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On the Gauss-Kronrod quadrature formula for a modified weight function of Chebyshev type

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Abstract

In this paper, we consider the Gauss-Kronrod quadrature formulas for a modified Chebyshev weight. Efficient estimates of the error of these Gauss–Kronrod formulae for analytic functions are obtained, using techniques of contour integration that were introduced by Gautschi and Varga (cf. Gautschi and Varga SIAM J. Numer. Anal. **20**, 1170–1186 1983). Some illustrative numerical examples which show both the accuracy of the Gauss–Kronrod formulas and the sharpness of our estimations are displayed. Though for the sake of brevity we restrict ourselves to the first kind Chebyshev weight, a similar analysis may be carried out for the other three Chebyshev type weights; part of the corresponding computations are included in a final appendix.

Keywords Gauss-Kronrod quadrature formulae · Chebyshev weight function · Contour integral representation · Remainder term for analytic functions · Error bound

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Dedicated to Prof. Sotirios E. Notaris on the occasion of his 60th birthday

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1 Introduction

Consider a positive measure $d\sigma$ on a real interval $[a, b]$ having infinitely many points of increase and finite moments of all orders. It is well known that the corresponding monic orthogonal polynomials $\{\pi_n\}$ satisfy a three-term recurrence relation

$$\begin{aligned} \pi_{-1}(t) &= 0, \quad \pi_0(t) = 1, \\ \pi_{k+1}(t) &= (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{1}$$

where $\alpha_k \in \mathbb{R}$, $\beta_k > 0$, and by convention $\beta_0 = \int_a^b d\sigma(t)$. In [11] Gautschi and Li considered a modification of the original measure, for a fixed integer $n \geq 1$, given by

$$d\widehat{\sigma}_n(t) = [\pi_n(t)]^2 \, d\sigma(t) \quad \text{on} \quad [a, b], \tag{2}$$

and studied the corresponding (monic) orthogonal polynomials $\widehat{\pi}_m = \widehat{\pi}_{m,n}$, $m = 0, 1, 2, \ldots$. As pointed out by the authors in [11], this kind of modifications of measures are useful, for instance, when dealing with constrained polynomial least squares approximation (see, e.g., [8]), or to provide additional interpolation points (the zeros of the induced polynomial $\{\widehat{\pi}_{n+1,n}\}$) in the process of extending Lagrange interpolation at the zeros of π_n (see [3]). Taking into account these and other applications, it seems natural to consider the numerical computation of integrals of the form

$$I_\sigma(f) = I(f; \sigma, n) = \int f(t) \, d\widehat{\sigma}_n(t)$$

by means of quadrature formulae; in particular, Gauss type rules are our main subject of interest. It is well known that the zeros and nodes of the Gauss rule can be efficiently computed by means of the eigenvalues and eigenvectors of the related tridiagonal Jacobi matrix, whose entries are given in terms of the above mentioned recursion coefficients. Then, the following problem arises in a natural way: given the recursion coefficients α_k, β_k for $d\sigma$, determine the recursion coefficients $\hat{\alpha}_k, \hat{\beta}_k$ for $d\hat{\sigma}_n$. Unfortunately, in general it is not feasible to get closed analytic expressions of the entries of the Jacobi matrix for the induced measure $d\widehat{\sigma}_n$ in terms of the corresponding for $d\sigma$; in this sense, in [11] a stable numerical algorithm is given. But in the particular case of the well-known four Chebyshev weights $d\sigma^{[i]}$, $i = 1, 2, 3, 4$, where

$$\begin{aligned} d\sigma^{[1]}(t) &= \frac{1}{\sqrt{1-t^2}}\, dt, \qquad d\sigma^{[2]}(t) = \sqrt{1-t^2}\, dt, \\ d\sigma^{[3]}(t) &= \sqrt{\frac{1+t}{1-t}}\, dt, \qquad d\sigma^{[4]}(t) = \sqrt{\frac{1-t}{1+t}}\, dt \end{aligned} \tag{3}$$

the related induced orthogonal polynomials are easily expressible as combinations of Chebyshev polynomials of the first kind T_k, i.e., orthogonal polynomials with respect to the Chebyshev weight $d\sigma = d\sigma^{[1]}$ (see [11, §3]). These results are very useful for the analysis of the error of the related quadrature formulas.

In the present paper, we focus on the first modified Chebyshev measure, namely

$$d\hat{\sigma}_n(t) = d\hat{\sigma}_n^{[1]}(t) = \left[\overset{\circ}{T}_n(t)\right]^2 d\sigma(t), \quad -1 < t < 1, \tag{4}$$

where

$$d\sigma(t) = \frac{1}{\sqrt{1-t^2}}\,dt \quad \text{and} \quad \overset{\circ}{T}_n(t) = 2^{1-n}T_n(t)\,, \tag{5}$$

with $\overset{\circ}{T}_n$ denoting the corresponding nth-degree monic Chebyshev polynomial. As we said above, for this, as well as for the other modified Chebyshev weights, it is feasible to get closed expressions of the entries of the Jacobi tridiagonal matrices in terms of the corresponding for the original Chebyshev ones. These are collected in previous papers as [11, Theorems 3.1–3.7] and [21, Section 2]. Such results will be useful, among other things, for computing the actual (sharp) value of the quadrature error in the numerical examples.

In this paper, we aim to obtain accurate estimates of the error of the Gauss–Kronrod quadrature formulas for analytic integrands related to this modification of the first kind Chebyshev measure; therefore, this partially completes the analysis started in [25], where those estimates were obtained for the ordinary Gauss quadrature formulas. In 1964, A. S. Kronrod, trying to estimate in a feasible way the error of the n-point Gauss-Legendre quadrature formula, developed the now called Gauss-Kronrod quadrature formula for the Legendre measure (cf.[15, 16]). For a general measure $d\sigma$ this formula has the form

$$\int_a^b f(t)d\sigma(t) = \sum_{\nu=1}^{n} W_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} W_\mu^* f(\tau_\mu^*) + R_n(f), \tag{6}$$

where τ_ν are the zeros of π_n, and the τ_μ^*, W_ν, W_μ^* are chosen such that (6) has maximum degree of exactness. It turns out that a necessary and sufficient condition for this to happen is that τ_μ^* be the zeros of the polynomial π_{n+1}^* (see [7, Corollary]), uniquely determined by the orthogonality relations

$$\int_a^b \pi_{n+1}^*(t)t^k\pi_n(t)\,d\sigma(t) = 0, \quad k = 0, 1, \ldots, n. \tag{7}$$

Observe that (7) implies that π_{n+1}^* is a polynomial orthogonal with respect to a variable-sign measure, from which the fact that its zeros be simple and belong to the interval (a, b) is not guaranteed in advance. Polynomials of this kind were considered for the first time by T. J. Stieltjes in 1894, for the Legendre measure $d\sigma(t) = dt$ on $[-1, 1]$. Stieltjes, in a letter to Hermite (see [1, vol 2, pp. 439–441]), conjectured that π_{n+1}^* has $n + 1$ real and simple zeros, all contained in $(-1, 1)$, and interlacing with the zeros of the nth-degree Legendre polynomial. Stieltjes' conjectures were proved by Szegő in 1935 (cf. [31]), not only for the Legendre but also for the Gegenbauer measure $d\sigma(t) = (1-t^2)^{\lambda-1/2}\,dt$ on $[-1, 1]$, when $0 < \lambda \leq 2$. After that, the polynomials π_{n+1}^*, now appropriately called Stieltjes polynomials, have apparently gone unnoticed until Kronrod's papers in 1964 (cf.[15, 16]). The connection between Stieltjes polynomials and Gauss-Kronrod formulae was pointed out by Mysovskih in [20], and independently by Barrucand in [2]. A nice and detailed survey of Kronrod rules in the last 50 years is provided by Notaris [24]. Numerically stable and effective procedures for calculating Gauss-Kronrod formulas are proposed in [17] and [4].

We consider here Gauss-Kronrod quadrature formulas for the modified Chebyshev weight function of the first kind $d\widehat{\sigma}_n = d\widehat{\sigma}_n^{[1]}$, that is,

$$I_\sigma(f) = I_n(f) + R_n(f), \tag{8}$$

where

$$I_\sigma(f) = \int_{-1}^{1} f(t) d\widehat{\sigma}_n^{[1]}(t), \quad I_n(f) = \sum_{\nu=1}^{n} W_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} W_\mu^* f(\tau_\mu^*),$$

under the assumption that all τ_ν, τ_μ^* belong to $[-1, 1]$. Our analysis of the error is based on its well-known representation in terms of an integral contour of a suitable *kernel*; namely, if we use a Gauss-Kronrod rule $I_n(f)$ with $2n+1$ nodes to approximate the value of the integral $I_\sigma(f)$ for a certain positive measure σ (hereafter, we assume the absolute continuity of the measure σ and, hence, that $d\sigma(t) = w(t)\,dt$) on the real interval $[-1, 1]$ and an analytic integrand f in a neighborhood Ω of this interval, the error of quadrature admits the following integral representation (see, e. g., [12])

$$R_n(f) = I_\sigma(f) - I_n(f) = \frac{1}{2\pi i} \oint_\Gamma K_n(z)\, f(z)\, dz\,, \tag{9}$$

where the *kernel* K_n is given by

$$K_n(z) = \frac{\varrho_n(z)}{\pi_n(z)\, \pi_{n+1}^*(z)}\,, \quad \varrho_n(z) = \int_{-1}^{1} \frac{\pi_n(t)\, \pi_{n+1}^*(t)}{z - t}\, w(t)\, dt\,, \tag{10}$$

with π_n denoting, as usual, the nth degree orthogonal polynomial with respect to w, π_{n+1}^* denoting the corresponding Stieltjes polynomial of degree $n+1$ for the modified Chebyshev weight, ϱ_n is the commonly called 2nd kind function associated to the nodal polynomial, and $\Gamma \subset \Omega$ is any closed smooth contour surrounding the real interval $[-1, 1]$. Elliptic contours $\mathcal{E}_\rho$ with foci at the points ± 1 and semi-axes given by $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$, with $\rho > 1$, are often considered as contours of integration, in order to get suitable estimations of the error of quadrature; this is due to the fact that they are the level curves for the conformal function which maps the exterior of $[-1, 1]$ onto the exterior of the unit circle $|z| > 1$ in the complex plane. In this sense, these elliptic level curves admit the expression

$$\mathcal{E}_\rho = \{z \in \mathbb{C} : |\phi(z)| = |z + \sqrt{z^2 - 1}| = \rho\}\,, \tag{11}$$

where $\rho > 1$ and the branch of $\sqrt{z^2 - 1}$ is taken so that $|\phi(z)| > 1$ for $|z| > 1$. On the other hand, the inverse function of ϕ, that is, the well-known Joukowsky transform, given by

$$z = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right),\ z \in \mathbb{C} \setminus [-1, 1]\,,\ |\xi| > 1\,, \tag{12}$$

will also be used in the subsequent sections.

The outline of the current paper is as follows. In Section 2 an explicit expression for the *kernel* (10) related to the induced Chebyshev weight $d\widehat{\sigma}_n^{[1]}$ is provided, which will be useful to get appropriate bounds for the error of the corresponding Gauss-Kronrod rules, which represents the main contribution of the paper. In addition, the accuracy of the obtained bounds is checked by means of some illustrative

numerical examples in Section 3. Finally, and for the sake of completeness, similar computations for the *kernels* corresponding to the other modified Chebyshev measures $d\sigma_n^{[i]}$, $i = 2, 3, 4$, are gathered in the final appendix.

To end this introduction, let us say that the problem of estimating the quadrature error for Gauss–type rules has been thoroughly studied in the literature; see the references [12, 18, 19, 22, 23], and [26–30], to only cite a few. See also [5] for a very recent survey of the error estimates of Gaussian type quadrature formulae for analytic functions on ellipses.

Let us finally point out that the seemingly restricted scope of our analysis is offset, in our opinion, by the extreme sharpness of the estimations shown in Section 3.

2 Error bounds of Gauss–Kronrod rules for the measure $d\widehat{\sigma}_n^{[1]}$

Hereafter, the (monic) orthogonal polynomials relative to the positive measure $d\widehat{\sigma}_n^{[1]}(t) = [\pi_n(t)]^2\, d\sigma^{[1]}(t)$, defined in (2), will be denoted simply by $\widehat{\pi}_{m,n}$, $m = 0, 1, 2, \ldots$. For simplicity, here and in the next section we only consider what may be referred to as the "diagonal" setting, that is, the case where $m = n$ and we simply denote $\widehat{\pi}_n = \widehat{\pi}_{n,n}$; on the other hand, π_{n+1}^* will denote the corresponding Stieltjes polynomial. Our first result gives the explicit expression of the *kernel* K_n in this case.

Lemma 1 *The kernel K_n is given by*

$$K_n(z) = -\frac{\pi(2\xi^{2n}+1)}{2^{2n-2}\xi^{2n-1}(\xi^{4n}-1)(\xi^2-1)}, \tag{13}$$

with ξ given by (12).

Proof By (10) the corresponding *kernel* is given by

$$K_n(z) = \frac{\varrho_n(z)}{\widehat{\pi}_n(z)\pi_{n+1}^*(z)}, \quad z \notin [-1, 1],$$

where

$$\widehat{\pi}_n(t) = \mathring{T}_n(t), \quad \pi_{n+1}^*(t) = (t^2-1)\mathring{U}_{n-1}(t), \quad \mathring{T}_n(t) = \frac{1}{2^{n-1}}T_n(t),$$

with U_k denoting as usual the Chebyshev polynomial of the second kind of degree k, and $\mathring{U}_k$ being the monic one, and

$$\varrho_n(z) = \int_{-1}^{1} \frac{\widehat{\pi}_n(z)\pi_{n+1}^*(z)}{z-t}\frac{\mathring{T}_n^2(t)\,dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{\mathring{T}_n(t)(t^2-1)\mathring{U}_{n-1}(t)}{z-t}\frac{\mathring{T}_n^2(t)\,dt}{\sqrt{1-t^2}},$$

that is,

$$\begin{aligned}
\varrho_n(z) &= \int_{-1}^{1} \frac{(t^2-1)\mathring{U}_{n-1}(t)}{z-t} \frac{\mathring{T}_n^3(t)\,dt}{\sqrt{1-t^2}} \\
&= -\int_0^{\pi} \frac{\cos^3(n\theta)(-\sin^2\theta)\frac{\sin n\theta}{\sin\theta} d\theta}{2^{4n-4}(z-\cos\theta)} \\
&= -\frac{1}{2^{4n-2}} \int_0^{\pi} \frac{(1+\cos 2n\theta)\sin 2n\theta \sin\theta}{z-\cos\theta} d\theta \\
&= -\frac{1}{2^{4n-1}} \int_0^{\pi} \frac{\cos(2n-1)\theta - \cos(2n+1)\theta + \frac{1}{2}(\cos(4n-1)\theta - \cos(4n+1)\theta)}{z-\cos\theta} d\theta \\
&= -\frac{1}{2^{4n-1}} \left(\int_0^{\pi} \frac{\cos(2n-1)\theta}{z-\cos\theta} d\theta - \int_0^{\pi} \frac{\cos(2n+1)\theta}{z-\cos\theta} d\theta \right. \\
&\quad \left. + \frac{1}{2}\int_0^{\pi} \frac{\cos(4n-1)\theta}{z-\cos\theta} d\theta - \frac{1}{2}\int_0^{\pi} \frac{\cos(4n+1)\theta}{z-\cos\theta} d\theta \right).
\end{aligned}$$

Thus, by using the identity (see [12, p. 1176])

$$\int_0^{\pi} \frac{\cos m\theta}{z-\cos\theta} d\theta = \frac{2\pi}{\xi^m(\xi-\xi^{-1})}, \tag{14}$$

we get

$$\begin{aligned}
\varrho_n(z) &= -\frac{\pi}{2^{4n-1}(\xi-\xi^{-1})} \left(\frac{2}{\xi^{2n-1}} - \frac{2}{\xi^{2n+1}} + \frac{1}{\xi^{4n-1}} - \frac{1}{\xi^{4n+1}} \right) \\
&= -\frac{\pi}{2^{4n-1}(\xi-\xi^{-1})} \left(\frac{2(\xi^2-1)}{\xi^{2n+1}} + \frac{\xi^2-1}{\xi^{4n+1}} \right) \\
&= -\frac{\pi(2\xi^{2n}+1)}{2^{4n-1}\xi^{4n}}.
\end{aligned} \tag{15}$$

Next, to compute the denominator of the *kernel* the following representation will be used (see [12, pp. 1176–1177])

$$\mathring{T}_n(z) = \frac{1}{2^{n-1}} T_n(z) = \frac{1}{2^n}\left(\xi^n + \frac{1}{\xi^n}\right), \quad \mathring{U}_{n-1}(z) = \frac{1}{2^{n-1}} U_{n-1}(z) = \frac{1}{2^{n-1}} \frac{\xi^n - \frac{1}{\xi^n}}{\xi - \frac{1}{\xi}}.$$

Therefore, we obtain

$$\begin{aligned}
\pi_{n+1}^*(z) &= (z^2-1)\mathring{U}_{n-1}(z) = \left[\frac{1}{4}\left(\xi + \frac{1}{\xi}\right)^2 - 1\right] \frac{1}{2^{n-1}} \frac{\xi^n - \frac{1}{\xi^n}}{\xi - \frac{1}{\xi}} \\
&= \frac{1}{2^{n+1}}\left(\xi - \frac{1}{\xi}\right)\left(\xi^n - \frac{1}{\xi^n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\pi_{n+1}^*(z)\widehat{\pi}_n(z) &= \frac{1}{2^{n+1}}\left(\xi - \frac{1}{\xi}\right)\left(\xi^n - \frac{1}{\xi^n}\right)\frac{1}{2^n}\left(\xi^n + \frac{1}{\xi^n}\right) \\
&= \frac{1}{2^{2n+1}}\left(\xi - \frac{1}{\xi}\right)\left(\xi^{2n} - \frac{1}{\xi^{2n}}\right).
\end{aligned} \tag{16}$$

Then, the proof of (13) easily follows. □

Now, we are in a position to obtain bounds for the error of the Gauss–Kronrod quadrature formula using (9) and (13). To do it, several methods will be employed.

2.1 L^∞–bound for the error

On the sequel, for a function g and a compact subset E of the complex plane, the L^∞–norm of g on E will be denoted by

$$\|g\|_E = \max_{z\in E} |g(z)|\,.$$

Now, from (9) and taking $\Gamma = \mathcal{E}_\rho$ for certain $\rho > 1$, we easily get that if f is analytic on $\mathcal{E}_\rho$ and its interior,

$$|R_m(f)| \leq \frac{l(\mathcal{E}_\rho)}{2\pi}\,\|K_m\|_{\mathcal{E}_\rho}\,\|f\|_{\mathcal{E}_\rho}\,, \tag{17}$$

where $l(\mathcal{E}_\rho)$ represents the length of the ellipse $\mathcal{E}_\rho$. If we denote by D_ρ the closed interior of $\mathcal{E}_\rho$, define

$$\rho_{max} = \sup\{\rho > 1 \,:\, f \ \text{ is analytic on } \ D_\rho\}\,.$$

Now, set

$$a_j = \frac{\rho^j + \rho^{-j}}{2},\ \ j \in \mathbb{N}\,. \tag{18}$$

Next, we have the following L^∞–bound for the error of the Gauss–Kronrod quadrature formula.

Theorem 1 *The error of the Gauss–Kronrod quadrature formula for* $d\widehat{\sigma}_n^{[1]}$ *is bounded by*

$$r_1(f) = \inf_{1<\rho<\rho_{\max}} \left[\frac{\pi(2\rho^{2n}+1)a_1\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n-2}\rho^{2n-1}(\rho^{4n}-1)(\rho^2-1)} \right], \tag{19}$$

where the expression of a_j *is given in* (18).

Proof From (13) and using polar coordinates and the Joukowsky transform (12), the modulus of the *kernel* in this case may be expressed in the form

$$|K_n(z)| = \frac{\pi\sqrt{4\rho^{4n}+4\rho^{2n}\cos 2n\theta+1}}{2^{2n-1}\cdot\rho^{4n}\sqrt{(a_2-\cos 2\theta)(a_{4n}-\cos 4n\theta)}}, \tag{20}$$

with the a_j given by (18), because

$$K_n(z) = -\frac{\pi(2\xi^{2n}+1)}{2^{2n-2}\xi^{4n}\left(\xi^{2n}-1/\xi^{2n}\right)(\xi-1/\xi)}$$

and

$$|\xi^k - 1/\xi^k| = \sqrt{2}\sqrt{a_{2k}-\cos 2k\theta}, \quad k\in\mathbb{N},$$
$$|2\xi^{2n}+1| = \sqrt{4\rho^{4n}+4\rho^{2n}\cos 2n\theta+1}.$$

Since the numerator and denominator of this expression obviously reach its maximum and minimum, respectively, at $\theta = 0$ for all $\rho > 1$, we can directly state that

$$\max_{\theta\in[0,2\pi]} |K_n(z)| = |K_n(0)| = |K_n(\pi)|\,, \quad \rho > 1.$$

On the other hand, the length of the ellipse can be estimated by (cf. [28])

$$l(\mathcal{E}_\rho) \leqslant 2\pi a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right),$$

and thus, (17) yields the bound (19). □

2.2 Error bounds based on an expansion of the remainder

If f is an analytic function in the interior of $\mathcal{E}_\rho$, for some $\rho > 1$, it admits the expansion

$$f(z) = {\sum_{k=0}^{\infty}}' \alpha_k T_k(z), \tag{21}$$

where α_k are given by

$$\alpha_k = \frac{1}{\pi}\int_{-1}^{1} (1-t^2)^{-1/2} f(t) T_k(t) dt.$$

The prime in the corresponding sum denotes that the first term is taken with the factor $1/2$. The series converges for each z in the interior of $\mathcal{E}_\rho$. In general, the Chebyshev-Fourier coefficients α_k in the expansion are unknown; however, Elliott [6] described a number of ways to estimate or bound them. In particular, under our assumptions the following upper bound will be useful,

$$|\alpha_k| \leq \frac{2}{\rho^k}\|f\|_{\mathcal{E}_\rho}\,. \tag{22}$$

The following result provides the desired bound for the error of quadrature.

Theorem 2 *The following bound for the error of the Gauss–Kronrod quadrature formula based on the expansion of the remainder is obtained for* $d\widehat{\sigma}_n^{[1]}$ *:*

$$r_2(f) = \inf_{1<\rho<\rho_{\max}} \left[\frac{\pi}{4^{n-1}} \cdot \frac{2\rho^{2n}+1}{2\rho^{2n}\left(\rho^{4n}-1\right)} \cdot \|f\|_{\mathcal{E}_\rho}\right]. \tag{23}$$

For the proof of this theorem, we need a result by D. B. Hunter [14, Lemma 5], which is included below, to make the paper self–contained.

Lemma 2 *With* ξ *and* z *as in* (12)*, we have:*

$$\int_{\mathcal{E}_\rho} \xi^{-k} T_j(z) dz = \begin{cases} i\pi, & j = 0, k = 1\,, \\ i\pi/2, & j > 0, k = j+1\,, \\ -i\pi/2, & j > 1, k = j-1\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Proof of Theorem 2 In the current case, the *kernel* is given by $K_n(z) = \frac{\varrho_n(z)}{\widehat{\pi}_n(z)\pi^*_{n+1}(z)}$, $z \notin [-1, 1]$, where we have (see (15))

$$\varrho_n(z) = -\frac{2\pi}{4^{2n}}\left(2\xi^{-2n} + \xi^{-4n}\right),$$

and (see (16))

$$\begin{aligned}\frac{1}{\widehat{\pi}_n(z)\pi^*_{n+1}(z)} &= 2\cdot 4^n\left(\xi^{2n} - \xi^{-2n}\right)^{-1}\left(\xi - \xi^{-1}\right)^{-1}\\ &= 2\cdot 4^n\xi^{-2n-1}\frac{1}{1-\xi^{-4n}}\frac{1}{1-\xi^{-2}}\\ &= 2\cdot 4^n\xi^{-2n-1}\sum_{p=0}^{\infty}\xi^{-4np}\sum_{q=0}^{\infty}\xi^{-2q}\\ &= 2\cdot 4^n\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\xi^{-4np-2q-2n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}K_n(z) &= -\frac{2\pi}{4^{2n}}2\cdot 4^n\left(2\xi^{-2n} + \xi^{-4n}\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\xi^{-4np-2q-2n-1}\\ &= -\frac{\pi}{4^{n-1}}\left[2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\xi^{-4np-2q-4n-1} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\xi^{-4np-2q-6n-1}\right]\\ &= -\frac{\pi}{4^{n-1}}\big[2\xi^{-4n-1} + 2\xi^{-4n-3} + 2\xi^{-4n-5} + \ldots\\ &\quad + \xi^{-6n-1} + \xi^{-6n-3} + \xi^{-6n-5} + \ldots\\ &\quad + 2\xi^{-8n-1} + 2\xi^{-8n-3} + 2\xi^{-8n-5} + \ldots\\ &\quad + \xi^{-10n-1} + \xi^{-10n-3} + \xi^{-10n-5} + \ldots\big].\end{aligned}$$

This way, the following shorter expression for K_n may be written,

$$K_n(z) = -\frac{\pi}{4^{n-1}}\sum_{k=0}^{\infty}\omega_{n,k}\xi^{-4n-k-1}, \tag{24}$$

where we have

$$\begin{aligned}\omega_{n,4kn} &= \omega_{n,4kn+2} = \cdots = \omega_{n,(4k+2)n-2} = 3k+2,\\ \omega_{n,(4k+2)n} &= \omega_{n,(4k+2)n+2} = \cdots = \omega_{n,(4k+4)n-2} = 3k+3,\\ \omega_{n,k} &= 0 \text{ for all other } k \in \mathbb{N}.\end{aligned}$$

The remainder term $R_n(f)$ can be represented in the form

$$R_n(f) = \frac{1}{2^{2n-2}}\sum_{k=0}^{\infty}\alpha_{4n+k}\epsilon_{n,k}, \tag{25}$$

where the coefficients $\epsilon_{n,k}$ are independent on f. Namely, using (21) and (24) in (9) we obtain

$$\begin{aligned} R_n(f) &= \frac{1}{2^{2n-2}} \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} \left({\sum_{k=0}^{\infty}}' \alpha_k T_k(z) \sum_{k=0}^{+\infty} \omega_{n,k} \xi^{-4n-k-1} \right) dz \\ &= \frac{1}{2^{2n-2}} \sum_{k=0}^{+\infty} \left(\frac{1}{2\pi i} {\sum_{j=0}^{+\infty}}' \alpha_j \int_{\mathcal{E}_\rho} T_j(z) \xi^{-4n-k-1} dz \right) \omega_{n,k}. \end{aligned}$$

Applying Lemma 2, this reduces to (25) with

$$\begin{aligned} \epsilon_{n,4kn} &= \frac{1}{2} \text{ for } \mathrm{k} \in \mathbb{N}_0, \\ \epsilon_{n,(4k+2)n} &= \frac{1}{4} \text{ for } \mathrm{k} \in \mathbb{N}_0, \\ \epsilon_{n,l} &= 0 \text{ for all other } l \in \mathbb{N}. \end{aligned}$$

Now we easily reach the following expression for the error of quadrature

$$R_n(f) = -\frac{1}{4^{n-1}} \left(\frac{1}{2} \cdot \sum_{k=0}^{\infty} \alpha_{4nk+4n} + \frac{1}{4} \cdot \sum_{k=0}^{\infty} \alpha_{4kn+6n} \right).$$

Then, inequality (22) yields

$$\begin{aligned} |R_n(f)| &\le \frac{\pi}{4^{n-1}} \cdot \|f\|_{\mathcal{E}_\rho} \cdot \left(\sum_{k=0}^{\infty} \frac{1}{\rho^{4nk+4n}} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{\rho^{4kn+6n}} \right). \\ &= \frac{\pi}{4^{n-1}} \cdot \|f\|_{\mathcal{E}_\rho} \cdot \left(\frac{1}{\rho^{4n}} + \frac{1}{2\rho^{6n}} \right) \sum_{k=0}^{\infty} \frac{1}{\rho^{4nk}} \\ &= \frac{\pi}{4^{n-1}} \cdot \|f\|_{\mathcal{E}_\rho} \cdot \left(\frac{1}{\rho^{4n}} + \frac{1}{2\rho^{6n}} \right) \left(\frac{1}{1-\rho^{-4n}} \right) \\ &= \frac{\pi}{4^{n-1}} \cdot \frac{2\rho^{2n}+1}{2\rho^{2n}\left(\rho^{4n}-1\right)} \cdot \|f\|_{\mathcal{E}_\rho}. \end{aligned}$$

Finally, the bound (23) is easily attained. □

2.3 L^1–bound for the error

From the integral expression (9), the error of quadrature may be bounded in the form

$$R_n(f) \le r_3(f) = \inf_{1<\rho<\rho_{\max}} \left[L^{[1]}(\mathcal{E}_\rho) \cdot \|f\|_{\mathcal{E}_\rho} \right], \tag{26}$$

where

$$L^{[1]}(\mathcal{E}_\rho) = \frac{1}{2\pi} \int_{\mathcal{E}_\rho} |K_n(z)| \, |dz|.$$

Now, we shall prove the following result.

Theorem 3 *In the case of the measure $d\widehat{\sigma}_n^{[1]}$, the error bound (26) takes the form*

$$r_3(f) = \inf_{1<\rho<\rho_{\max}} \left(\frac{\pi}{2^{2n-1}\rho^{2n}} \sqrt{\frac{4\rho^{4n}+5}{\rho^{8n}-1}} \cdot \|f\|_{\mathcal{E}_\rho} \right). \tag{27}$$

Proof From (20), the modulus of the *kernel* is given by

$$|K_n(z)| = \frac{\pi\sqrt{4\rho^{4n}+4\rho^{2n}\cos 2n\theta+1}}{2^{2n-1}\cdot\rho^{4n}\sqrt{(a_2-\cos 2\theta)(a_{4n}-\cos 4n\theta)}}.$$

It is easy to check that $|dz| = (1/\sqrt{2})\cdot\sqrt{a_2-\cos 2\theta}\,d\theta$ (cf. [14]), which yields

$$\begin{aligned} L^{[1]}(\mathcal{E}_\rho) &= \frac{1}{\rho^{4n}2^{2n}\sqrt{2}}\int_0^{2\pi}\sqrt{\frac{4\rho^{4n}+4\rho^{2n}\cos 2n\theta+1}{a_{4n}-\cos 4n\theta}}d\theta \\ &= \frac{1}{\rho^{4n}2^{2n-1}\sqrt{2}}\int_0^{\pi}\sqrt{\frac{4\rho^{4n}+4\rho^{2n}\cos 2n\theta+1}{a_{4n}-\cos 4n\theta}}d\theta. \end{aligned}$$

Applying the Cauchy-Schwarz inequality in L^2 to the last expression, we obtain

$$L^{[1]}(\mathcal{E}_\rho) \le \frac{\sqrt{\int_0^\pi d\theta}}{\rho^{4n}2^{2n-1}\cdot\sqrt{2}}\sqrt{(4\rho^{4n}+1)\cdot I_0 + 4\rho^{2n}\cdot I_1},$$

where, using from [13, 3.613] that

$$\int_0^\pi \frac{\cos mx\,dx}{a^2-2a\cos x+1} = \frac{\pi}{a^m(a^2-1)},$$

we obtain the explicit expressions for the integrals

$$\begin{aligned} I_0 &= \int_0^\pi \frac{d\theta}{a_{4n}-\cos 4n\theta} = \int_0^\pi \frac{d\theta}{\frac{1}{2}\left(\rho^{4n}+\frac{1}{\rho^{4n}}\right)-\cos 4n\theta} = \frac{2\pi\rho^{4n}}{\rho^{8n}-1}, \\ I_1 &= \int_0^\pi \frac{\cos 2n\theta\,d\theta}{a_{4n}-\cos 4n\theta} = \int_0^\pi \frac{\cos 2n\theta\,d\theta}{\frac{1}{2}\left(\rho^{4n}+\frac{1}{\rho^{4n}}\right)-\cos 4n\theta} = \frac{2\pi\rho^{2n}}{\rho^{8n}-1}. \end{aligned}$$

Then, we get

$$\begin{aligned} L^{[1]}(\mathcal{E}_\rho) &\le \frac{\sqrt{\pi}}{\rho^{2n}\sqrt{2}\cdot 2^{2n-1}}\sqrt{(4\rho^{4n}+1)\frac{2\pi}{\rho^{8n}-1}+4\cdot\frac{2\pi}{\rho^{8n}-1}} \\ &= \frac{\pi}{2^{2n-1}\rho^{2n}}\sqrt{\frac{4\rho^{4n}+5}{\rho^{8n}-1}}. \end{aligned}$$

Finally, the bound (27) easily follows. □

3 Numerical results

Throughout this section, several numerical experiments are displayed to illustrate the results in previous section. In this sense, the obtained error bounds $r_1(f)$, $r_2(f)$ and $r_3(f)$ have been tested for the three following characteristic examples (commonly used in the literature on numerical integration):

$$f_0(z) = e^{\omega z^2},\ \omega > 0;\ \ f_1(z) = e^{\cos(\omega z)},\ \omega > 0;\ \ f_2(z) = \frac{e^{e^z}}{(a+z)^k (b+z)^l (c+z)^m},$$

where $a < -1$, $c \le b \le a$ and $k \in \mathbb{N}$, $l,\ m \in \mathbb{N}_0$, and it is easy to check that the following properties are satisfied,

$$\max_{z\in\mathcal{E}_\rho} |f_0(z)| = e^{\omega a_1^2}, \quad \max_{z\in\mathcal{E}_\rho} |f_1(z)| = e^{\cosh \omega b_1},$$

$$\max_{z\in\mathcal{E}_\rho} |f_2(z)| = \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^l |c+a_1|^m},$$

with $a_1 = \frac{\rho+\rho^{-1}}{2}$ and $b_1 = \frac{\rho-\rho^{-1}}{2}$.

It is clear that the functions $f_0(z)$ and $f_1(z)$ are entire, so $\rho_{\max} = \infty$ in both cases. Otherwise, for $f_2(z)$ the condition $a < -1$, $c \le b \le a$ means that the function f is analytic inside the elliptical contour $\mathcal{E}_{\rho_{\max}}$, for a certain $\rho_{\max} > 1$, where $|a| = \frac{1}{2}(\rho_{\max} + \rho_{\max}^{-1})$. We use some values of parameters a, b, c which have been used in literature (see, e. g., [30]); in particular, $a = -1.408333333333333$, $b = -1.892857142857143$, $c = -2.408695652173913$, $k = 1$, $l = 5$, $m = 10$, which means that $\rho_{\max} = 2.4$.

In order to compute the actual (sharp) error bound for the quadrature formula

$$\int_{-1}^{1} f(t) d\widehat{\sigma}_n^{[1]}(t) \approx \sum_{\nu=1}^{n} W_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} W_\mu^* f(\tau_\mu^*), \tag{28}$$

we use [21, Theorem 4.1], which provides explicit formulas for all coefficients W_ν, W_μ^* and nodes τ_ν, τ_μ^*, $\nu = 1, \ldots, n$, $\mu = 1, \ldots, n+1$. To proceed analogously with the other Chebyshev measures $d\sigma_n^{[i]}$, $i = 2, 3, 4$, it is possible to use the numerically stable and effective methods [4, 17] (see also [9] along with [10]).

First of all and though it is a well-known fact that the Gauss–Kronrod quadrature formula is a refinement of the classical Gauss rule (up to the extent that the value given by the former is commonly used to estimate the error of the latter), let us previously include a small table (Table 1) comparing the actual estimations of the error of both quadrature rules for some examples corresponding to the integrand f_0. We denote by "Error GF" and "Error GKF" the errors of the classical Gauss formula and the Gauss–Kronrod rule, respectively. The numerical examples displayed below clearly show that the number of precision digits of the latter is approximately the

Table 1 The values of the actual error of Gauss and Gauss–Kronrod rules for the integrand f_0 and some values of n and ω in the case of $d\widehat{\sigma}_n^{[1]}$

n, ω	Error GF	Error GKF
6, 1	2.596(−9)	6.325(−19)
10, 1	7.833(−18)	7.408(−36)
20, 1	1.060(−41)	1.913(−83)
6, 5	3.702(−4)	1.280(−9)
10, 5	6.473(−10)	5.606(−21)
20, 5	8.020(−27)	1.334(−54)

double of that of the former. The results corresponding to the Gauss rule are taken from [25, Table 4.3].

Now, we are concerned with showing the sharpness of the error estimations (19), (23), and (27). The results are displayed in Tables 2, 3, and 4, where "Error" means the actual (sharp) error and $I_\sigma(f)$ represents the exact value of the integral $\int_{-1}^{1} f(t)\, d\widehat{\sigma}_n(t)$.

Tables 2–4 above show how sharp the bounds of the quadrature error obtained in Section 2 are; namely, the average deviation from the actual value of the error does not exceed one precision digit. At the same time, the high accuracy of the Gauss–Kronrod rules, especially in the case of the entire integrands f_0 and f_1, is clearly shown.

Table 2 The values of the derived bounds $r_1(f_0), r_2(f_0), r_3(f_0)$, compared with the actual error for some values of n, ω

n, ω	$r_1(f_0)$	$r_2(f_0)$	$r_3(f_0)$	Error(f_0)	$I_\sigma(f_0)$
6, 0.1	3.5167(−30)	3.5094(−30)	3.9236(−30)	4.014(−31)	1.614(−3)
8, 0.1	2.2645(−42)	2.2610(−42)	2.5278(−42)	2.243(−43)	1.009(−4)
10, 0.1	5.3086(−55)	5.3019(−55)	5.9277(−55)	4.710(−56)	6.303(−6)
15, 0.1	5.5441(−88)	6.5395(−88)	6.1933(−88)	4.023(−89)	6.156(−9)
20, 0.1	1.9363(−122)	1.9351(−122)	2.053(−122)	1.218(−123)	6.011(−12)
6, 1	5.6540(−18)	5.5346(−18)	6.1879(−18)	6.325(−19)	2.690(−3)
8, 1	3.6164(−26)	3.5596(−26)	3.9798(−26)	3.531(−27)	1.681(−4)
10, 1	8.4563(−35)	8.3492(−35)	9.3347(−35)	7.408(−36)	1.051(−5)
15, 1	8.7864(−58)	8.7137(−58)	9.7422(−58)	6.322(−59)	1.026(−8)
20, 1	3.0626(−82)	3.0433(−82)	3.4025(−82)	1.913(−83)	1.002(−11)
6, 5	1.2657(−8)	1.1307(−8)	1.2642(−8)	1.280(−9)	6.160(−2)
8, 5	4.7840(−14)	4.4080(−14)	4.9283(−14)	4.347(−15)	3.843(−3)
10, 5	6.7644(−20)	6.3353(−20)	7.0831(−20)	5.606(−21)	2.402(−4)
15, 5	6.5739(−36)	6.2973(−36)	7.0406(−36)	4.566(−37)	2.345(−7)
20, 5	2.1975(−53)	2.1294(−53)	2.3807(−53)	1.334(−54)	2.290(−10)

Table 3 The values of the derived bounds $r_1(f_1), r_2(f_1), r_3(f_1)$, compared with the actual error for some values of n, ω

n, ω	$r_1(f_1)$	$r_2(f_1)$	$r_3(f_1)$	$Error(f_1)$	$I_\sigma(f_1)$
5, 0.1	4.8294(−34)	4.8277(−34)	5.3976(−34)	4.386(−35)	1.664(−2)
10, 0.1	1.4537(−72)	1.4534(−72)	1.6249(−72)	8.832(−74)	1.625(−5)
15, 0.1	3.5576(−112)	3.5569(−112)	3.9768(−112)	1.709(−113)	1.587(−8)
20, 0.1	1.9294(−152)	1.9290(−152)	2.362(−152)	7.846(−154)	1.549(−11)
5, 1	2.5892(−14)	2.5029(−14)	2.7983(−14)	2.251(−15)	1.336(−2)
10, 1	5.8519(−33)	5.7125(−33)	6.3868(−33)	3.440(−34)	1.305(−5)
15, 1	1.1319(−52)	1.1093(−52)	1.2403(−52)	5.270(−54)	1.274(−8)
20, 1	4.9479(−73)	4.8619(−73)	5.4357(−73)	1.961(−74)	1.244(−11)
5, 5	4.2332(−4)	2.1976(−4)	2.4355(−4)	1.537(−5)	5.807(−3)
10, 5	3.7490(−11)	2.3566(−11)	2.6347(−11)	1.107(−12)	5.993(−6)
15, 5	6.4478(−19)	4.3223(−19)	4.8325(−19)	1.672(−20)	5.845(−9)
20, 5	4.4589(−27)	3.1462(−27)	3.5176(−27)	9.235(−29)	5.708(−12)

Appendix. Computing the kernel for the other modified Chebyshev weights

In this appendix the computations for the *kernels* corresponding to the modifications of the other Chebyshev measures $d\sigma^{[i]}$, $i = 2, 3, 4$, given in (3) are gathered. Let us first recall the expression of the corresponding monic Chebyshev polynomials,

$$\begin{aligned} \pi_n^{[2]}(t) &= \overset{\circ}{U}_n(t) = 2^{-n} U_n(t), \\ \pi_n^{[3]}(t) &= \overset{\circ}{V}_n(t) = 2^{-n} V_n(t), \qquad \pi_n^{[4]}(t) = \overset{\circ}{W}_n(t) = 2^{-n} W_n(t) \end{aligned} \tag{29}$$

Table 4 The values of the derived bounds $r_1(f_2), r_2(f_2), r_3(f_2)$, compared with the actual error for some values of n

n	$r_1(f_2)$	$r_2(f_2)$	$r_3(f_2)$	$Error(f_2)$	$I_\sigma(f_2)$
6	9.1674(−8)	7.0078(−8)	7.8345(−8)	2.874(−9)	3.240(−4)
10	1.0317(−15)	8.2467(−16)	9.2201(−16)	1.607(−17)	1.241(−6)
15	5.0823(−26)	4.1125(−26)	4.5979(−26)	4.045(−28)	1.212(−9)
20	1.8773(−36)	1.5255(−36)	1.7056(−36)	9.828(−39)	1.183(−12)
25	6.1335(−47)	4.9943(−47)	5.5838(−47)	2.388(−49)	1.155(−15)
30	1.8734(−57)	1.5274(−57)	1.7077(−58)	5.800(−60)	1.128(−18)
40	1.5589(−78)	1.2729(−78)	1.4231(−78)	3.423(−81)	1.076(−24)

and their well–known trigonometric representations

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$
$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \qquad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}.$$

Finally, taking into account the following connection between the orthogonal polynomials corresponding to the modified Chebyshev measures of the third and fourth kind, namely,

$$\hat{\pi}_{m,n}^{[4]}(t) = (-1)^m\, \hat{\pi}_{m,n}^{[3]}(-t), \quad m = 0, 1, 2, \ldots,$$

cf. [11, (3.15)] and [21, (2.21)], for $n \geq 1$, the results corresponding to the modified measure $d\sigma_n^{[4]}$ may be easily obtained from the corresponding for $d\sigma_n^{[3]}$. Thus, in this appendix we focus on the cases $i = 2, 3$.

Let us mention that throughout this appendix we continue dealing with the diagonal setting $m = n$ and using the abbreviate notation for the corresponding orthogonal and Stieltjes polynomials for the modified measures, that is, $\widehat{\pi}_n^{[i]}$ and $\pi_{n+1}^{[i]*}$, respectively.

As we will see below, after computing the kernels our main finding will be that the argument θ of the point where each of these kernels attains its maximum modulus remains constant for ρ is big enough; e.g., there exists some $\rho^* > 0$ such that for $\rho \geq \rho^*$, the argument of the extremum will be $\theta = \theta_0 =$ const.

Explicit expressions for the *kernel* $K_n^{[2]}(z)$ In the case of

$$d\widehat{\sigma}_n^{[2]}(t) = \mathring{U}_n^2(t)\sqrt{1-t^2}\,dt$$

the *kernel* is given by

$$K_n^{[2]}(z) = \frac{\varrho_n^{[2]}(z)}{\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z)}, \quad z \notin [-1, 1],$$

where $\widehat{\pi}_n^{[2]}(z) = \mathring{T}_n(z) = \frac{1}{2^{n-1}}T_n(z)$ and, thus, $\pi_n^{[2]}(\cos\theta) = \frac{1}{2^{n-1}}\cos n\theta$.

By introducing the substitution $z = \cos\theta$, $\pi_{n+1}^{[2]*}(z)$ can be expressed in the form

$$\pi_{n+1}^{[2]*}(\cos\theta) = \frac{1}{2^n}\left(\cos(n+1)\theta - \frac{1}{2}\cos(n-1)\theta - \frac{1}{2^2}\cos(n-3)\theta - \ldots \frac{1}{2^{\frac{n}{2}}}\cos\theta\right),$$

if n is even [21, (4.23)]. Then we have,

$$\begin{aligned}\varrho_n^{[2]}(\cos\theta) &= \frac{1}{2^{3n-1}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(\cos\theta)\cos n\theta\sin^2(n+1)\theta}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{3n-1}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(\cos\theta)\left[\frac{1}{2}\cos n\theta-\frac{1}{4}\cos(3n+2)\theta-\frac{1}{4}\cos(n+2)\theta\right]}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{3n}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(cos\theta)\cos n\theta}{z-\cos\theta}d\theta-\frac{1}{2^{3n+1}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(\cos\theta)\cos(3n+2)\theta}{z-\cos\theta}d\theta\\ &\quad-\frac{1}{2^{3n+1}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(\cos\theta)\cos(n+2)\theta}{z-\cos\theta}d\theta = J_1-J_2-J_3,\end{aligned}$$

where

$$\begin{aligned}J_1 &= \frac{1}{2^{3n}}\int_0^\pi \frac{\pi_{n+1}^{[2]*}(\cos\theta)\cos n\theta}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n}}\int_0^\pi \frac{\left[\cos(n+1)\theta-\frac{1}{2}\cos(n-1)\theta-\ldots\frac{1}{2^{n/2}}\cos\theta\right]\cos n\theta}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n+1}}\int_0^\pi \frac{\cos(2n+1)\theta+\cos\theta-\frac{1}{2}\left(\cos(2n-1)\theta+\cos\theta\right)-\ldots\frac{1}{2^{n/2}}\left(\cos(n+1)\theta+\cos(n-1)\theta\right)}{z-\cos\theta}d\theta.\end{aligned}$$

By using again (14), the above integral takes its final form

$$\begin{aligned}J_1 &= \frac{1}{2^{4n}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+1}}+\frac{1}{\xi}-\frac{1}{2}\left(\frac{1}{\xi^{2n-1}}+\frac{1}{\xi}\right)-\frac{1}{2^2}\left(\frac{1}{\xi^{2n-3}}+\frac{1}{\xi^3}\right)-\cdots\right.\\ &\quad\left.-\frac{1}{2^{n/2}}\left(\frac{1}{\xi^{n+1}}+\frac{1}{\xi^{n-1}}\right)\right]\\ &= \frac{1}{2^{4n}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+1}}+\frac{1}{\xi}-\sum_{l=0}^{\frac{n-2}{2}}\left(\frac{1}{2}\right)^{l+1}\left(\xi^{-2n+2l+1}+\xi^{-2l-1}\right)\right].\end{aligned}$$

By applying similar techniques, J_2 and J_3 are given by

$$J_2 = \frac{1}{2^{4n+1}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{4n+3}}+\frac{1}{\xi^{2n-1}}-\sum_{l=0}^{\frac{n-2}{2}}\left(\frac{1}{2}\right)^{l+1}\left(\xi^{-4n+2l-1}+\xi^{-2n-2l-3}\right)\right],$$

$$J_3 = \frac{1}{2^{4n+1}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+3}}+\frac{1}{\xi}-\sum_{l=1}^{\frac{n}{2}}\left(\frac{1}{2}\right)^{l}\left(\xi^{-2n+2l-3}+\xi^{-2l-1}\right)\right].$$

On the other hand, the denominator $\left(\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z)\right)$ is given by

$$\begin{aligned}\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z) &= \frac{1}{2^n}\left(\xi^n+\frac{1}{\xi^n}\right)\left[\mathring{T}_{n+1}(z)-\frac{1}{8}\mathring{T}_{n-1}(z)-\frac{1}{8^2}\mathring{T}_{n-3}(z)-\cdots-\frac{1}{8^{\frac{n}{2}}}\mathring{T}_1(z)\right]\\ &= \frac{1}{2^{2n+1}}\left(\xi^n+\frac{1}{\xi^n}\right)\left[\xi^{n+1}+\frac{1}{\xi^{n+1}}-\frac{1}{2}\left(\xi^{n-1}+\frac{1}{\xi^{n-1}}\right)-\cdots-\frac{1}{2^{\frac{n}{2}}}\left(\xi+\frac{1}{\xi}\right)\right].\end{aligned}\tag{30}$$

Therefore, the *kernel* may be expressed now as

$$K_n^{[2]}(z) = \frac{\varrho_n^{[2]}(z)}{\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z)} = \frac{J_1 - J_2 - J_3}{\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z)}. \tag{31}$$

As announced above, now we claim that the argument θ of the extremum point of a kernel stabilizes for ρ sufficiently large. Although statements of this kind are clearly false in general, in our cases they are justified by a simple result which was shown in the recent survey paper (cf. [5, Theorem 4.1]). To make the paper self-contained, the statement of this result is included.

Theorem 4 *Let* $Q(\rho,\theta) = \sum_{i=0}^{n} q_i(\theta)\rho^{n-i}$ *and* $R(\rho,\theta) = \sum_{i=0}^{m} r_i(\theta)\rho^{m-i}$ *be continuous functions in two variables that are polynomials in* ρ. *Assume that* $R(\rho,\theta) > 0$ *in the whole region* $\rho > K$ *and consider the function* $f(\rho,\theta) = \frac{Q(\rho,\theta)}{R(\rho,\theta)}$, *and denote by* $p_0(\theta)$ *the leading coefficient of* $P(\rho,\theta)$, *as a polynomial in* ρ, *such that* $f(\rho,\theta_0) - f(\rho,\theta) = \frac{P(\rho,\theta)}{R(\rho,\theta)R(\rho,\theta_0)}$, *with*

$$P(\rho,\theta) = Q(\rho,\theta_0)R(\rho,\theta) - Q(\rho,\theta)R(\rho,\theta_0), \tag{32}$$

and a certain value θ_0 *of* θ.

If the following properties hold:

(i) $p_0(\theta) > 0$ *for* $\theta \in [\alpha,\beta] \setminus \{\theta_0\}$, *where* $p_0(\theta)$ *is the leading coefficient of* $P(\rho,\theta)$ *(see (32) above) as a polynomial in* ρ, *and*

(ii) $q_i(\theta) - q_i(\theta_0) = O(p_0(\theta))$ *and* $r_i(\theta) - r_i(\theta_0) = O(p_0(\theta))$ *for* θ *in a neighborhood of* θ_0, *for each* $i = 1,\ldots,n$,

then there is a constant ρ^* *such that for each* $\rho \geq \rho^*$ *we have* $\max_{\alpha\leq\theta\leq\beta} f(\rho,\theta) = f(\rho,\theta_0)$.

Indeed, the expression above for $f(\rho,\theta_0) - f(\rho,\theta) = \frac{P(\rho,\theta)}{R(\rho,\theta)R(\rho,\theta_0)}$ shows that it suffices to prove that $P(\rho,\theta)$ is positive for all $\theta \neq \theta_0$, whenever ρ is large enough. For the complete proof see [5].

Our aim now is to apply Theorem 4 to the kernel in (31).

After a little calculation we obtain

$$J_1 - J_2 - J_3 = \frac{1}{2^{4n}} \frac{\pi}{\xi - \xi^{-1}} \frac{1}{\xi^{4n+3}} \cdot A,$$

where

$$\begin{aligned} A = {} & \frac{1}{2}\xi^{4n+2} - \frac{1}{2}\xi^{2n+4} + \xi^{2n+2} - \frac{1}{2}\xi^{2n} - \frac{1}{2} \\ & + \xi^{n+1}(\xi^{2n} - 2\xi^{2n+2} + 1) \sum_{l=0}^{\frac{n-2}{2}} \left(\frac{1}{2}\right)^{l+2} \left(\xi^{-n+2l+1} + \xi^{n-2l-1}\right). \end{aligned}$$

After expanding the sum above and simplifying, we obtain

$$A = -\left(\frac{1}{2}\right)^{\frac{n+2}{2}} \xi^{3n+2} - 3\left(\frac{1}{2}\right)^{\frac{n+2}{2}} \xi^{3n} - 3\left(\frac{1}{2}\right)^{\frac{n}{2}} \xi^{3n-2} - 3\left(\frac{1}{2}\right)^{\frac{n-2}{2}} \xi^{3n-4} + \ldots,$$

and so

$$J_1 - J_2 - J_3 = -\frac{1}{2^{4n+\frac{n}{2}+1}} \frac{\pi}{\xi - \xi^{-1}} \frac{1}{\xi^{4n+3}} \cdot A_1, \tag{33}$$

where

$$A_1 = \sum_{i=0}^{\frac{3n+2}{2}} c_i \xi^{2i},$$

and

$$c_{\frac{3n+2}{2}} = 1, \quad c_{\frac{3n}{2}} = 3, \quad c_{\frac{3n-2}{2}} = 6, \quad c_{\frac{3n-4}{2}} = 12, \ldots . \tag{34}$$

Now, in a straightforward way we can compute, in view of $\xi = \rho e^{\mathrm{i}\theta} = \rho(\cos\theta + \mathrm{i}\cos\theta)$,

$$\begin{aligned}
Q(\rho,\theta) \equiv |A_1|^2 &= \left| \sum_{i=0}^{\frac{3n+2}{2}} c_i \xi^{2i} \right|^2 = \left| c_0 + c_1\xi^2 + c_2\xi^4 + \cdots + c_{\frac{3n+2}{2}} \xi^{3n+2} \right|^2 \\
&= \left| c_0 + c_1(\rho^2\cos 2\theta + \mathrm{i}\rho^2 \sin 2\theta) + \cdots + c_{\frac{3n+2}{2}} (\rho^{3n+2}\cos(3n+2)\theta + \mathrm{i}\rho^{3n+2}\sin(3n+2)\theta) \right|^2 \\
&= \Big| c_0 + c_1\rho^2\cos 2\theta + c_2\rho^4\cos 4\theta + \cdots + c_{\frac{3n+2}{2}}\rho^{3n+2}\cos(3n+2)\theta \\
&\quad + \mathrm{i}\left(c_1\rho^2\sin 2\theta + c_2\rho^4\sin 4\theta + \cdots + c_{\frac{3n+2}{2}}\rho^{3n+2}\sin(3n+2)\theta \right) \Big|^2 \\
&= \left(c_0 + c_1\rho^2\cos 2\theta + c_2\rho^4\cos 4\theta + \cdots + c_{\frac{3n+2}{2}}\rho^{3n+2}\cos(3n+2)\theta \right)^2 \\
&\quad + \left(c_1\rho^2\sin 2\theta + c_2\rho^4\sin 4\theta + \cdots + c_{\frac{3n+2}{2}}\rho^{3n+2}\sin(3n+2)\theta \right)^2 \\
&= c_0^2 + c_1^2\rho^4 + c_2^2\rho^8 + \cdots + c_{\frac{3n+2}{2}}^2 \rho^{6n+4} \\
&\quad + 2\sum_{i=0}^{\frac{3n}{2}} \sum_{j=i+1}^{\frac{3n+2}{2}} c_i c_j \rho^{2i+2j} \cos 2i\theta \cos 2j\theta \\
&\quad + 2\sum_{i=1}^{\frac{3n}{2}} \sum_{j=i+1}^{\frac{3n+2}{2}} c_i c_j \rho^{2i+2j} \sin 2i\theta \sin 2j\theta \\
&= \sum_{i=0}^{\frac{3n+2}{2}} c_i^2 \rho^{4i} + 2c_0 \sum_{j=1}^{\frac{3n+2}{2}} c_j \rho^{2j} \cos 2j\theta + 2\sum_{i=1}^{\frac{3n}{2}} \sum_{j=i+1}^{\frac{3n+2}{2}} c_i c_j \rho^{2i+2j} \cos 2(i-j)\theta.
\end{aligned} \tag{35}$$

Furthermore, from (30) we have that

$$\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z) = \frac{1}{2^{2n+1}\xi^{2n+1}} \left(\xi^{2n}+1\right)\left(\xi^{2n+2} + 1 - \frac{1}{2}\left(\xi^{2n}+\xi^2\right) - \cdots - \frac{1}{2^{\frac{n}{2}}}\left(\xi^{n+2}+\xi^n\right)\right),$$

i.e.,

$$\widehat{\pi}_n^{[2]}(z)\pi_{n+1}^{[2]*}(z) = \frac{1}{2^{2n+1}\xi^{2n+1}} \cdot B, \tag{36}$$

where

$$B = \sum_{i=0}^{2n+1} d_i \xi^{2i},$$

and

$$d_0 = d_{2n+1} = 1,\ d_1 = d_{2n} = -\frac{1}{2},\ d_2 = d_{2n-1} = -\frac{1}{4},\ d_3 = d_{2n-2} = -\frac{1}{8}, \ldots. \tag{37}$$

Now, in analogous way as in (35), we have

$$\begin{aligned} R(\rho,\theta) \equiv |B|^2 &= \left|\sum_{i=0}^{2n+1} d_i \xi^{2i}\right|^2 = \left|d_0 + d_1\xi^2 + c_2\xi^4 + \cdots + d_{2n+1}\xi^{4n+2}\right|^2 \\ &= \sum_{i=0}^{2n+1} d_i^2\rho^{4i} + 2d_0\sum_{j=1}^{2n+1} d_j\rho^{2j}\cos 2j\theta + 2\sum_{i=1}^{2n}\sum_{j=i+1}^{2n+1} d_i d_j \rho^{2i+2j}\cos 2(i-j)\theta. \end{aligned} \tag{38}$$

Therefore, we are looking for the maximum of (see (31); (33), (36))

$$\left|K_n^{[2]}(z)\right|^2 = \frac{\pi^2}{2^{5n}\rho^{4n+4}} \cdot \frac{1}{\left|\xi - \xi^{-1}\right|^2} \cdot \frac{Q(\rho,\theta)}{R(\rho,\theta)},$$

where $Q(\rho,\theta)$, $R(\rho,\theta)$ are given by (35), (38), respectively.

Since numerical results performed by us clearly show that $\left|K_n^{[2]}(z)\right|$ attains its maximum value at $\theta = 0$ (also at $\theta = \pi$) for ρ large enough, we are going to apply Theorem 4 with $\theta_0 = 0$.

Since $\frac{1}{\left|\xi-\xi^{-1}\right|^2}$ attains its maximum at $\theta = 0$ (see the second equality after (20), with $k = 1$), it remains to prove that $\frac{Q(\rho,\theta)}{R(\rho,\theta)}$ attains its maximum at $\theta = 0$ for ρ large enough, i.e., that $P(\rho,\theta)$ in (32) is positive for all $\theta \neq 0$, whenever ρ is large enough. On the basis of (35), (38), we calculate $P(\rho,\theta)$ in (32),

$$\begin{aligned} P(\rho,\theta) &= Q(\rho,\theta_0)R(\rho,\theta) - Q(\rho,\theta)R(\rho,\theta_0) \\ &= \Big[c_0^2 + c_1^2\rho^4 + c_2^2\rho^8 + \cdots + c_{\frac{3n+2}{2}}^2\rho^{6n+4} \\ &\quad + \left(2c_{\frac{3n}{2}}c_{\frac{3n+2}{2}}\cos 2\theta_0\rho^{6n+2} + \ldots\right) + \left(2c_0c_{\frac{3n+2}{2}}\cos(3n+2)\theta_0\rho^{3n+2} + \ldots\right)\Big] \\ &\quad \times \Big[d_0^2 + d_1^2\rho^4 + d_2^2\rho^8 + \cdots + d_{2n+1}^2\rho^{8n+4} \\ &\quad + \left(2d_{2n}d_{2n+1}\cos 2\theta\rho^{8n+2} + \ldots\right) + \left(2d_0d_{2n+1}\cos(4n+2)\theta\rho^{4n+2} + \ldots\right)\Big] \\ &\quad - \Big[c_0^2 + c_1^2\rho^4 + c_2^2\rho^8 + \cdots + c_{\frac{3n+2}{2}}^2\rho^{6n+4} \\ &\quad + \left(2c_{\frac{3n}{2}}c_{\frac{3n+2}{2}}\cos 2\theta\rho^{6n+2} + \ldots\right) + \left(2c_0c_{\frac{3n+2}{2}}\cos(3n+2)\theta\rho^{3n+2} + \ldots\right)\Big] \\ &\quad \times \Big[d_0^2 + d_1^2\rho^4 + d_2^2\rho^8 + \cdots + d_{2n+1}^2\rho^{8n+4} \\ &\quad + \left(2d_{2n}d_{2n+1}\cos 2\theta_0\rho^{8n+2} + \ldots\right) + \left(2d_0d_{2n+1}\cos(4n+2)\theta_0\rho^{4n+2} + \ldots\right)\Big] \\ &= 2c_{\frac{3n+2}{2}}^2 d_{2n}d_{2n+1}\cos 2\theta\rho^{14n+6} + 2c_{\frac{3n}{2}}c_{\frac{3n+2}{2}}\cos 2\theta_0 d_{2n+1}^2\rho^{14n+6} \\ &\quad -2c_{\frac{3n+2}{2}}^2 d_{2n}d_{2n+1}\cos 2\theta_0\rho^{14n+6} - 2c_{\frac{3n}{2}}c_{\frac{3n+2}{2}}\cos 2\theta d_{2n+1}^2\rho^{14n+6} + \ldots. \end{aligned}$$

For $\theta_0 = 0$, we have

$$\begin{aligned} P(\rho,\theta) &= Q(\rho,0)R(\rho,\theta) - Q(\rho,\theta)R(\rho,0) \\ &= \left(-2c^2_{\frac{3n+2}{2}} d_{2n} d_{2n+1}(1-\cos 2\theta) + 2c_{\frac{3n}{2}} c_{\frac{3n+2}{2}} d^2_{2n+1}(1-\cos 2\theta)\right)\rho^{14n+6} + \dots . \end{aligned}$$

So, we obtain the leading coefficient (the coefficient of ρ^{14n+6}) in $P(\rho,\theta)$,

$$p_0(\theta) = 2c_{\frac{3n+2}{2}} d_{2n+1}\left(c_{\frac{3n}{2}} d_{2n+1} - c_{\frac{3n+2}{2}} d_{2n}\right)(1-\cos 2\theta),$$

what, by (34), (37), reduces to

$$p_0(\theta) = 7(1-\cos 2\theta) > 0, \quad \theta \neq 0\ (\theta \neq \pi), \tag{39}$$

so the condition (i) of Theorem 4 is fulfilled.

All the coefficients $q_i(\theta) - q_i(0)$ of $Q(\rho,\theta)$, and $r_i(\theta) - r_i(0)$ of $R(\rho,\theta)$, are sums with finite many summands of the form $\eta_m(1-\cos 2m\theta)$, $m \in \mathbb{N}$, $\eta_m \in \mathbb{R}$, and are thus $O(1-\cos 2\theta)$, $\theta \to 0$, so the condition (ii) of Theorem 4 is fulfilled as well, since (cf. [12, p. 1177])

$$1 - \cos 2m\theta = 2\sin^2 m\theta \le 2m^2 \sin^2\theta = m^2(1-\cos 2\theta). \tag{40}$$

In this way, we have proved the next corollary of Theorem 4.

Corollary 1 *There exists some* ρ^{**} (> 1) *such that*

$$\max_{\theta\in[0,2\pi]} \left|K_n^{[2]}(z)\right| = \left|K_n^{[2]}(0)\right| = \left|K_n^{[2]}(\pi)\right|, \quad \rho > \rho^{**}.$$

Remark 1 We shall illustrate the above assertion about the coefficients of $q_i(\theta) - q_i(0)$ of Q and $r_i(\theta) - r_i(0)$ of R by two examples. Namely, using (34), (37), (39), (40), we have

$$\begin{aligned} |q_{6n}(\theta) - q_{6n}(0)| &= \left|2c_{\frac{3n-2}{2}} c_{\frac{3n+2}{2}}(\cos 4\theta - 1)\right| \le \frac{48}{7} p_0(\theta), \\ |r_6(\theta) - r_6(0)| &= |2d_0 d_3(\cos 6\theta - 1) + 2d_1 d_2(\cos 2\theta - 1)| \\ &\le 2\cdot 9|d_0|\cdot|d_3|(1-\cos 2\theta) + 2|d_1|\cdot|d_2|(1-\cos 2\theta) = \frac{5}{14} p_0(\theta). \end{aligned}$$

Remark 2 As one of the reviewers pointed out, our former proof of Corollary 1 (and Corollary 2 below) was based on [25, Lemma 5.1], which was formulated for more general cases than those dealt with in that paper. Though that proof, as presented in [25, Lemma 5.1], was not totally correct, further results in [25], in particular Theorem 3.1 there are correct. Those proofs may be easily fixed by using Theorem 4 (cf. [5, Theorem 4.1]), in an analogous way to that followed in the proof of Corollary 1 in the current paper.

Explicit expressions for the *kernel* $K_n^{[3]}(z)$ In the case of

$$d\widehat{\sigma}_n^{[3]}(t) = \mathring{V}_n^2(t)\sqrt{\frac{1+t}{1-t}}\,dt$$

the *kernel* is given by

$$K_n^{[3]}(z) = \frac{\varrho_n^{[3]}(z)}{\widehat{\pi}_n^{[3]}(z)\pi_{n+1}^{[3]*}(z)}, \quad z \notin [-1, 1],$$

where $\widehat{\pi}_n^{[3]}(z) = \mathring{T}_n(z) = \frac{1}{2^{n-1}} T_n(z)$ and, thus, $\pi_n^{[3]}(\cos\theta) = \frac{1}{2^{n-1}} \cos n\theta$.

As above, the substitution $z = \cos\theta$ allows to express polynomial $\pi_{n+1}^{[3]*}$ in the form (cf. [21, (4.38)]):

$$\begin{aligned}&\pi_{n+1}^{[3]*}(\cos\theta)\\ &= \frac{1}{2^n}\left[\cos(n+1)\theta - \frac{1}{2}\cos n\theta - \frac{3}{2^2}\cos(n-1)\theta + \cdots + (-1)^{n-1}\frac{3}{2^n}\cos\theta + (-1)^n\frac{3}{2^{n+2}}\right].\end{aligned}$$

Now we have

$$\begin{aligned}\varrho_n^{[3]}(\cos\theta) &= \frac{1}{2^{3n-1}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(cos\theta)\cos n\theta\big(1+\cos(2n+1)\theta\big)}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n-1}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(\cos\theta)\big[\cos n\theta + \frac{1}{2}\cos(3n+1)\theta + \frac{1}{2}\cos(n+1)\theta\big]}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n-1}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(\cos\theta)\cos n\theta}{z-\cos\theta}d\theta + \frac{1}{2^{4n}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(\cos\theta)\cos(3n+1)\theta}{z-\cos\theta}d\theta\\ &\quad + \frac{1}{2^{4n}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(\cos\theta)\cos(n+1)\theta}{z-\cos\theta}d\theta = \widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3,\end{aligned}$$

where

$$\begin{aligned}\widehat{J}_1 &= \frac{1}{2^{4n-1}}\int_0^\pi \frac{\pi_{n+1}^{[3]*}(\cos\theta)\cos n\theta}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n-1}}\int_0^\pi \frac{\big[\cos(n+1)\theta - \frac{1}{2}\cos n\theta \cdots + (-1)^n\frac{3}{2^{n+2}}\big]\cos n\theta}{z-\cos\theta}d\theta\\ &= \frac{1}{2^{4n-1}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+1}} + \frac{1}{\xi} - \frac{1}{2} - \frac{1}{2\xi^{2n}} - \frac{3}{2^2}\left(\frac{1}{\xi^{2n-1}} + \frac{1}{\xi}\right)\right.\\ &\quad \left. + \frac{3}{2^3}\left(\frac{1}{\xi^{2n-2}} + \frac{1}{\xi^2}\right) - \frac{3}{2^4}\left(\frac{1}{\xi^{2n-3}} + \frac{1}{\xi^3}\right)\cdots - \frac{3(-1)^{n-1}}{2^n}\left(\frac{1}{\xi^{n+1}} + \frac{1}{\xi^{n-1}}\right) + \frac{3(-1)^n}{2^{n+2}}\frac{1}{\xi^n}\right]\\ &= \frac{1}{2^{4n-1}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+1}} + \frac{1}{\xi} - \frac{1}{2} - \frac{1}{2\xi^{2n}} + \frac{3(-1)^n}{2^{n+2}\xi^n} + 3\sum_{k=1}^{n-1}\frac{(-1)^k}{2^{k+1}}\big(\xi^{-2n+k} + \xi^{-k}\big)\right],\end{aligned}$$

and, proceeding analogously, $\widehat{J}_2$ and $\widehat{J}_3$ are given by

$$\begin{aligned}\widehat{J}_2 &= \frac{1}{2^{4n}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{4n+2}} + \frac{1}{\xi^{2n}} - \frac{1}{2\xi^{4n+1}} - \frac{1}{2\xi^{2n+1}} + \frac{3(-1)^n}{2^{n+2}\xi^{3n+1}}\right.\\ &\quad \left. + 3\sum_{k=2n}^{3n-2}\frac{(-1)^k}{2^{k+2-2n}}\big(\xi^{-6n+k} + \xi^{-k-2}\big)\right],\end{aligned}$$

$$\widehat{J}_3 = \frac{1}{2^{4n}}\frac{\pi}{\xi-\xi^{-1}}\left[\frac{1}{\xi^{2n+2}} - \frac{1}{2\xi} - \frac{1}{2\xi^{2n+1}} + 1 + 3\sum_{k=2}^{n}\frac{(-1)^{k+1}}{2^k}\big(\xi^{-2n-2+k} + \xi^{-k}\big)\right].$$

On the other hand, the denominator $\widehat{\pi}_{n,n}^{[3]}(z)\pi_{n+1,n}^{[3]*}(z)$ is given by

$$\widehat{\pi}_n^{[3]}(z)\pi_{n+1}^{[3]*}(z) = \frac{1}{2^{2n+1}}\left(\xi^n + \frac{1}{\xi^n}\right)\left[\left(\xi^{n+1} + \frac{1}{\xi^{n+1}}\right) - \frac{1}{2}\left(\xi^n + \frac{1}{\xi^n}\right) - \frac{3}{2^2}\left(\xi^{n-1} + \frac{1}{\xi^{n-1}}\right) + \cdots + \frac{3(-1)^n}{2^{n+2}}\right]$$

and, hence, the *kernel* admits the expression

$$K_n^{[3]}(z) = \frac{\varrho_n^{[3]}(z)}{\widehat{\pi}_n^{[3]}(z)\pi_{n+1}^{[3]*}(z)} = \frac{\widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3}{\widehat{\pi}_{n,n}^{[3]}(z)\pi_{n+1}^{[3]*}(z)}.$$

Now, proceeding analogously as in the proof of previous Corollary 1, we can establish the following result. We omit the details.

Corollary 2 *There exists some* ρ^* (> 1) *such that*

$$\max_{\theta\in[0,2\pi)} \left|K_n^{[3]}(z)\right| = \left|K_n^{[3]}(0)\right|, \quad \rho > \rho^*.$$

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Declarations

Conflict of interest The authors declare no competing interests.

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