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Brachistochronic Rigid Body General Motion

The time interval minimization of rigid body motion with constant mechanical energy has been considered in this paper. Generalized coordinates are Cartesian's coordinates of mass center and the Euler's angles, which are specified at the initial and the final position. The problem has been solved by the application of the Pontryagin's principle. Finite difference method has been applied in order to obtain the solution of the two-point boundary value problem.

Keywords: *brachistochronic motion, rigid body, Pontryagin's principle, finite difference.*

1. INTRODUCTION

The well-known Bernoulli's problem of the brachistochrone [1] has set in 1996 the foundations of numerous researches in analytical mechanics. Let us mention the paper [2] in which the differential equations of motion of a material system were formulated and given problems of motion of a rigid body solved. By the use of the classical calculus of variations the differential equations of the brachistochronic motion of a non-conservative mechanical system were obtained [3]. In order to obtain finite solutions of the problem mentioned, it is necessary to solve two-point boundary value problem for a system of ordinary differential equations or, even in particular cases, to solved complicate algebraic problems. The most detailed review of scientific results in the field of brachistohronic motion is given in the monograph [4].

Having in mind the equivalence of some problems of the classical calculus of variations with the problems of optimal control, the problems mentioned can be formulated and solved applying the Pontryagin's principle [5]. The aim of this paper is to obtain equations of the brachistochronic general motion of the rigid body, which is the extension of the research in [6]. The analytical solution of the plane motion and the realisation of the control are given in [6]. In this paper it was necessary to apply the corresponding numerical methods [7].

2. THE FORMULATION OF THE PROBLEM OF OPTIMAL CONTROL

The problem of determining the equations of motion of a rigid body in the gravitational field between two given positions in a minimal time and with invariable mechanical energy is considered. Having given the initial and the final values of the generalized coordinates (the coordinates of the mass centre x_c, y_c, z_c and the Eulerian angles ψ, θ and φ), but not the generalized velocities, the projections v_k of the velocity vector and the projections ω_k of the angular velocity on the principal inertial axes of the rigid body, can be taken as controls. The problem of the optimal control has the form:

$$\begin{gathered}
\dot{x}_c = v_1 \\
\dot{y}_c = v_2 \\
\dot{z}_c = v_3 \\
\dot{\psi} = \omega_1 \sin\varphi \sin^{-1}\theta + \omega_2 \cos\varphi \sin^{-1}\theta \\
\dot{\theta} = \omega_1 \cos\varphi - \omega_2 \sin\varphi \\
\dot{\varphi} = \omega_3 - \omega_1 \sin\varphi \operatorname{ctg}\theta - \omega_2 \cos\varphi \operatorname{ctg}\theta \\
\frac{1}{2}(v_1{}^2 + v_2{}^2 + v_3{}^2 + i_1{}^2\omega_1{}^2 + i_2{}^2\omega_2{}^2 + i_3{}^2\omega_3{}^2) - g y_c = 0 \\
t_0 = 0,\ x_c(0) = x_{c0},\ y_c(0) = y_{c0},\ z_c(0) = z_{c0}, \\
\psi(0) = \psi_0,\ \theta(0) = \theta_0,\ \varphi(0) = \varphi_0 \\
t_1 = ?,\ x_c(t_1) = x_{c1},\ y_c(t_1) = y_{c1},\ z_c(t_1) = z_{c1}, \\
\psi(t_1) = \psi_1,\ \theta(t_1) = \theta_1,\ \varphi(t_1) = \varphi_1 \\
J = \int_{t_0}^{t_1} \mathrm{d}t \to \min\,, \qquad (1)
\end{gathered}$$

where g is the acceleration of gravity, while i_k denotes the radii of inertia with respect to the principal central axes.

The corresponding problems of the singular controls [8], too, reduces to the problem (1), if one takes for the controls the corresponding projections of the principal vector and the principal moment of the system about the mass centre of the forces acting on the body.

3. ANALYSIS OF THE SOLUTIONS

In order to solve the problem (1), let us write the Pontryagin's function [5]:

$$\begin{gathered}
H = \lambda_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \\
+\mu_1(\omega_1 \sin\varphi \sin^{-1}\theta + \omega_2 \cos\varphi \sin^{-1}\theta) + \\
+\mu_2(\omega_1 \cos\varphi - \omega_2 \sin\varphi) + \\
+\mu_3(\omega_3 - \omega_1 \sin\varphi \operatorname{ctg}\theta - \omega_2 \cos\varphi \operatorname{ctg}\theta) \qquad (2)
\end{gathered}$$

and the system of differential equations for the co-state variables λ_k and μ_k:

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$$\dot{\lambda}_1 = 0$$
$$\dot{\lambda}_2 = -\rho g$$
$$\dot{\lambda}_3 = 0$$
$$\dot{\mu}_1 = 0$$
$$\dot{\mu}_2 = \mu_1(\omega_1 \sin\varphi + \omega_2 \cos\varphi)\sin^{-2}\theta\cos\theta -$$
$$-\mu_3(\omega_1 \sin\varphi + \omega_2 \cos\varphi)\sin^{-2}\theta$$
$$\dot{\mu}_3 = \mu_1 \sin^{-1}\theta(-\omega_1\cos\varphi + \omega_2 \sin\varphi) +$$
$$+\mu_2(\omega_1 \sin\varphi + \omega_2\cos\varphi) +$$
$$+\mu_3 \mathrm{ctg}\theta(\omega_1\cos\varphi - \omega_2 \sin\varphi)\,, \tag{3}$$

where ρ is the multiplier of the constraint from the problem (1).

As the controls are from an open set, the conditions of optimality [5] become:

$$\lambda_1 - \rho v_1 = 0$$
$$\lambda_2 - \rho v_2 = 0$$
$$\lambda_3 - \rho v_3 = 0$$
$$\mu_1 \sin\varphi\sin^{-1}\theta + \mu_2\cos\varphi -$$
$$-\mu_3\sin\varphi\mathrm{ctg}\theta - \rho i_1{}^2\omega_1 = 0$$
$$\mu_1\cos\varphi\sin^{-1}\theta - \mu_2\sin\varphi -$$
$$-\mu_3\cos\varphi\mathrm{ctg}\theta - \rho i_2{}^2\omega_2 = 0$$
$$\mu_3 - \rho i_3{}^2\omega_3 = 0\,. \tag{4}$$

The multiplier of the constraint is determined from the conditions [5]:

$$H = 0,\ \lambda_0 = -1, \tag{5}$$

so that its value is:

$$\rho = \frac{1}{2gy_\mathrm{c}}\,, \tag{6}$$

and the corresponding optimal controls are:

$$v_1 = 2g\lambda_1 y_\mathrm{c}$$
$$v_2 = 2g\lambda_2 y_\mathrm{c}$$
$$v_3 = 2g\lambda_3 y_\mathrm{c}$$
$$\omega_1 = (\mu_1\sin\varphi\sin^{-1}\theta + \mu_2\cos\varphi -$$
$$-\mu_3\sin\varphi\mathrm{ctg}\theta)2gy_\mathrm{c}i_1{}^{-2}$$
$$\omega_2 = (\mu_1\cos\varphi\sin^{-1}\theta - \mu_2\sin\varphi -$$
$$-\mu_3\cos\varphi\mathrm{ctg}\theta)2gy_\mathrm{c}i_2{}^{-2}$$
$$\omega_3 = 2gy_\mathrm{c}i_3{}^{-2}\mu_3\,. \tag{7}$$

Before we start solving the two-point boundary value problem, having put (7) in (1), let us note the following:

a) the path of the mass centre is a plane curve, as in virtue of (1), (3) and (7),

$$\frac{\mathrm{d}x_\mathrm{c}}{\mathrm{d}z_\mathrm{c}} = \mathrm{const}\,, \tag{8}$$

so that, without loss in generality, we can take:

$$z_\mathrm{c}(t) = 0 \quad t\in[t_0, t_1]\,; \tag{9}$$

b) the equation of motion:

$$y_\mathrm{c}(t) = \frac{g}{p^2}(1-\cos pt) \tag{10}$$

has the same form as in the case of a brachistochronic motion of the particle or in the case of a plane motion of the rigid body [6];

c) if the ellipsoid of inertia is reduced to a sphere ($i_1 = i_2 = i_3 = i$), the vector of angular velocity, in virtue of (1), (3) and (7), has a constant direction. In the general case of the ellipsoid of inertia, the direction of the angular velocity is variable.

4. THE ANALYTICAL SOLUTION IN THE CASE OF A SPHERE OF INERTIA

The direction of the angular velocity vector being constant, the change of orientation of the rigid body results from a rotation around the axis of an also constant direction. The direction and the angle can be determined by the theory of finite rotations [9]. Namely, the knowledge of Euler's angles at the beginning and at the end of the interval of motion allows to calculate the vectors of the finite rotations: $\vec{\Theta}_0$, which corresponds to the rotation between the position $\psi = \theta = \varphi = 0$ and the initial position, $\vec{\Theta}_1$, which corresponds to the rotation between the position $\psi = \theta = \varphi = 0$ and the final position, as well as $\vec{\Theta}$ for the rotation between the initial and the final positions. For that purpose, we use the relation between Euler's angles and the vector of the finite rotations [9]:

$$\vec{\Theta}_s = 2(\sin\frac{\theta_s}{2}\cos\frac{\psi_s-\varphi_s}{2}\vec{i} + \sin\frac{\theta_s}{2}\sin\frac{\psi_s-\varphi_s}{2}\vec{j} +$$
$$+\cos\frac{\theta_s}{2}\sin\frac{\psi_s+\varphi_s}{2}\vec{k})(\cos\frac{\theta_s}{2}\sin\frac{\psi_s+\varphi_s}{2})^{-1}, s = 0,1 \tag{11}$$

as well as the rule of the subtraction of rotations [9]:

$$\vec{\Theta} = (\vec{\Theta}_1 - \vec{\Theta}_0 + \frac{1}{2}\vec{\Theta}_0\times\vec{\Theta}_1)(1+\frac{1}{4}\vec{\Theta}_0\cdot\vec{\Theta}_1)^{-1}\,. \tag{12}$$

By calculating the vector $\vec{\Theta}$, the unit vector $\vec{e}$ of the axis of finite rotation, as well as the angle χ are also determined, as:

$$\vec{\Theta} = 2\vec{e}\,\mathrm{tg}\frac{\chi}{2}\,. \tag{13}$$

In that way, the problem (1) is reduces to the problem:

$$\dot{x}_\mathrm{c} = v_1$$
$$\dot{y}_\mathrm{c} = v_2$$
$$\dot{\alpha} = \omega$$
$$\frac{1}{2}(v_1{}^2 + v_2{}^2 + i^2\omega^2) - gy_\mathrm{c} = 0$$
$$t_0 = 0,\ x_\mathrm{c}(0) = x_{\mathrm{c}0},\ y_\mathrm{c}(0) = y_{\mathrm{c}0},\ \alpha(0) = 0$$
$$t_1 = ?,\ x_\mathrm{c}(t_1) = x_{\mathrm{c}1},\ y_\mathrm{c}(t_1) = y_{\mathrm{c}1},\ \alpha(t_1) = \chi$$
$$J = \int_{t_0}^{t_1}\mathrm{d}t \to \min \tag{14}$$

which has an analytical solution, as in the case of brachistochronic plane motion [6], which will not be considered here.

5. THE NUMERICAL SOLUTIONS

Let us solve the problem (1) for given:

$$x_{c0} = 0,\ y_{c0} = 1m,\ \psi_0 = 0, \theta_0 = \frac{\pi}{6},\ \varphi_0 = 0$$

$$x_{c1} = 1m,\ y_{c1} = 1m,\ \psi_1 = \frac{\pi}{3},\ \theta_1 = \frac{\pi}{3},\ \varphi_1 = \frac{\pi}{3}$$

$$i_1 = 5.5m,\ i_2 = 3.5m,\ i_3 = 1.5m\ . \qquad (15)$$

As t_1 is not prescribed, by the introduction of the dimensionless time t':

$$0 \le t' \le 1\ ,\ t = Tt',\ t_1 = T \qquad (16)$$

in virtue of (1), (3), (6), (7) and (16) we have the following system of differential equations:

$$\begin{aligned}(x_c)' &= T\dot{x}_c\\ (y_c)' &= T\dot{y}_c\\ (\psi)' &= T\dot{\psi}\\ (\theta)' &= T\dot{\theta}\\ (\varphi)' &= T\dot{\varphi}\\ (\lambda_1)' &= 0\\ (\lambda_2)' &= T\dot{\lambda}_2\\ (\mu_1)' &= 0\\ (\mu_2)' &= T\dot{\mu}_2\\ (\mu_3)' &= T\dot{\mu}_3\\ (T)' &= 0\end{aligned} \qquad (17)$$

where $()'$ denotes differentiation with respect to t'. The problem is now reduced to the case of prescribed interval of the independent variable, whereas one has to add the condition at the initial point to the boundary conditions (15):

$$\frac{1}{2}({v_{10}}^2 + {v_{20}}^2 + {i_1}^2{\omega_{10}}^2 + {i_2}^2{\omega_{20}}^2 + {i_3}^2{\omega_{30}}^2) - $$
$$- gy_{c0} = 0\ , \qquad (18)$$

where:

$$\begin{aligned}v_{10} &= 2g\lambda_{10}y_{c0}\\ v_{20} &= 2g\lambda_{20}y_{c0}\\ v_{30} &= 2g\lambda_{30}y_{c0}\end{aligned}$$

$$\omega_{10} = (\mu_{10}\sin\varphi_0\sin^{-1}\theta_0 + \mu_{20}\cos\varphi_0 -$$
$$-\mu_{30}\sin\varphi_0\operatorname{ctg}\theta_0)2gy_{c0}{i_1}^{-2}$$
$$\omega_{20} = (\mu_{10}\cos\varphi_0\sin^{-1}\theta_0 - \mu_{20}\sin\varphi_0 -$$
$$-\mu_{30}\cos\varphi_0\operatorname{ctg}\theta_0)2gy_{c0}{i_2}^{-2}$$
$$\omega_{30} = 2gy_{c0}{i_3}^{-2}\mu_{30}\ . \qquad (19)$$

The problem is solved by the use of the program [7] based upon the method of finite differences with tolerance of relative error of 10^{-6}. The path of the mass centre, as well as, the laws of change of the Euler's angles is presented in the Figure 1 and in the Figure 2.

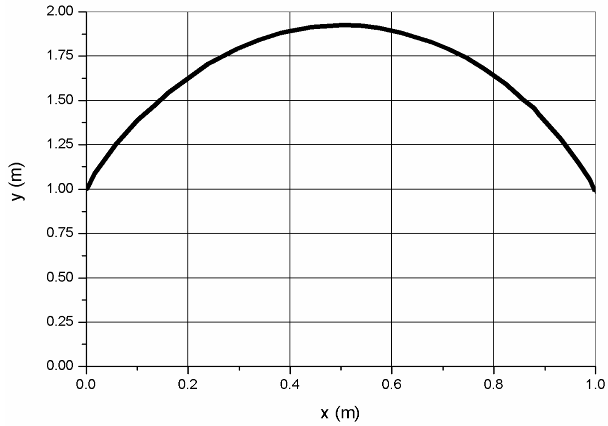

Figure 1. Path of the mass centre

The time of motion, t_1 = 0.959330s, is also calculated.

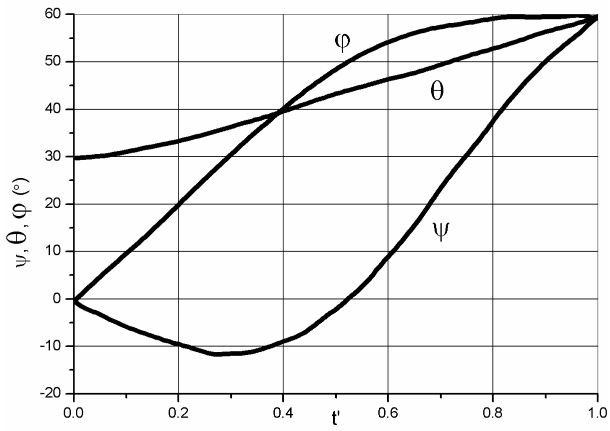

Figure 2. Laws of change of the Euler's angles

6. CONCLUSION

The difficulties which arise from numerical methods are the limiting factor for the application of the theory of optimal control in mechanics. The fact that no general algorithm and no corresponding program for the solution of two-point boundary value problems of the principle of maximum do not exist today makes every successfully solved problem of this kind valuable. Some recently solved problems of optimal control of mechanical systems [4] confirm that opinion.

The fact that the control belongs to an open set, as well as the smoothness of the solution obtained in this paper, made possible a relatively simple application of numerical methods. Meanwhile, the problems in which the mechanical systems with controls from a closed set are considered demand a more complicated treatment.

Some particular results of this paper can be found in PhD thesis [8].

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БРАХИСТОХРОНО ОПШТЕ КРЕТАЊЕ КРУТОГ ТЕЛА

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Разматра се минимизација времена кретања крутог тела уз неизмењену вредност механичке енергије. За генералисане координате узете су координате центра маса и Ојлерови углови, чије су вредности задате на почетку и крају интервала кретања. Задатак је решен применом Понтрјагиновог принципа максимума. Нумеричко решење двотачкастог граничног проблема добијено је методом коначних разлика за системе обичних диференцијалних једначина.