), then the necessary condition $\partial H/\partial u=0$ requires that the coefficient of u in H be (identically) zero on the optimal state-space trajectory. Accordingly, all higher-order derivatives of $\partial H/\partial u$ which are taken with respect to the independent variable x must be zero as well:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{\partial H}{\partial u}\right]=0;\quad k=0,1,2,\ldots. \tag{22}$$

It is then found that

$$\begin{aligned}
&\frac{\partial H}{\partial u}=\lambda_p,\\
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial H}{\partial u}\right]=\frac{\mathrm{d}\lambda_p}{\mathrm{d}x}\\
&\qquad=-\frac{\partial\Psi}{\partial p}-\lambda_y-\lambda_v\frac{\partial\Psi_v}{\partial p}-\lambda_m\frac{\partial\Psi_m}{\partial p},\\
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial H}{\partial u}\right]=-\frac{\partial^2\Psi}{\partial p^2}\frac{\mathrm{d}p}{\mathrm{d}x}-\frac{\partial^2\Psi}{\partial v\partial p}\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{\mathrm{d}\lambda_y}{\mathrm{d}x}\\
&-\frac{\mathrm{d}\lambda_v}{\mathrm{d}x}\frac{\partial\Psi_v}{\partial p}-\lambda_v\left[\frac{\partial^2\Psi_v}{\partial y\partial p}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial^2\Psi_v}{\partial p^2}\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{\partial^2\Psi_v}{\partial v\partial p}\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{\partial^2\Psi_v}{\partial m\partial p}\frac{\mathrm{d}m}{\mathrm{d}x}\right]\\
&\quad-\frac{\mathrm{d}\lambda_m}{\mathrm{d}x}\frac{\partial\Psi_m}{\partial p}-\lambda_m\left[\frac{\partial^2\Psi_m}{\partial p^2}\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{\partial^2\Psi_m}{\partial v\partial p}\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{\partial^2\Psi_m}{\partial m\partial p}\frac{\mathrm{d}m}{\mathrm{d}x}\right]\\
&\quad=\left[-\frac{\partial^2\Psi}{\partial p^2}-\lambda_v\frac{\partial^2\Psi_v}{\partial p^2}-\lambda_m\frac{\partial^2\Psi_m}{\partial p^2}\right]u\\
&\quad-\left[\frac{\partial^2\Psi_v}{\partial v\partial p}\Psi_v+\frac{\partial^2\Psi_v}{\partial m\partial p}\Psi_m+\frac{\partial^2\Psi_v}{\partial y\partial p}p-\frac{\partial\Psi_v}{\partial y}-\frac{\partial\Psi_v}{\partial p}\frac{\partial\Psi_v}{\partial v}-\frac{\partial\Psi_m}{\partial p}\frac{\partial\Psi_v}{\partial m}\right]\lambda_v\\
&\quad-\left[\frac{\partial^2\Psi_m}{\partial v\partial p}\Psi_v+\frac{\partial^2\Psi_m}{\partial m\partial p}\Psi_m-\frac{\partial\Psi_v}{\partial p}\frac{\partial\Psi_m}{\partial v}-\frac{\partial\Psi_m}{\partial p}\frac{\partial\Psi_m}{\partial m}\right]\lambda_m-\left[-\frac{\partial\Psi_v}{\partial p}\frac{\partial\Psi}{\partial v}+\frac{\partial^2\Psi}{\partial v\partial p}\Psi_v\right]
\end{aligned} \tag{23}$$

by means of Eqs. (12) and (20), where the dependences of the functions indicated in these equations have been inferred from the forms of the functions assumed for Ψ_v, Ψ_m, and Ψ.

Thus, for $k = 0$, 1, and 2, Eq. (22) through (23) yield

$$\lambda_p \equiv 0, \tag{24}$$

$$\lambda_y = -\frac{\partial \Psi}{\partial p} - \lambda_v \frac{\partial \Psi_v}{\partial p} - \lambda_m \frac{\partial \Psi_m}{\partial p}, \tag{25}$$

$$\begin{aligned} u \;=\; & \left\{\left[\frac{\partial^2 \Psi_v}{\partial v \partial p}\Psi_v + \frac{\partial^2 \Psi_v}{\partial m \partial p}\Psi_m + \frac{\partial^2 \Psi_v}{\partial y \partial p} p - \frac{\partial \Psi_v}{\partial y} - \frac{\partial \Psi_v}{\partial p}\frac{\partial \Psi_v}{\partial v} - \frac{\partial \Psi_m}{\partial p}\frac{\partial \Psi_v}{\partial m}\right]\lambda_v \right. \\ & + \left[\frac{\partial^2 \Psi_m}{\partial v \partial p}\Psi_v + \frac{\partial^2 \Psi_m}{\partial m \partial p}\Psi_m - \frac{\partial \Psi_v}{\partial p}\frac{\partial \Psi_m}{\partial v} - \frac{\partial \Psi_m}{\partial p}\frac{\partial \Psi_m}{\partial m}\right]\lambda_m \\ & \left. + \left[-\frac{\partial \Psi_v}{\partial p}\frac{\partial \Psi}{\partial v} + \frac{\partial^2 \Psi}{\partial v \partial p}\Psi_v\right]\right\} \Big/ \left[-\frac{\partial^2 \Psi}{\partial p^2} - \lambda_v \frac{\partial^2 \Psi_v}{\partial p^2} - \lambda_m \frac{\partial^2 \Psi_m}{\partial p^2}\right] \\ \triangleq\; & \Gamma(y, p, v, m, \lambda_v, \lambda_m). \end{aligned} \tag{26}$$

It is observed that the result $\lambda_p \equiv 0$ fulfills the boundary conditions imposed upon λ_p, so these conditions can be excluded from further consideration. Furthermore, λ_y does not appear (except for its derivative) in any of the remaining governing relations. As a result, the expression for λ_y (though valid) is not required nor immediately useful. Hence, it is expedient to consider the case of $k = 2$ in Eq. (22) in order to determine the control law for u. The control u represents the singular optimal control of the first order. The Kelley necessary condition [8] for optimality of the first-order singular control u

$$(-1)^n \frac{\partial}{\partial u}\left(\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[\frac{\partial H}{\partial u}\right]\right) \geq 0, \quad n = 1, \tag{27}$$

with $n \in \mathbb{N}$ denoting the order of the singular optimal control, yields

$$\frac{\partial^2 \Psi}{\partial p^2} + \lambda_v \frac{\partial^2 \Psi_v}{\partial p^2} + \lambda_m \frac{\partial^2 \Psi_m}{\partial p^2} \geq 0. \tag{28}$$

Assembled all together, the relevant governing relations for determination of the singular control and the optimal state-space trajectory are

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}x} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = \Gamma(y, p, v, m, \lambda_v, \lambda_m) \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \Psi_v(y, p, v, m), \\ & \frac{\mathrm{d}m}{\mathrm{d}x} = \Psi_m(p, v, m), \qquad \frac{\mathrm{d}\lambda_v}{\mathrm{d}x} = \Omega_v(y, p, v, m, \lambda_v, \lambda_m), \qquad \frac{\mathrm{d}\lambda_m}{\mathrm{d}x} = \Omega_m(y, p, v, m, \lambda_v, \lambda_m) \end{aligned} \tag{29}$$

with the accompanying boundary conditions

$$\begin{aligned} & y(x_0) = y_0, \qquad v(x_0) = v_0, \qquad m(x_0) = m_0, \\ & y(0) = 0, \qquad \lambda_v(0) = 0, \qquad \lambda_m(0) = 0. \end{aligned} \tag{30}$$

Although, in a general case, solving the two-point boundary value problem of the minimum principle is very complex, it is not like that in this case. If this problem is solved using the shooting method [31], by choosing the values

$$v(0) = v_f, \qquad p(0) = p_f, \qquad m(0) = m_f, \tag{31}$$

and solving the corresponding Cauchy problem defined by Eqs. (29) and (30), the following dependences can be established in a numerical form

$$\begin{aligned} & y_0 = f_y(v_f, p_f, m_f) \\ & v_0 = f_v(v_f, p_f, m_f) \\ & m_0 = f_m(v_f, p_f, m_f). \end{aligned} \tag{32}$$

Each of these dependences can be graphically represented in a v_f, p_f, m_f-space, and at the intersection of these surfaces there is a solution of the system of nonlinear equation (32). The fact that the missing conditions (31), in contrast to the costate variables, represent physical quantities (speed, slope, and mass) considerably simplifies the estimation of the intervals

$$\begin{aligned} & v_f^{\star} \leq v_f \leq v_f^{\star\star} \\ & p_f^{\star} \leq p_f \leq p_f^{\star\star} \\ & m_f^{\star} \leq m_f \leq m_f^{\star\star} \end{aligned} \tag{33}$$

in which the solutions of the system of Eq. (32) should be examined. Solving the system (32) for the unknown values v_f, p_f and m_f, the task posed is completed. The described procedure can be implemented in Mathematica 6.0 [33] as follows:

(*** This defines functions that return the values $y(x_0)$, $v(x_0)$, and $m(x_0)$ as functions of numerical values for $p(x)$, $v(x)$, and $m(x)$ at $x = 0$ ***)

```
fy1[mf_?NumberQ,pf_?NumberQ,vf_?NumberQ]:=First[y[x1]
/.NDSolve[{y'[x]==p[x],p'[x] ==u,v'[x] ==Fv, m'[x] ==Fm,
lv'[x] == Flv, lm'[x]==Flm, y[0]==0,p[0]==pf,v[0]==vf,
m[0]==mf,lv[0]==0,lm[0]==0},
{y,p,v,m,lv,lm},{x,0,x0}]]

fv1[mf_?NumberQ,pf_?NumberQ,vf_?NumberQ]:=First[v[x1]
/.NDSolve[{y'[x]==p[x],p'[x] ==u,v'[x] ==Fv, m'[x] ==Fm,
lv'[x] ==Flv, lm'[x]==Flm, y[0]==0,p[0]==pf,v[0]==vf,
m[0]==mf,lv[0]==0,lm[0]==0},
{y,p,v,m,lv,lm},{x,0,x0}]]

fm1[mf_?NumberQ,pf_?NumberQ,vf_?NumberQ]:=First[m[x1]
/.NDSolve[{y'[x]==p[x],p'[x] ==u,v'[x] ==Fv, m'[x] ==Fm,
lv'[x] ==Flv, lm'[x]==Flm, y[0]==0,p[0]==pf,v[0]==vf,
m[0]==mf,lv[0]==0,lm[0]==0},
{y,p,v,m,lv,lm},{x,0,x0}]]
```

(***This makes plots of functions in Eq. (32)***)

```
Show[ContourPlot3D[fy1[mf,pf,vf]==y0,
{mf, mfmin,mfmax},{pf,pfmin,pfmax},
{vf,vfmin,vfmax},
AxesLabel->{Subscript[m,f],
Subscript[p,f], Subscript[v,f]}],

ContourPlot3D[fv1[mf,pf,vf]==y0,
{mf,mfmin,mfmax },{pf,pfmin,pfmax},
{vf,vfmin,vfmax},
AxesLabel ->{Subscript[m,f],
Subscript[p,f], Subscript[v,f]}],

ContourPlot3D[fm1[mf,pf,vf]==y0,
{mf,mfmin,mfmax},{pf,pfmin,pfmax},
{vf,vfmin,vfmax},
AxesLabel ->{Subscript[m,f],
 Subscript[p,f], Subscript[v,f]}]]
```

(***This numerically solves the equation system (32) for m_f, p_f, and v_f ***)

```
s1=FindRoot[{fy1[mf,pf,vf]==y0,
fv1[mf,pf,vf]==v0,fm1[mf,pf,vf]==m0},
{{mf,mf*,mf**},{pf,pf*,pf**},{vf,vf*,vf**}}]
```

4. Numerical example

Consider a variable mass particle (see Fig. 2) of initial mass m_0 and initial speed v_0 at the position $M_0(x_0, y_0)$, which should reach the coordinate origin in the shortest time sliding along a smooth curve in a vertical plane in a uniform gravitational field.

Let the particle mass variation be proportional to the particle mass

$$\dot{m} = k_m m \tag{34}$$

where k_m is a specified constant. The acquired or expelled mass is assumed to be at rest relative to a stationary reference frame as it enters or leaves the particle, respectively, and so the relative velocity $\vec{v}_r$ is determined from

$$\vec{v}_r = -\vec{v} \tag{35}$$

so that the Meshchersky reactive force reads

$$\vec{\Phi} = -k_m m v \vec{t}\,. \tag{36}$$

The particle is also acted upon by the resistance force proportional to the square of the particle speed

$$\vec{F}_w = -k_v v^2 \vec{t}\,, \tag{37}$$

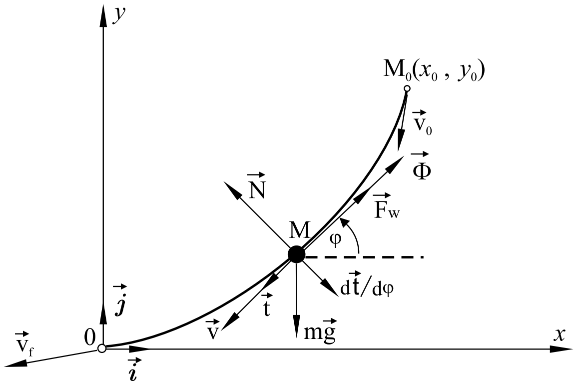

Fig. 2. Variable mass particle in a vertical plane.

where k_v is a specified positive constant. The projections of the resultant of active forces (1) are

$$F_t^a = mg\sin\varphi - k_v v^2, \qquad F_n^a = mg\cos\varphi \tag{38}$$

where $\vec{g} = -g\vec{j}$ is the gravitational acceleration. Now, the differential equation (29) of the two-point boundary value problem take the form

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= p,\\
\frac{\mathrm{d}p}{\mathrm{d}x} &= g(1+p^2)\left(-m + m^2k_m\lambda_m + mk_m\lambda_v v + 2v^2k_v\lambda_v\right)/[v^2(-m + m^2k_m\lambda_m - mv\lambda_v k_m - v^2\lambda_v k_v)],\\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{gp}{v} + \sqrt{1+p^2}\left(\frac{k_v}{m}v + k_m\right),\\
\frac{\mathrm{d}m}{\mathrm{d}x} &= -\frac{k_m m}{v}\sqrt{1+p^2},\\
\frac{\mathrm{d}\lambda_v}{\mathrm{d}x} &= -\lambda_v\left(\frac{gp}{v^2} + \frac{k_v}{m}\sqrt{1+p^2}\right) - \frac{\lambda_m k_m m}{v^2}\sqrt{1+p^2} + \frac{\sqrt{1+p^2}}{v^2},\\
\frac{\mathrm{d}\lambda_m}{\mathrm{d}x} &= \lambda_v\frac{k_v v}{m^2}\sqrt{1+p^2} + \frac{\lambda_m k_m}{v}\sqrt{1+p^2},
\end{aligned} \tag{39}$$

where the first-order singular optimal control (26) has the form

$$u = g(1+p^2)(-m + m^2k_m\lambda_m + mk_m\lambda_v v + 2v^2k_v\lambda_v)/[v^2(-m + m^2k_m\lambda_m - mv\lambda_v k_m - v^2\lambda_v k_v)]. \tag{40}$$

The two-point boundary value problem has been solved for the following values of the parameters

$$m_0 = 1\ \mathrm{kg}, \qquad v_0 = 5\ \frac{\mathrm{m}}{\mathrm{s}}, \qquad x_0 = 6\ \mathrm{m}, \qquad y_0 = 2\ \mathrm{m}, \qquad k_m = 0.2\ \mathrm{s}^{-1}, \qquad k_v = 0.1\ \frac{\mathrm{kg}}{\mathrm{m}}. \tag{41}$$

The missing boundary values m_f, p_f, and v_f are obtained by numerically solving the equation system (32), where the solutions are sought in the region

$$\begin{aligned}
&1.25\ \mathrm{kg} \le m_f \le 1.3\ \mathrm{kg}\\
&4.2\ \frac{\mathrm{m}}{\mathrm{s}} \le v_f \le 4.3\ \frac{\mathrm{m}}{\mathrm{s}}\\
&-0.35 \le p_f \le -0.32
\end{aligned} \tag{42}$$

which has been defined using the graphic representation of the solution (the intersection of the surfaces (32)) in Fig. 3.

Accordingly, one obtains $m_f = 1.279580584$ kg, $v_f = 4.268122289\ \frac{\mathrm{m}}{\mathrm{s}}$, $p_f = -0.320713340$ and the brachistochrone $y = y(x)$ as well as the other solutions of the system (39) are shown in Fig. 4.

It is also possible to determine the time $t = t(x)$ of the variable-mass particle motion along the brachistochrone by numerically solving the differential equations

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\sqrt{1+p^2}}{v}, \tag{43}$$

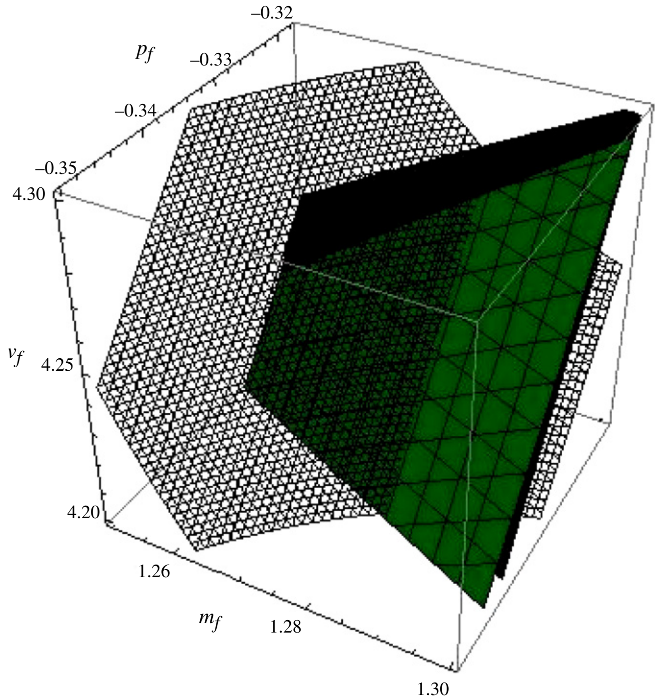

Fig. 3. Plots of the surfaces $y_0 = f_y(v_f, p_f, m_f)$, $v_0 = f_v(v_f, p_f, m_f)$, $m_0 = f_m(v_f, p_f, m_f)$.

Table 1
Numerical values of the parameters of brachistochrone curves for $k_m = 0.2\ \text{s}^{-1}$, $k_v = 0.1$ kg/m, $x_0 = 6$ m, $y_0 = 2$ m, $m_0 = 1$ kg, and various values of the speed v_0.

v_0 (m/s)	m_f (kg)	φ_f	v_f (m/s)	t_f (s)
2	1.377432185	−0.549166312	3.255706264	1.601105149
3	1.343049286	−0.478235438	3.541402971	1.474713080
5	1.279580584	−0.310349890	4.268122289	1.232661779
7	1.228451502	−0.153855836	5.073097336	1.028772174
9	1.189568059	−0.032037950	5.906776905	0.867951328
12	1.148433096	0.089382734	7.207456809	0.691992439

with $t_0 = 0$. This equation are obtained from Eqs. (3) and (11). Based on this, the descent time for the example considered is $t_f = 1.232661779$ s. Further, having in mind the solutions of the system (39), the Kelley condition (27)

$$K \equiv -\frac{\partial}{\partial u}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial H}{\partial u}\right]\right) = (m - m^2 k_m \lambda_m + m v \lambda_v k_m + v^2 \lambda_v k_v)/[m v (1 + p^2)^{3/2}] \geq 0 \tag{44}$$

can be also tested, as shown in Fig. 5.

Finally, using the solution of the system (39) and the expression (40), the optimal control can be completely defined as a function of the coordinate x, as shown in Fig. 6.

The obtained curve can be used as a reference curve for investigating effects of the values of parameters m_0, v_0, k_m, and k_v on brachitochronic motion of a particle. Thus, Tables 1–4 show changes in the values of parameters m_f, v_f, φ_f, and t_f depending on various values of parameters m_0, v_0, k_m, and k_v. Based on these data, the corresponding brachistochrone curves are presented in Figs. 7–10. Each of these figures shows Bernoulli's brachistochrone ($k_m = 0$, $k_v = 0$, $v_0 = 5$ m/s, $m_0 = m_f = 1$ kg, $\varphi_f = -0.188543550$, $v_f = 8.014986239$ m/s, $t_f = 0.914748210$ s) indicated by a dashed line. In Figs. 7–10 an arrow with designation of a corresponding parameter indicates how brachistochrone curves distribute with the growing value of the parameter designated in figures.

Observing Figs. 7–10 as well as numerical values in Tables 1–4, the following conclusions can be drawn:

- Increase of speed v_0 indirectly implies decrease of time t_f and increase of the value of speed v_f. Also, decrease of time t_f directly implies decrease of mass m_f. In Fig. 7 it is observable that with increase of speed v_0, brachistochrone curves tend to assume the shape of a straight line;

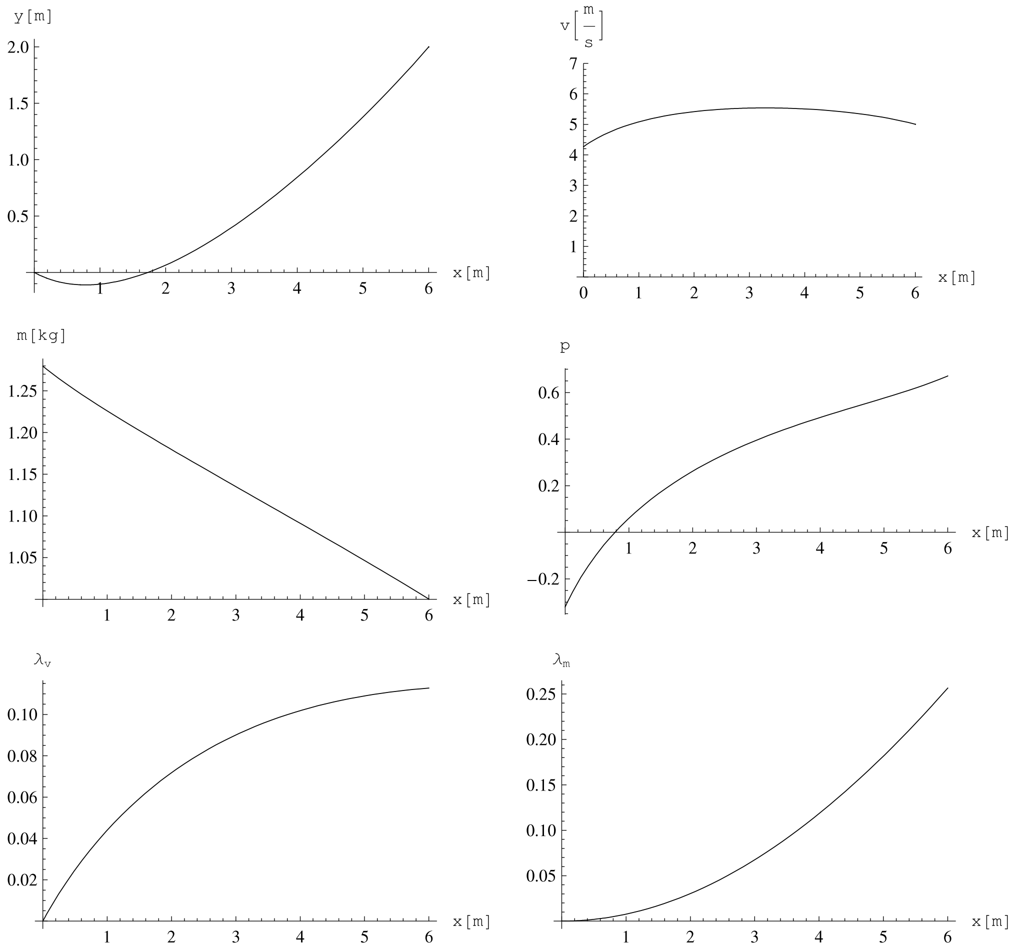

Fig. 4. Diagrams of functions which are solutions of the system (37).

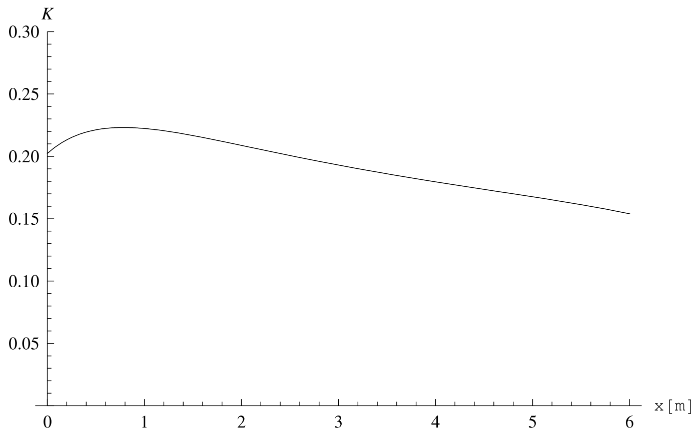

Fig. 5. Testing of the Kelley condition ($K > 0$).

- Increase of mass m_0 directly causes increase of mass m_f. Larger m_0 mass means increase of the amount of work done by gravity force during the particle descent, which affects increase of the value of speed v_f, thereby decrease of time t_f. Also, it is observable in Fig. 8 that with increase of mass m_0 brachistochrone curves assume the shape very close to the shape of Bernoulli's brachistochrone;

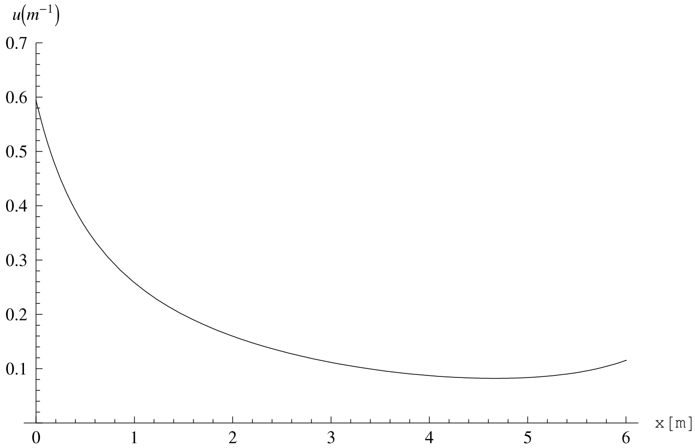

Fig. 6. Optimal control $u = u(x)$.

Table 2
Numerical values of the parameters of brachistochrone curves for $k_m = 0.2\ \mathrm{s}^{-1}$, $k_v = 0.1$ kg/m, $x_0 = 6$ m, $y_0 = 2$ m, $v_0 = 5$ m/s, and various values of the mass m_0.

m_0	m_f (kg)	φ_f	v_f (m/s)	t_f (s)
1	1.279580584	−0.310349890	4.268122289	1.232661779
2	2.495049659	−0.274725424	5.283262526	1.105807264
4	4.929125790	−0.255724639	5.967100648	1.044336438
6	7.364083584	−0.249342837	6.226707331	1.024275727
9	11.016917337	−0.245092848	6.409373208	1.011037274

Table 3
Numerical values of the parameters of brachistochrone curves for $m_0 = 1$ kg, $k_v = 0.1$ kg/m, $x_0 = 6$ m, $y_0 = 2$ m, $v_0 = 5$ m/s, and various values of the parameter k_m.

k_m s^{-1}	m_f (kg)	φ_f	v_f (m/s)	t_f (s)
−0.5	0.600919495	−0.147380849	5.934700234	1.018588596
−0.2	0.803239573	−0.209545605	5.302425610	1.095511308
0.2	1.279580584	−0.310349890	4.268122289	1.232661779
0.5	1.987354486	−0.398100930	3.410018316	1.373608698
0.7	2.843620027	−0.458700678	2.849109575	1.492968429

Table 4
Numerical values of the parameters of brachistochrone curves for $m_0 = 1$ kg, $k_m = 0.2\ \mathrm{s}^{-1}$, $x_0 = 6$ m, $y_0 = 2$ m, $v_0 = 5$ m/s, and various values of the parameter k_v.

k_v (kg/m)	m_f (kg)	φ_f	v_f (m/s)	t_f (s)
0.1	1.279580584	−0.310349890	4.268122289	1.232661779
0.3	1.411348022	−0.398587425	2.585733644	1.722726458
0.5	1.531226434	−0.438986393	2.040623850	2.130345026
0.6	1.586551811	−0.452882132	1.879570639	2.307814944

- It is evident that with increase of the positive value of parameter k_m, the value of mass m_f rises. However, for $k_m > 0$ the Meshchersky reactive force $\overrightarrow{\Phi}$ represents the resistance force, therefore increase of the value for k_m causes increase of effects of the resistance forces, thereby decrease of speed v_f and increase of time of brachistochronic motion t_f. In the case of $k_m < 0$, the Meshchersky reactive force $\overrightarrow{\Phi}$ represents the driving force. This fact implies that with decrease of the negative value of parameter k_m, the value of v_f increases and the value of t_f decreases;
- With increase of the value of parameter k_v, the value of speed v_f is decreased, and the time of brachistochronic motion t_f is increased. This is a consequence of the fact that with increase of the value of coefficient k_v the effect of the resistance force $\overrightarrow{F}_w$ strengthens. Also, increase of time t_f directly implies increase of mass m_f. In Fig. 10 it is observable that increase of the value of parameter k_v results in change of the brachistochrone curve concavity and that it has inflection points.

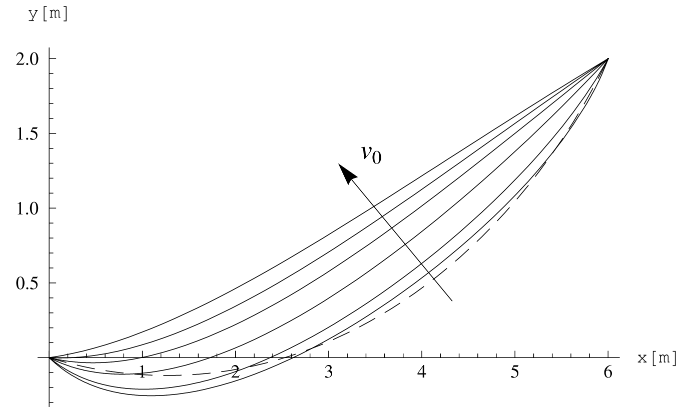

Fig. 7. Brachistochrones for various values of the parameter v_0.

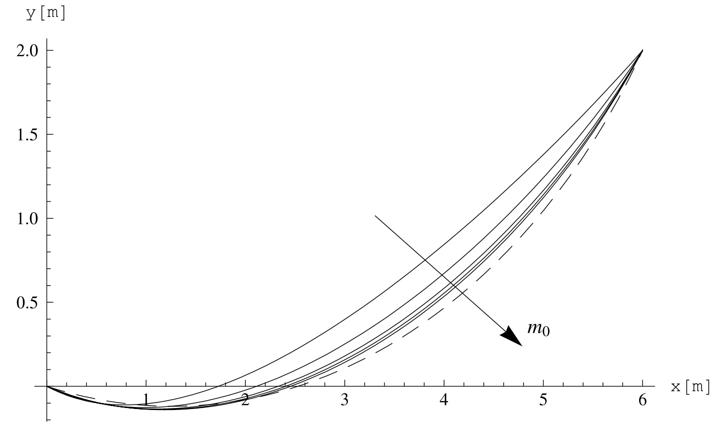

Fig. 8. Brachistochrones for various values of the parameter m_0.

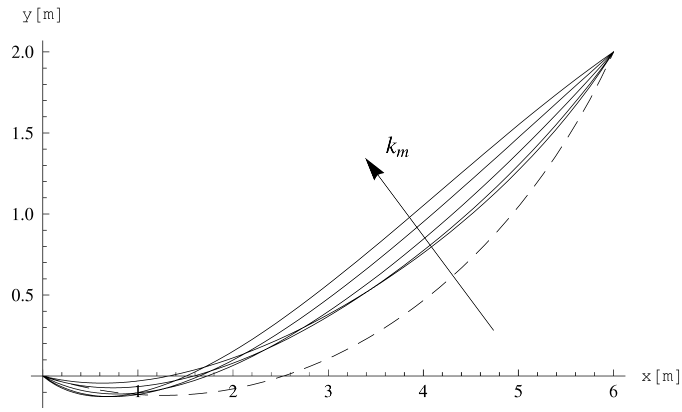

Fig. 9. Brachistochrones for various values of the parameter k_m.

In regard to (12), the optimal control $u(x)$ at inflection points equals zero. Taking into account this fact, the x-coordinate of inflection points can be determined based on the graph of $u(x)$, which is shown in Fig. 11 for the case of $k_v = 0.5$ kg/m.

5. Conclusions

In this paper, the problem of brachistochronic motion of a variable-mass particle has been formulated within the framework of Pontryagin's minimum principle and singular optimum control theory, and a numerical solution to the resulting two-point boundary value problem (for cases of continuous particle mass accretion and depletion) has been

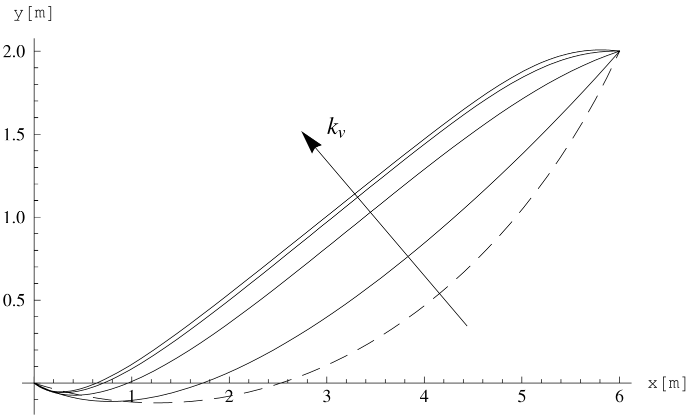

Fig. 10. Brachistochrones for various values of the parameter k_v.

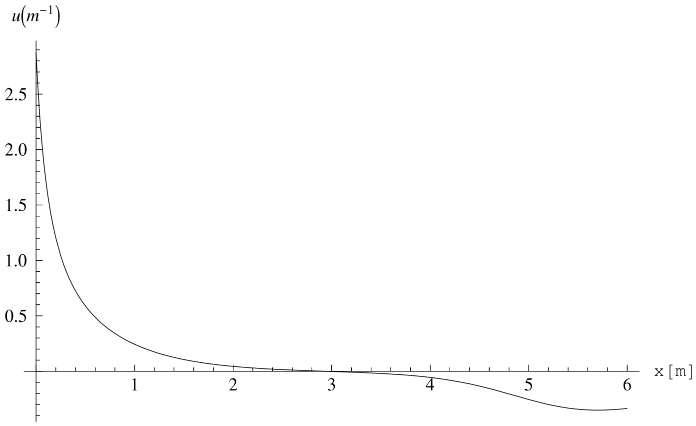

Fig. 11. Optimal control $u = u(x)$ for $k_v = 0.5$ kg/m.

presented. Due to the fact that the solution of the considered problem in the paper is based on fairly general assumptions with respect to active and reactive forces, the cases of the brachistochrone with viscous friction and the brachistochrone in the field of central forces, considered in the papers cited in Section 1, can be observed as special cases of our paper. In regard to this, it is sufficient to take $m(x) = \text{const.}$, $\lambda_m(x) \equiv 0$, $\Psi_m(p, v, m) \equiv 0$, exclude from the further considerations the last equations in the systems (12) and (20), and adapt the structure of the function Ψ_v in accordance with active forces considered in above mentioned papers. By slightly changing the boundary conditions, the Refs. [29,30] can be also covered by our paper. Even in the case of generalizations of the brachistochrone problem where analytical solutions were obtained, there are a certain number of unknown constants in the solutions. These constants must be numerically determined from corresponding relations (see, for example, [11,12]). Taking the above observations into account, it seems that numerical computations are necessary (to some extent) in order to obtain a complete solution to any generalization of the classical brachistochrone problem. This is the reason why in this paper a numerical procedure has been formed, enabling a complete solution to the posed brachistochrone problem and the problems similar to it. The procedure can be also generalized to the brachistochronic motion of a nonholonomic rheonomic material system [34].

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