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Research Article

A Counterexample on a Theorem by Khojasteh, Goodarzi, and Razani

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In the paper by Khojasteh et al. (2010), the authors tried to generalize Branciari's theorem, introducing the new integral type contraction mappings. In this note we give a counterexample on their main statement (Theorem 2.9). Also we give a comment explaining what the mistake in the proof is, and suggesting what conditions might be appropriate in generalizing fixed point results to cone spaces, where the cone is taken from the infinite dimensional space.

1. Introduction

In the paper [1], Branciari proved the following fixed point theorem with integral-type contraction condition.

Theorem 1.1. *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$, and $f : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\int_0^{d(f(x),f(y))} \phi(t)\,\mathrm{d}t \le \alpha \int_0^{d(x,y)} \phi(t)\,\mathrm{d}t, \tag{1.1}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is nonnegative measurable mapping, having finite integral on each compact subset of $[0, +\infty)$ such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)\mathrm{d}t > 0$. Then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

There are many generalizations of fixed point results to the so-called cone metric spaces, introduced by several Russian authors in mid-20th. These spaces are re-introduced by Huang and Zhang [2]. In the same paper, the notion of convergent and Cauchy sequences are given.

Definition 1.2. Let E be a Banach space. By Θ we denote the zero element of E. A subset P of E is called a cone if

(1) P is closed, nonempty, and $P \neq \{\Theta\}$;

(2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

(3) $P \cap (-P) = \{\Theta\}$.

Given a cone $P \subseteq E$, we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ and $x \neq y$, whereas $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

We say that P is a solid cone if and only if $\operatorname{int} P \neq \emptyset$.

Let P be a solid cone in E and let $\leq$ be the corresponding partial ordering on E with respect to P. We say that P is a normal cone if and only if there exists a real number $K > 0$ such that $\Theta \leq x \leq y$ implies

$$\|x\| \leq K\|y\| \tag{1.2}$$

for each $x, y \in P$. The least positive K satisfying (1.2) is called the normal constant of P.

Definition 1.3. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:

(1) $\Theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \Theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.4. Let (X, d) be a solid cone metric space, let $x \in X$, and let (x_n) be a sequence in X. Then

(1) (x_n) converges to x if for every $c \in \operatorname{int} P$ there exists a positive integer N such that for all $n \geq N d(x_n, x) \ll c$. We denote this by $\lim x_n = x$ or $x_n \to x$;

(2) (x_n) is a cone Cauchy sequences if for every $c \in \operatorname{int} P$ there exists a positive integer N such that for all $m, n \geq N d(x_m, x_n) \ll c$;

(3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

In the paper [3] Khojasteh et al. tried to generalize Branciari fixed point result to the cone metric spaces. They introduce the concept of integration along the interval $[a, b] = \{ta + (1 - t)b \mid 0 \leq t \leq 1\} \subseteq P$ as a limit of Cauchy sums.

Definition 1.5 (see [3]). Let P be a normal solid cone, and let $\phi : P \to P$. We say that ϕ is integrable on $[a, b]$ if and only if the following sums:

$$L = \sum_{i=0}^{n-1} \phi(x_i) \|x_i - x_{i+1}\|,$$
$$U = \sum_{i=0}^{n-1} \phi(x_{i+1}) \|x_i - x_{i+1}\| \tag{1.3}$$

converge to the same element of P, where $[x_k, x_{k+1})$ form a partition of $[a, b)$. Clearly, $[a, b)$ stands for $[a, b] \setminus \{b\}$. This element is denoted by

$$\int_a^b \phi \, \mathrm{d}_p. \tag{1.4}$$

We say that ϕ is subadditive if and only if for any $a, b \in P$ there holds

$$\int_\Theta^{a+b} \phi \, \mathrm{d}_p \le \int_\Theta^a \phi \, \mathrm{d}_p + \int_\Theta^b \phi \, \mathrm{d}_p. \tag{1.5}$$

Using this concept, they stated the following statement (Theorem 2.9 in [3]).

Theorem 1.6 (see [3]). *Let (X, d) be a complete cone metric space and let P be a normal cone. Suppose that $\phi : P \to P$ is a nonvanishing map which is subadditive cone integrable on each $[a, b] \subseteq P$ and such that for each $\varepsilon \gg 0$, $\int_0^\varepsilon \phi d_p \gg 0$. If $f : X \to X$ is a map such that for all $x, y \in X$,*

$$\int_\Theta^{d(f(x), f(y))} \phi \, \mathrm{d}_p \le \alpha \int_\Theta^{d(x,y)} \phi \, \mathrm{d}_p, \tag{1.6}$$

for some $\alpha \in (0, 1)$, then f has a unique fixed point in X.

However, the last statement is not true. This will be proved in the next section.

2. Constructing the Counterexample

Consider the Banach space

$$E = C_{\mathbf{R}}[0, 1] = \{x : [0, 1] \longrightarrow \mathbf{R} \mid x \text{ is continuous}\}, \tag{2.1}$$

with the norm $\|x\| = \sup_{0 \le t \le 1} |x(t)|$, and the cone

$$P = \{x \in E \mid \forall t \in [0, 1] \ x(t) \ge 0\}. \tag{2.2}$$

It is obvious that P is a normal solid cone with normal constant equals to 1.

Consider the set

$$X = \{x \in E \mid x(0) = 1, \quad x(1) = 0\}, \tag{2.3}$$

and the mapping $d : X \times X \to P$ given by

$$d(x,y)|_t = d(x,y)(t) = \left|x(t) - y(t)\right|, \quad \forall t \in [0,1] \text{ or more simply } d(x,y) = \left|x - y\right|. \tag{2.4}$$

Proposition 2.1. (a) $\operatorname{int} P = \{x \in P \mid \exists \delta > 0$ *for all* $t \in [0,1]$ $x(t) \geq \delta\}$.

(b) (X,d) *is a cone metric space.*

(c) *A sequence* $x_n \in X$ *is convergent (in* (X,d)*) to* x *if and only if* $\|x_n - x\| \to 0$*. Also* x_n *is Cauchy sequence (in* (X,d)*) if and only if* x_n *is a Cauchy sequence with respect to norm in* $X \subseteq E$*.*

(d) (X,d) *is a complete cone metric space.*

Proof. (a) and (b) obvious.

(c) Let $x_n \to x$ (in (X,d)), and let $\varepsilon > 0$. Then, the function $\varepsilon(t) \equiv \varepsilon \in \operatorname{int} P$, and we have that for all $n \geq n_0$ and for all $t \in [0,1]$ there holds

$$|x_n(t) - x(t)| = d(x_n, x) \leq \varepsilon(t) \equiv \varepsilon. \tag{2.5}$$

Let $\|x_n - x\| \to 0$, and let $c(t) \in \operatorname{int} P$. Then there exists $\delta > 0$ such that for all $t \in [0,1]$, $c(t) \geq \delta$. Also for all $n \geq n_0$ there holds

$$d(x_n, x) = |x_n(t) - x(t)| \leq \frac{\delta}{2}. \tag{2.6}$$

Hence, $c(t) - d(x_n(t),\ x(t)) \geq \delta/2$ implying $d(x_n, x) \ll c$.

The similar argument proves the second part of the statement concerning Cauchy sequences.

(d) The set X can be represented as $X = \Lambda_0^{-1}(\{1\}) \cap \Lambda_1^{-1}(\{0\})$, where $\Lambda_t : E \to \mathbf{R}$ is the bounded linear functional given by $\Lambda_t(x) = x(t)$. Therefore, X is a closed subset of E and hence complete in the norm. By part (c) of this proposition, it implies that X is a complete cone metric space. □

Let $e : [0,1] \to \mathbf{R}$ denote the function identically equal to 1. Consider the mapping $\phi : P \to P$ given by

$$\phi(x) = \frac{1}{\|x\|} \int_0^1 x(t)\, \mathrm{d}t \cdot e, \tag{2.7}$$

for $x \neq \Theta$ and $\phi(\Theta) = \Theta$.

Proposition 2.2. (a)ϕ *is integrable on every segment* $[a,b] \subseteq P$ *and* $\int_{\Theta}^{x} \phi \mathrm{d}_p = \int_0^1 x(t)\,\mathrm{d}t \cdot e$.

(b)ϕ *is a nonvanishing subadditive function such that for all* $\varepsilon \gg 0$ *there holds* $\int_0^{\varepsilon} \phi \mathrm{d}_p \gg 0$.

Proof. (a) The integrability of ϕ on $[a,b] \not\ni \Theta$ follows immediately from its continuity. Further, let x_k be a partition of $[\Theta, x]$. Then $x_j = t_j x$ for some partition t_j of $[0,1]$, and we have

$$\begin{aligned}\lim L &= \lim \sum_{j=0}^{n-1} \phi(t_j x)\|x\|(t_{j+1}-t_j) \\ &= \lim \sum_{j=0}^{n-1} \left(\int_0^1 x(t)\,\mathrm{d}t \cdot e\right)(t_{j+1}-t_j) = \int_0^1 x(t)\,\mathrm{d}t \cdot e,\end{aligned} \tag{2.8}$$

and similarly $\lim U = \int_0^1 x(t)\mathrm{d}t \cdot e$.

(b) Follows from the part (a). □

Proposition 2.3. *Let the space* (X,d) *be defined by* (2.1) *and* (2.3). *Let* $F : E \to E$ *be given by* $(Fx)(t) = x(2t)$ *for* $0 \le t \le 1/2$, *and* $(Fx)(t) = x(1)$, *otherwise, and let* $f = F|_X$.

The space (X,d) *together with the mappings* f *and* ϕ *satisfies all assumptions of Theorem 1.6. On the other hand,* f *has no fixed point.*

Proof. We only have to check the inequality (1.6). Note that for all $z \in X$ and all $t \ge 1/2$ we have $(F(z))(t) = 0$. Also, note that F is a linear mapping, and $F(|x|) = |F(x)|$. Therefore $d(F(x),F(y)) = |F(x)-F(y)| = |F(x-y)| = F(|x-y|) = F(d(x,y))$. Thus (1.6) becomes

$$\int_{\Theta}^{F(d(x,y))} \phi\,\mathrm{d}_p \le \alpha \int_{\Theta}^{d(x,y)} \phi\,\mathrm{d}_p. \tag{2.9}$$

Taking into account Proposition 2.2, part (a), we have for all $z \in P$

$$\begin{aligned}\int_{\Theta}^{F(z)} \phi\,\mathrm{d}_p &= \int_0^1 (F(z))(t)\,\mathrm{d}t \cdot e = \int_0^{1/2} z(2t)\,\mathrm{d}t \cdot e \\ &= \frac{1}{2}\int_0^1 z(s)\,\mathrm{d}s \cdot e = \frac{1}{2}\int_{\Theta}^{z} \phi\,\mathrm{d}_p.\end{aligned} \tag{2.10}$$

Putting $z = d(x,y)$, we obtain

$$\int_{\Theta}^{F(d(x,y))} \phi\,\mathrm{d}_p = \frac{1}{2}\int_{\Theta}^{d(x,y)} \phi\,\mathrm{d}_p, \tag{2.11}$$

which completes the proof of the first statement.

On the other hand, f has no fixed point. Namely, if we suppose that x is a fixed point for f, it means that $x(t) \equiv 0$ for all $t > 1/2$, and moreover $x(t) \equiv 0$ for all $t > 1/4$, and also for all $t > 1/2^n$, by induction. By continuity of x, it follows that $x(0) = 0$ implying $x \notin X$! □

3. A Comment

The mistake in the proof of Theorem 1.6 given in [3] is in the following. The authors from $\int_0^{d(x_{n+1},x_n)} \phi \, \mathrm{d}_p \to 0$ conclude that $d(x_{n+1},x_n) \to 0$ also, which is unjustifiable. The original Branciari's proof [1] deals with one-dimensional integral, and such conclusion is valid due to the implication

$$\varepsilon > 0 \Longrightarrow \int_0^{\varepsilon} \phi(t) \, \mathrm{d}t > 0 \tag{3.1}$$

and the existence of the total ordering on $\mathbf{R}$. However, in infinite dimensional case, such conclusion invokes continuity of the function inverse to $x \mapsto \int_0^x \phi \, \mathrm{d}_p$. Even for the linear mappings this is not always true, but only under additional assumption that initial mapping is bijective. This asserts the well known Banach open mapping theorem. In the absence of some generalization of the open mapping theorem to nonlinear case, it is necessary to include continuity of the inverse function in the assumptions, as it was done in [4].

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