Research article

Longtime evolution and stationary response of a stochastic tumor-immune system with resting T cells

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Abstract: In this paper, we take the resting T cells into account and interpret the progression and regression of tumors by a predator-prey like tumor-immune system. First, we construct an appropriate Lyapunov function to prove the existence and uniqueness of the global positive solution to the system. Then, by utilizing the stochastic comparison theorem, we prove the moment boundedness of tumor cells and two types of T cells. Furthermore, we analyze the impact of stochastic perturbations on the extinction and persistence of tumor cells and obtain the stationary probability density of the tumor cells in the persistent state. The results indicate that when the noise intensity of tumor perturbation is low, tumor cells remain in a persistent state. As this intensity gradually increases, the population of tumors moves towards a lower level, and the stochastic bifurcation phenomena occurs. When it reaches a certain threshold, instead the number of tumor cells eventually enter into an extinct state, and further increasing of the noise intensity will accelerate this process.

Keywords: prey-predator like system; tumor-immune model; persistence and extinction; stationary probability density

1. Introduction

A tumor is a new organism formed by the proliferation of local tissue in the body under the action of various oncogene factors. In recent years, an increasing interest has been focused on the role of the immune system in fighting a tumor growth [1–6]. T cells in the immune system are one of the important components in resisting tumor. There are two population of T cells; one is the helper T cells expressed by CD4+T proteins, they cannot kill tumor cells directly, but once they are stimulated by the macrophage or their cognate antigen, they can produce a kind of cytokine such as interleukin -2 (IL-2

) and send biochemical signals to a special type of effector cell called cytotoxic T lymphocyte CD8+ T cells [7]. Cytotoxic T lymphocyte cells are the other important population in the immune response, they can eliminate or kill the infected cells by mounting a cytotoxic reaction after they are activated by the helper T cells [8,9]. Therefore, the roles of the helper T cells and the cytotoxic T cells are distinct, they perform the complementary functions to eliminate the tumor.

It is a key task to identify the components of the cells to understand how the immune system works to overcome the tumor. However, due to the complexity of cell dynamics, most work concentrates on the discussion about the interaction between the tumor cells and the cytotoxic T lymphocytes, but neglects the role played by the helper T cells in tumor elimination. In recent years, some researchers have proposed that the helper T cells are a vital and essential component in tumor-immune response [10–13], and built a system of ODEs to describe the population in a system consist of the helper T cells, cytotoxic T cells and the tumor cells. The representative work can be found in Refs. [7,14–17].

Furthermore, according to the biochemical features of the tumor cells and the immune cells, some models use a prey-predator interaction model. Immune cells are the predators, who attack and destroy the tumor cells. That is, the helper T cells in a resting state interact with antigens to release cytokines and stimulate the cytotoxic T cells. As follows, the cytotoxic T cells in a hunting state attack or destroy the tumor cells. While tumor cells are the prey, which are eliminated or killed by the immune cells. Predator hunting cells and predator resting cells provide a useful model to learn the dynamical behaviors such as equilibration, stable solutions, longtime evolution and so on. In fact, a system with a component of the resting T cells provide insight to learn the tumor growth. For example, Sarkar [18] in 2005 considered a prey-predator like system with hunting and resting cells, and investigated the stability of the positive equilibrium from the analytical and numerical perspective, they obtained a threshold value for the rate of predation of the tumor cells by the hunting cells and discussed the control of malignant tumor growth in deterministic case. In 2014, Kaur [14] analyzed tumor growth progression and regression in a prey-predator system, their analysis indicated that the tumor growth will persist from a recurring state to a dormant state with the increase of the resting cells. In 2018, Dritschel [7] developed a mathematical model to examine the role of the helper T cells in an anti-tumor immune response, their work revealed that the immunoediting depends highly on the infiltration rates of the helper and cytotoxic T cells. In 2023, Dehingia [16] proposed a modified system consisting of tumor cells, resting helper T cells and hunting cytotoxic T cells, he discussed the stability and Hopf bifurcation of this system.

The works mentioned above regarding the prey-predator like systems considering two kinds of T cells and the tumor cells are mainly investigated in the deterministic condition. However, as is widely recognized, a variety of random factors in the cell environment should be taken into account. That is, the biochemical proliferation of the cells may undergo stochastic variations influenced by factors such as nutrient availability, temperature, radiation, oxygen supply, and gene expression [19–25]. Therefore, an idealized tumor-immune system should be modeled stochastically.

In this paper, the novelty of our efforts will focus on the special tumor-immune system including hunting and resting T cells, at the same time, stochastic fluctuations on the proliferation parameters will be explored, in order to understand the impact of the noise on the cells dynamics, as well as give insight on how to eliminate tumors and how to improve cancer treatment.

2. Stochastic prey-predator like tumor-immune system

Based on the deterministic prey-predator like tumor-immune model in [14], we study a modified model to take the impact of random perturbations on the cell evolution into account, and use standard Brownian motions to characterize it in the tumor-immune model. The cells'dynamics is possible to be changed in the case of stochastic perturbation, so it is necessary to have a relevant research from the point of random variables. On the other hand, there are plenty of valuable results about this model in the deterministic case, we expect to give some new insights by studying a stochastic one. The model is given by a set of differential equations in a form of the follow

$$
\begin{cases}
dT(t) & = \left(\alpha_1 T \left(1 - \dfrac{T}{K_1}\right) - \beta_1 TH\right)dt + \xi_1 T dB_1(t), \\
dH(t) & = (\gamma HR - \delta_1 H - \beta_2 HT)dt + \xi_2 H dB_2(t), \\
dR(t) & = \left(\alpha_2 R \left(1 - \dfrac{R}{K_2}\right) - \gamma HR - \delta_2 R + \dfrac{kTR}{T + \eta}\right)dt + \xi_3 R dB_3(t),
\end{cases}
\tag{2.1}
$$

where $T(t), H(t)$ and $R(t)$ are the population of tumor cells, predator hunting T cells and predator resting T cells in the given physiologic space, respectively. α_1 and α_2 represent the inherent growth rates of tumor cells and resting T cells. K_1, K_2 indicate the maximum carrying capacity of the environment. β_1 and β_2 represent the rate at which hunting T cells kill tumor cells and the rate at which tumor cells deplete hunting T cells. δ_1 and δ_2 indicate the apoptosis rates of hunting T cells and resting T cells. γ represents the activation rate of resting T cells to hunting T cells, k is the proliferation rate of resting T cells, η is the half-saturation coefficient of proliferation term. $B_1(t), B_2(t), B_3(t)$ are mutually independent standard Brownian motions, and ξ_1, ξ_2, ξ_3 represent noise intensity of them.

3. Dynamical evolution and properties

3.1. Existence and uniqueness of the positive solution

Since $T(t), H(t), R(t)$ represent the quantities of tumor cells, hunting T cells, and resting T cells respectively, it is necessary to verify that the solutions of the system (2.1) are non-negative. Therefore, in this section, we begin with the following theorem on the global existence and non-negativity of solutions to the system (2.1).

Theorem 3.1. *For any given initial values* $(T(0), H(0), R(0)) \in R_+^3$, *there exists a global solution* $(T(t), H(t), R(t))$ *of system (2.1) such that*

$$
P\left\{(T(t), H(t), R(t)) \in R_+^3, \forall t \geq 0\right\} = 1.
\tag{3.1}
$$

Proof. Due to the coefficients of the system (2.1) satisfying the local Lipschitz condition, for a given initial value $(T(0), H(0), R(0)) \in R_+^3$, the system has an unique local positive solution $(T(t), H(t), R(t))$ for $t \in [0, \tau_e)$ a.s., where τ_e represents the explosion time. In order to confirm the local solutions to be global ones, it is necessary to prove that $\tau_e = \infty$.

Let $m_0 \geq 1$ be sufficiently large such that $T(0) \in [\frac{1}{m_0}, m_0], H(0) \in [\frac{1}{m_0}, m_0], R(0) \in [\frac{1}{m_0}, m_0]$. For $m > m_0$, we define

$$
\tau_m = \inf\left\{t \in [0, \tau_e) : T(t) \notin (\frac{1}{m}, m) \text{ or } H(t) \notin (\frac{1}{m}, m) \text{ or } R(t) \notin (\frac{1}{m}, m)\right\},
\tag{3.2}
$$

where we set inf $\emptyset = \infty$. By the definition of τ_m we observe that τ_m increases with m approaching to the infinity and $\tau_m < \tau_e$, therefore

$$\lim_{m \to \infty} \tau_m = \tau_\infty \leq \tau_e.$$

Now, we prove that $\tau_\infty = \infty$ is true almost surely first. Moreover, a proof by contradiction is applied, that is, assume the condition does not hold. If so, There are a constant $T_1 \geq 0$ and another constant $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T_1\} > \varepsilon.$$

Thus, there exists an integer $m_1 \geq m_0$ such that for all $m \geq m_1$, we have

$$P\{\tau_m \leq T_1\} \geq \varepsilon. \tag{3.3}$$

Let us define a Lyapunov function $V : Int(R_+^3) \to R_+$:

$$V(T, H, R) = (T + 1 - \log T) + (H + 1 - \log H) + (R + 1 - \log R).$$

Applying Itô's formula, we can differentiate the function V

$$\begin{aligned}
&\mathcal{L}[V(T, H, R)] \\
&= \left(1 - \frac{1}{T}\right)\left[\alpha_1 T \left(1 - \frac{T}{K_1}\right) - \beta_1 TH\right] + \frac{1}{2}\xi_1^2 + \left(1 - \frac{1}{H}\right)[\gamma HR - \delta_1 H - \beta_2 HT] \\
&\quad + \frac{1}{2}\xi_2^2 + \left(1 - \frac{1}{R}\right)\left[\alpha_2 R\left(1 - \frac{R}{K_2}\right) - \gamma HR - \delta_2 R + \frac{kTR}{T + \eta}\right] + \frac{1}{2}\xi_3^2 \\
&\leq \left(\delta_1 + \delta_2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\xi_3^2\right) + \left(\alpha_1 + \frac{\alpha_1}{K_1} + \beta_2\right)T + (\beta_1 + \gamma)H + \left(\alpha_2 + k + \frac{\alpha_2}{K_2}\right)R,
\end{aligned}$$

where $\mathcal{L}[\cdot]$ is the drift term of the $[\cdot]$ derivative. Denote

$$A = \delta_1 + \delta_2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\xi_3^2$$

$$B = 2\max\left\{\alpha_1 + \frac{\alpha_1}{K_1} + \beta_2, \beta_1 + \gamma, \alpha_2 + k + \frac{\alpha_2}{K_2}\right\}$$

then we can rewrite the above expression as:

$$\begin{aligned}
\mathcal{L}[V(T, H, R)] &\leq A + B[(T + 1 - logT) + (H + 1 - logH) + (R + 1 - logR)] \\
&= A + BV(T, H, R).
\end{aligned} \tag{3.4}$$

Integrating on both sides from 0 to $\tau_m \wedge T_1$ of the inequality (3.4) and then taking the expectation for

every terms, we have:

$$EV(T(\tau_m \wedge T_1), H(\tau_m \wedge T_1), R(\tau_m \wedge T_1))$$

$$\leq V(T(0), H(0), R(0)) + AT_1 + BE \int_0^{\tau_m \wedge T_1} V(T(t), H(t), R(t))dt$$

$$= V\big(T(0), H(0), R(0)\big) + AT_1 + BE \int_0^{T_1} \mathrm{I}_{[0,\tau_m]}(t)V\big(T(t), H(t), R(t)\big)dt$$

$$\leq V(T(0), H(0), R(0)) + AT_1$$

$$+ BE \int_0^{T_1} V\big(T(\tau_m \wedge T_1), H(\tau_m \wedge T_1), R(\tau_m \wedge T_1)\big)dt$$

$$= V(T(0), H(0), R(0)) + AT_1$$

$$+ B \int_0^{T_1} EV(T(\tau_m \wedge T_1), H(\tau_m \wedge T_1), R(\tau_m \wedge T_1))dt,$$

where $I_A(\cdot)$ represents the indicator function of set A, $\tau_m \wedge T_1 = \min(\tau_m, T_1)$. Utilizing the Gronwall inequality [26], we can establish the following result:

$$EV(T(\tau_m \wedge T_1), H(\tau_m \wedge T_1), R(\tau_m \wedge T_1)) \leq (V(T(0), H(0), R(0)) + AT_1)e^{BT_1}. \qquad (3.5)$$

Let $\Omega_m = \{\omega \in \Omega : \tau_m = \tau_m(\omega) \leq T_1\} : (m \geq m_1)$, based on the Eq (3.3), we have:

$$P(\Omega_m) \geq \varepsilon. \qquad (3.6)$$

Observe that for every ω in the set $\omega \in \Omega_m$, it holds that at least one of $T(\tau_m, \omega), H(\tau_m, \omega), R(\tau_m, \omega)$ is equal to m or $\frac{1}{m}$. Therefore,

$$V(T(\tau_m, \omega), H(\tau_m, \omega), R(\tau_m, \omega)) \geq (m - 1 - \log m) \wedge (\frac{1}{m} - 1 + \log m).$$

Combining Eqs (3.5) and (3.6), we obtain:

$$\varepsilon(m - 1 - \log m) \wedge (\frac{1}{m} - 1 + \log m) \leq E[V(T(\tau_m, \omega), H(\tau_m, \omega), R(\tau_m, \omega))]$$

$$\leq (V(T(0), H(0), R(0)) + AT_1)e^{BT_1}.$$

On the other hand,

$$\lim_{m \to \infty} (m - 1 - \log m) \wedge (\frac{1}{m} - 1 + \log m) = \infty.$$

Let $m \to \infty$, we have $\infty \leq Ke^{2bT} < \infty$, which implies that τ_∞ must be equal to infinity. Considering $\tau_\infty \leq \tau_e$, we conclude that $\tau_e = \infty$ is indeed equal to infinity. $\qquad \square$

3.2. Moment boundedness

Based on the existence of the positive solution for every cell population, this section focuses on the asymptotic estimation of the moments of $T(t)$, $H(t)$, and $R(t)$.

Definition 1. *p-order statistical moments of the solutions to the system (2.1) are bounded. there exists a function $r = r(x_0, t_0)$ such that when $|x_0| \leq m$, if for any $m > 0$, it holds that*

$$E |x(t; t_0, x_0)|^p \leq r(x_0, t_0), \quad t \geq t_0$$

If the function $r(x_0, t_0)$ mentioned here is independent of t_0, then the solution to the system (2.1) is said to have uniformly bounded p-order moments.

Theorem 3.2. *For any $p > 1$, we have*

$$\limsup_{t \to \infty} ET^p(t) \leq \left[\frac{\alpha_1 + \dfrac{(p-1)}{2}\xi_1^2}{\dfrac{\alpha_1}{K_1}} \right]^p, \tag{3.7}$$

$$\limsup_{t \to \infty} ER^p(t) \leq \left[\frac{A_1 + \dfrac{(p-1)}{2}\xi_3^2}{\dfrac{\alpha_2}{K_2}} \right]^p. \tag{3.8}$$

For any $p \in (0, 1 + 2\delta_1/\xi_2^2)$, we can find a positive constant L_1 such that

$$\limsup_{t \to \infty} EH^p(t) \leq L_1. \tag{3.9}$$

Proof. we are going to prove the moment boundedness of $T(t)$ first. Consider the following auxiliary procedure $\varphi(t)$

$$\begin{cases} d\varphi(t) &= \left[\alpha_1 \varphi \left(1 - \dfrac{\varphi}{K_1} \right) \right] dt + \xi_1 \varphi dB_1(t), \\ \varphi(0) &= T_0 > 0, \end{cases} \tag{3.10}$$

where $B_1(t)$ is the standard Brownian motions defined in system (2.1). Using the comparison theorem [27], we can obtain that $0 \leq T(t) \leq \varphi(t)$ $(t \geq 0)$. By applying Itô's formula, the drift coefficient in the differential formula of $\varphi^p(t)$ can be expressed as

$$\mathcal{L}[\varphi(t)^p] = p\varphi^{p-1}\left[\alpha_1 \varphi \left(1 - \frac{\varphi}{K_1} \right) \right] + \frac{1}{2}p(p-1)\varphi^{p-2}\xi_1^2\varphi^2$$

$$\tag{3.11}$$

$$= p\left(\alpha_1 + \frac{(p-1)}{2}\xi_1^2 \right)\varphi^p - p\frac{\alpha_1}{K_1}\varphi^{p-1}.$$

Taking expectations on both sides of Eq (3.11) and applying the Holder's inequality [26], we have

$$\frac{dE\varphi^p(t)}{dt} = p\left(\alpha_1 + \frac{(p-1)}{2}\xi_1^2\right)E\varphi^p - p\frac{\alpha_1}{K_1}E\varphi^{p+1}$$

$$\leq p\left(\alpha_1 + \frac{(p-1)}{2}\xi_1^2\right)E\varphi^p - p\frac{\alpha_1}{K_1}\left[E\left(\varphi^p\right)\right]^{\frac{p+1}{p}}.$$

Therefore,

$$ET^p(t) \leq E\varphi^p(t) \leq \left[(T_0^p)^{-\frac{1}{p}}e^{-\left(\alpha_1 + \frac{(p-1)}{2}\xi_1^2\right)t} + \frac{\frac{\alpha_1}{K_1}}{\alpha_1 + \frac{(p-1)}{2}\xi_1^2}\left(1 - e^{-\left(\alpha_1 + \frac{(p-1)}{2}\xi_1^2\right)t}\right)\right]^{-p}.$$

Thus, for any $p > 1$, we have

$$\limsup_{t\to\infty} ET^p(t) \leq \left[\frac{\alpha_1 + \frac{(p-1)}{2}\xi_1^2}{\frac{\alpha_1}{K_1}}\right]^p. \tag{3.12}$$

Next, we will demonstrate the moment boundedness of $R(t)$. Similarly, consider the following auxiliary process $\psi(t)$

$$\begin{cases} d\psi(t) & = \left(\alpha_2\psi(t)\left(1 - \frac{\psi}{K_2}\right) - \delta_2\psi + k\psi\right)dt + \xi_3\psi dB_3(t), \\ \psi(0) & = R_0 > 0, \end{cases} \tag{3.13}$$

where $B_3(t)$ is the standard Brownian motions defined in system (2.1). Then, we get $0 \leq R(t) \leq \psi(t)$ for $t \geq 0$. Notice that $\psi(t)$ follows the similar structure as $\varphi(t)$, so we can obtain:

$$ER^p(t) \leq E\psi^p(t) \leq \left[(R_0^p)^{-\frac{1}{p}}e^{-\left(A_1 + \frac{(p-1)}{2}\xi_3^2\right)t} + \frac{\frac{\alpha_2}{K_2}}{A_1 + \frac{(p-1)}{2}\xi_3^2}\left(1 - e^{-\left(A_1 + \frac{(p-1)}{2}\xi_3^2\right)t}\right)\right]^{-p},$$

where $A_1 = \alpha_2 + k - \delta_2$. Therefore, for any $p > 1$, we have

$$\limsup_{t\to\infty} ER^p(t) \leq \left[\frac{A_1 + \frac{(p-1)}{2}\xi_3^2}{\frac{\alpha_2}{K_2}}\right]^p. \tag{3.14}$$

Finally, we will establish the moment boundedness of $H(t)$. Consider the following two auxiliary processes, $y(t)$ and $z(t)$

$$\begin{cases} dy(t) & = (\gamma y(t)z(t) - \delta_1 y(t))\,dt + \xi_2 y(t)dB_2(t), \\ dz(t) & = \left(\alpha_2 z(t)\left(1 - \frac{z(t)}{K_2}\right) - \gamma y(t)z(t) - \delta_2 z(t) + kz(t)\right)dt + \xi_3 z(t)dB_3(t), \\ y(0) & = y_0 > 0, \\ z(0) & = z_0 > 0, \end{cases} \tag{3.15}$$

where $B_2(t), B_3(t)$ are mutually independent standard Brownian motions defined in system (2.1). Consider the function $V_1(t)$ defined as $V_1(t) = (y(t) + z(t))^q$. According to the Itô's formula, we get

$$
\begin{aligned}
\mathcal{L}[(y + z)^q] &= q(y + z)^{q-1}[\gamma yz - \delta_1 y] + \frac{q(q - 1)}{2}(y + z)^{q-2}\xi_2^2 y^2 \\
&\quad + q(y + z)^{q-1}\left[\alpha_2 z\left(1 - \frac{z}{K_2}\right) - \gamma yz - \delta_2 z + kz\right] + \frac{q(q - 1)}{2}(y + z)^{q-2}\xi_3^2 z^2 \\
&\leq q(y + z)^{q-2}\left[-\left(\delta_1 - \frac{q - 1}{2}\xi_2^2\right)y^2 + yz\left(\alpha_2 + k - \frac{\alpha_2}{K_2}z\right)\right. \\
&\quad \left. + z^2\left(\alpha_2 + k + \frac{q - 1}{2}\xi_3^2 - \frac{\alpha_2}{K_2}z\right)\right].
\end{aligned}
$$

The condition $q \in (0, 1 + 2\delta_1/\xi_2^2)$ implies $\delta_1 - \frac{q - 1}{2}\xi_2^2 > 0$, then we can find a positive constant κ sufficiently small such that

$$
\delta_1 - \frac{q - 1}{2}\xi_2^2 - \frac{\kappa}{q} > 0. \tag{3.16}
$$

Define $U(t) = e^{\kappa t}(y(t) + z(t))^q$, we get

$$
\begin{aligned}
\mathcal{L}[e^{\kappa t}(y + z)^q] &= \kappa e^{\kappa t}(y(t) + z(t))^q \\
&\quad + e^{\kappa t}\left\{q(y + z)^{q-1}[\gamma yz - \delta_1 y] + \frac{q(q - 1)}{2}(y + z)^{q-2}\xi_2^2 y^2\right. \\
&\quad \left. + q(y + z)^{q-1}\left[\alpha_2 z\left(1 - \frac{z}{K_2}\right) - \gamma yz - \delta_2 z + kz\right] + \frac{q(q - 1)}{2}(y + z)^{q-2}\xi_3^2 z^2\right\} \\
&\leq q e^{\kappa t}(y + z)^{q-2}W(y, z),
\end{aligned}
$$

where

$$
\begin{aligned}
W(y, z) &= -\left(\delta_1 - \frac{q - 1}{2}\xi_2^2 - \frac{\kappa}{q}\right)y^2 + yz\left(\frac{2\kappa}{q} + \alpha_2 + k - \frac{\alpha_2}{K_2}z\right) \\
&\quad + z^2\left(\frac{\kappa}{q} + \alpha_2 + k + \frac{q - 1}{2}\xi_3^2 - \frac{\alpha_2}{K_2}z\right).
\end{aligned}
$$

As $\delta_1 - \frac{q - 1}{2}\xi_2^2 - \frac{\kappa}{q} > 0$ and $\frac{\alpha_2}{K_2} > 0$, it implies that

$$
\lim_{y^2 + z^2 \to +\infty} W(y, z) = -\infty.
$$

Combining the continuity of $q(y + z)^{(q-2)}W(y, z)$, we can find a positive constant L satisfies:

$$
L = \sup_{y,z \in R_+} \{q(y + z)^{q-2}W(y, z)\} < +\infty.
$$

Therefore,

$$
\mathcal{L}[e^{\kappa t}(y + z)^q] \leq e^{\kappa t}L. \tag{3.17}
$$

By integrating both sides of the inequality (3.17) from 0 to t and taking the expectation, we have

$$\limsup_{t \to \infty} E(y + z)^q \le \frac{L}{\kappa}.$$

By applying the comparison theorem [27], we have $H(t) \le y(t)$ and $R(t) \le z(t)$. Let $L_1 = \dfrac{L}{\kappa}$ such that

$$\limsup_{t \to \infty} EH^q(t) \le \limsup_{t \to \infty} E(y + z)^q \le L_1.$$

$\square$

4. Extinction and persistence

4.1. Extinction

In this section, we will analyze the extinction of the resting T cells $R(t)$ and tumor cells $T(t)$ in system (2.1), and provide numerical examples for verification.

Theorem 4.1. *When $\delta + \dfrac{\xi^2}{4} - \alpha_2 - k > 0$, the resting T cells will eventually become extinct, i.e.,*

$$\lim_{t \to \infty} R(t) = 0 \ a.s.$$

Here, $\delta = \min\{\delta_1, \delta_2\}, \xi^2 = \min\{\xi_2^2, \xi_3^2\}$.

Proof. We denote $\Psi(t) = H(t) + R(t)$. Taking the logarithm of $\Psi(t)$ and applying Itô's formula yields

$$
\begin{aligned}
d \log \Psi(t) &= \left\{ \frac{1}{\Psi}\left[\gamma HR - \delta_1 H - \beta_2 HT + \alpha_2 R\left(1 - \frac{R}{K_2}\right) - \gamma HR \right. \right. \\
&\quad \left. \left. -\delta_2 R + \frac{kTR}{T + \eta} \right] - \frac{\xi_2^2 H^2 + \xi_3^2 R^2}{2\Psi^2} \right\} dt + \frac{\xi_2 H}{\Psi} dB_2(t) + \frac{\xi_3 R}{\Psi} dB_3(t) \\
&\le \left\{ \frac{1}{\Psi}(\alpha_2 R + kR - \delta_1 H - \delta_2 R) - \frac{\xi_2^2 H^2 + \xi_3^2 R^2}{2\Psi^2} \right\} dt \\
&\quad + \frac{\xi_2 H}{\Psi} dB_2(t) + \frac{\xi_3 R(t)}{\Psi(t)} dB_3(t) \\
&\le \left(\alpha_2 + k - \delta - \frac{\xi^2}{4} \right) dt + \frac{\xi_2 H}{\Psi} dB_2(t) + \frac{\xi_3 R}{\Psi} dB_3(t).
\end{aligned}
\tag{4.1}
$$

By integrating both sides of the inequality (4.1) from 0 to t and dividing by t, we obtain

$$\frac{\log \Psi(t) - \log \Psi(0)}{t} \le \left(\alpha_2 + k - \delta - \frac{\xi^2}{4} \right) + \frac{1}{t}\int_0^t \frac{\xi_2 H(s)}{\Psi(s)} dB_2(t) + \frac{1}{t}\int_0^t \frac{\xi_3 R(s)}{\Psi(s)} dB_3(t).$$

Combining the strong law of large numbers [26], we have

$$\lim_{t \to \infty} \frac{1}{t}\int_0^t \frac{\xi_2 H(s)}{\Psi(s)} dB_2(t) \le \lim_{t \to \infty} \xi_2 \frac{B_2(t)}{t} = 0 \ a.s.$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\xi_3 R(s)}{\Psi(s)} dB_3(t) \le \lim_{t\to\infty} \xi_3 \frac{B_3(t)}{t} = 0 \ a.s.$$

Therefore, if $\delta + \frac{\xi^2}{4} - \alpha_2 - k > 0$, we can derive

$$\limsup_{t\to\infty} \frac{\log \Psi(t)}{t} \le \alpha_2 + k - \delta - \frac{\xi^2}{4} < 0 \ a.s. \tag{4.2}$$

As a result, we conclude that $\lim_{t\to\infty} \Psi(t) = 0 \ a.s..$ Combining with the positivity of $R(t)$ and $H(t)$, we have $\lim_{t\to\infty} R(t) = 0 \ a.s..$ □

Theorem 4.2. *If $\alpha_1 - \frac{1}{2}\xi_1^2 < 0$, tumor cells will eventually go extinct, such that*

$$\lim_{t\to\infty} T(t) = 0 \ a.s.$$

Proof. By combining the Itô' formula, we have

$$d \log T(t) = \left[\alpha_1 \left(1 - \frac{T}{K_1}\right) - \beta_1 H - \frac{1}{2}\xi_1^2\right] dt + \xi_1 dB_1(t)$$

$$\tag{4.3}$$

$$\le \left(\alpha_1 - \frac{1}{2}\xi_1^2\right) dt + \xi_1 dB_1(t).$$

Integrating the inequality (4.3) from 0 to t and dividing by t, results in

$$\frac{1}{t} \log T(t) \le \frac{1}{t} \log T(0) + \alpha_1 - \frac{1}{2}\xi_1^2 + \frac{\xi_1 B_1(t)}{t}.$$

As t approaches ∞, by combining the strong law of large numbers [26], we obtain that

$$\lim_{t\to\infty} \frac{\xi_1 B_1(t)}{t} = 0 \ a.s. \tag{4.4}$$

Therefore, if $\alpha_1 - \frac{1}{2}\xi_1^2 < 0$, we can derive

$$\limsup_{t\to\infty} \frac{1}{t} \log T(t) \le \alpha_1 - \frac{1}{2}\xi_1^2 < 0 \ a.s.$$

This implies that

$$\lim_{t\to\infty} T(t) = 0 \ a.s. \tag{4.5}$$

In other words, as t approaches ∞, the tumor cell population $T(t)$ tends to 0.

□

Next, we will use the Milstein method [28] to illustrate our main theoretical results. The discretized equations for the system (2.1) are given as follows:

$$T_{k+1} = T_k + \left(\alpha_1 T_k \left(1 - \frac{T_k}{K_1}\right) - \beta_1 T H\right) T_k \Delta t + \xi_1 T_k \sqrt{\Delta t} W_{1,k} + \frac{1}{2}\xi_1^2 T_k \left(W_{1,k}^2 - 1\right) \Delta t,$$

$$H_{k+1} = H_k + (\gamma HR - \delta_1 H - \beta_2 HT) H_k \Delta t + \xi_2 H_k \sqrt{\Delta t} W_{2,k} + \frac{1}{2}\xi_2^2 H_k (W_{2,k}^2 - 1)\Delta t, \tag{4.6}$$

$$R_{k+1} = R_k + \left(\alpha_2 R \left(1 - \frac{R}{K_2}\right) - \gamma HR - \delta_2 R + \frac{kTR}{T + \eta}\right) R_k \Delta t + \xi_3 R_k \sqrt{\Delta t} W_{3,k} + \frac{1}{2}\xi_3^2 R_k (W_{3,k}^2 - 1)\Delta t.$$

Here, $\Delta t > 0$ represents the time increment, $\xi_i (i = 1, 2, 3)$ denotes the noise intensity, and $W_{i,k}(i = 1, 2, 3.\ k = 1, 2, ...)$ follows a standard normal distribution. The parameter values are shown in Table 1.

Table 1. Biological significance and values of parameters.

Parameter	Biological significance	Value	Source
α_1	The intrinsic growth rate of tumor cells	$0.18/day$	[29]
α_2	The intrinsic growth rate of resting T cells	$0.0245/day$	[30]
β_1	The ratio of hunting T cells killing tumor cells	$1.101 \times 10^{-7}/cells/day$	[30]
β_2	The ratio of tumor cells consume hunting T cells	$3.422 \times 10^{-10}/cells/day$	[30]
δ_1	The rate of apoptosis of hunting T cells	$0.0412/day$	[30]
δ_2	The rate of apoptosis in resting T cells	$0.002/day$	estimate
$1/K_1$	The environmental carrying capacity of tumor cells	$2 \times 10^{-9}/cells$	[29]
$1/K_2$	The environmental carrying capacity of resting T cells	$1 \times 10^{-9}/cells$	[30]
γ	The activation rate of resting T cells on hunting T cells	$4.2 \times 10^{-9}/cells/day$	[30]
k	The proliferation rate of resting T cells	$0.1245/day$	[29]
η	The half-saturation coefficient of proliferation factor	$2.019 \times 10^7 cells$	[29]

The initial values for tumor cells, hunting T cells, and resting T cells are given as $T(0) = 5 \times 10^9 (cells)$, $H(0) = 4 \times 10^6 (cells)$, $R(0) = 3 \times 10^8 (cells)$, respectively. First, verify the extinction of resting T cells $R(t)$ with $\xi_1 = 2$, $\xi_2 = 2$, $\xi_3 = 1$. Calculate $\delta + \frac{\xi^2}{4} - \alpha_2 - k = 0.103 > 0$. According to Theorem 4.1, resting T cells will eventually go extinct, which is consistent with the results shown in Figure 1. When $\xi_1 = 0.8$, $\xi_2 = 2$, $\xi_3 = 2$, the calculation yields $\alpha_1 - \frac{1}{2}\xi_1^2 = -0.14 < 0$. According to

Theorem 4.2, tumor cells $T(t)$ will go extinct. As shown in Figure 2, tumor cells eventually converge to zero.
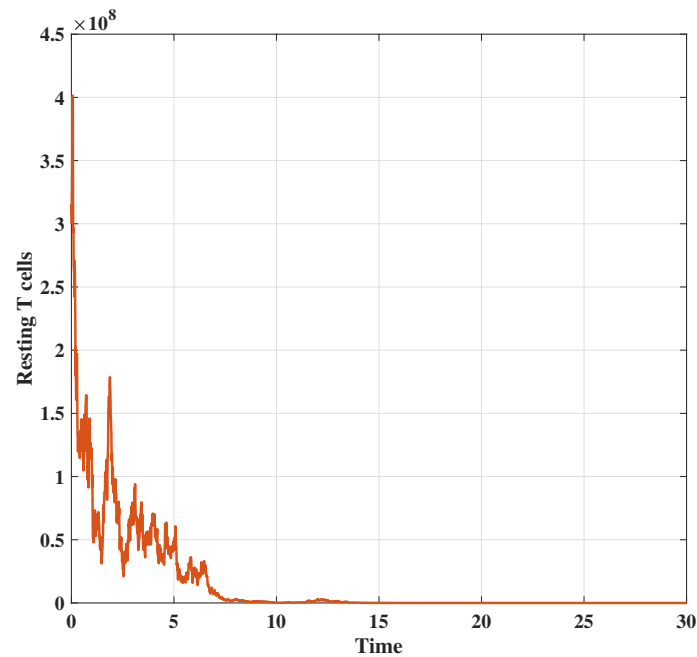

Figure 1. Simulation of extinction of resting T cells $R(t)$ with values of $\xi_1 = 2, \xi_2 = 2, \xi_3 = 1$.
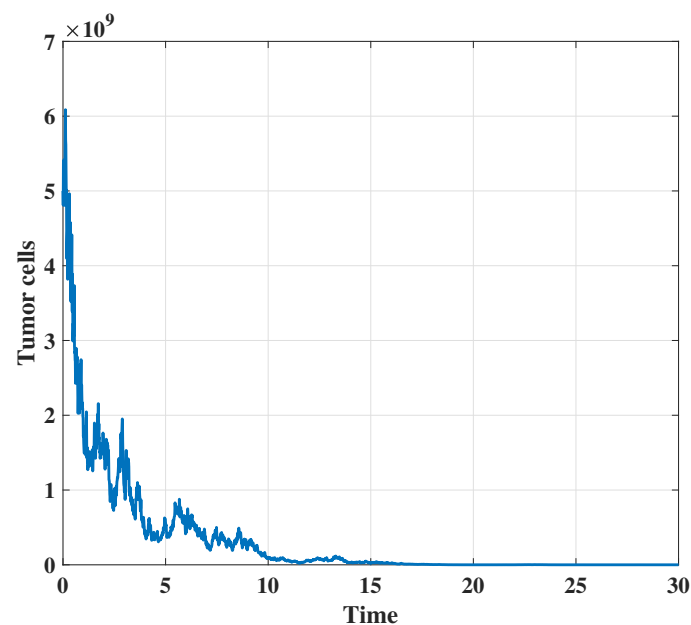

Figure 2. Simulation of extinction of tumor cells $T(t)$ with values of $\xi_1 = 0.8, \xi_2 = 2, \xi_3 = 2$.

Next, let's discuss the effect of stochastic perturbation on tumor cell proliferation by fixing $\xi_2 = 2, \xi_3 = 2$, and varying the noise intensity ξ_1 while ensuring the extinction condition for $T(t)$. We take

$\xi_1 = 0.8, 1.5, 2$ as examples, and the result are shown in Figure 3. From Figure 3, it can be observed that increasing the noise intensity ξ_1 accelerates the extinction of tumor cells. This implies that stochastic environmental perturbations suppress the growth of tumor cells.
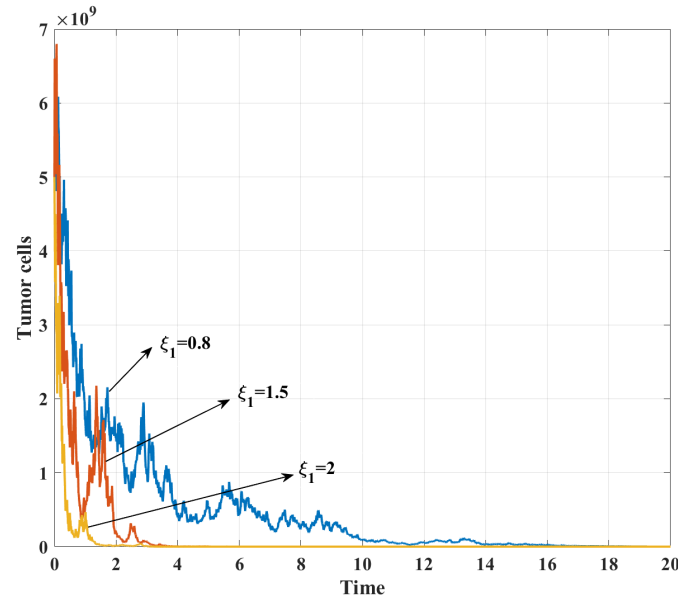

Figure 3. Simulation of extinction of tumor cells $T(t)$ under fixed $\xi_2 = 2$ and $\xi_3 = 2$, with different values of ξ_1 : 0.8, 1.5 and 2.

4.2. Persistence

Lemma 1 ([31]). *Assuming $Z \in [\Omega \times [0, +\infty), R^+]$, if there exist positive constants η, t_0 and η_0 such that*

$$logZ(t) \leq or \geq \eta t - \eta_0 \int_0^t Z(s)ds + \sum_{i=0}^n \sigma_i B_i \quad (\forall t \geq t_0) \tag{4.7}$$

then we have

$$\limsup_{t\to\infty}\langle Z\rangle_t \leq \frac{\eta}{\eta_0} \quad or \quad \limsup_{t\to\infty}\langle Z\rangle_t \geq \frac{\eta}{\eta_0} \quad a.s. \tag{4.8}$$

Lemma 2 ([32]). *Let $T(t), H(t), R(t)$ be the solution of stochastic system (2.1) initial conditions $(T(0), H(0), R(0))$*

1) if $\alpha_1 - \frac{1}{2}\xi_1^2 > 0$, then

$$\limsup_{t\to\infty} \frac{1}{t} \log T(t) \leq 0 \quad a.s. \tag{4.9}$$

2) if $\delta_2 + \frac{1}{2}\xi_3^2 - \alpha_2 - k > 0$, then

$$\limsup_{t\to\infty} \frac{1}{t} \log H(t) \leq 0 \quad a.s. \tag{4.10}$$

3) if $\alpha_2 - \delta_2 + k - \frac{1}{2}\xi_3^2 > 0$, then

$$\limsup_{t\to\infty} \frac{1}{t}\log R(t) \le 0 \quad a.s. \tag{4.11}$$

Proof. Construct the following auxiliary procedures:

$$\begin{cases} dT_1(t) &= \left(\alpha_1 T_1\left(1 - \dfrac{T_1}{K_1}\right)\right)dt + \xi_1 T_1 dB_1(t), \\ dH_1(t) &= (\gamma H_1 R_1 - \delta_1 H_1)\,dt + \xi_2 H_1 dB_2(t), \\ dR_1(t) &= \left(\alpha_2 R_1\left(1 - \dfrac{R_1}{K_2}\right) - \delta_2 R_1 + kR_1\right)dt + \xi_3 R_1 dB_3(t), \end{cases} \tag{4.12}$$

where $B_1(t), B_2(t), B_3(t)$ are mutually independent standard Brownian motions defined in system (2.1). The comparison theorem [27] leads to $0 \le T(t) \le T_1(t), 0 \le H(t) \le H_1(t), 0 \le R(t) \le R_1(t)$. Using the Itô's formula to $\log T_1(t), \log H_1(t)$, and $\log R_1(t)$, then integrating both sides from 0 to t, we get

$$\log T_1(t) - \log T_1(0) = \left(\alpha_1 - \frac{1}{2}\xi_1^2\right)t - \frac{\alpha_1}{K_1}\int_0^t T_1(s)ds + \xi_1 B_1(t),$$

$$\log H_1(t) - \log H_1(0) = \left(-\delta_1 - \frac{1}{2}\xi_2^2\right)t + \gamma\int_0^t R_1(s)ds + \xi_2 B_2(t),$$

$$\log R_1(t) - \log R_1(0) = \left(\alpha_2 - \delta_2 + k - \frac{1}{2}\xi_3^2\right)t - \frac{\alpha_2}{K_2}\int_0^t R_1(s)ds + \xi_3 B_3(t).$$

For convenience to write, we define the notation $\langle x(t) \rangle = \frac{1}{t}\int_0^t x(s)ds$. Then, we can obtain

$$\begin{aligned} \frac{\log T_1(t) - \log T_1(0)}{t} &= \left(\alpha_1 - \frac{1}{2}\xi_1^2\right) - \frac{\alpha_1}{K_1}\langle T_1(t)\rangle + \frac{\xi_1 B_1(t)}{t}, \\ \frac{\log H_1(t) - \log H_1(0)}{t} &= \left(-\delta_1 - \frac{1}{2}\xi_2^2\right) + \gamma\langle R_1(t)\rangle + \frac{\xi_2 B_2(t)}{t}, \\ \frac{\log R_1(t) - \log R_1(0)}{t} &= \left(\alpha_2 - \delta_2 + k - \frac{1}{2}\xi_3^2\right) - \frac{\alpha_2}{K_2}\langle R_1(t)\rangle + \frac{\xi_3 B_3(t)}{t}. \end{aligned} \tag{4.13}$$

According to the strong law of large numbers [26], we get

$$\lim_{t\to\infty}\frac{\xi_1 B_1(t)}{t} = 0 \; a.s. \quad \lim_{t\to\infty}\frac{\xi_2 B_2(t)}{t} = 0 \; a.s. \quad \lim_{t\to\infty}\frac{\xi_3 B_3(t)}{t} = 0 \; a.s. \tag{4.14}$$

For sufficiently small $\varepsilon > 0$, when $\alpha_1 - \frac{1}{2}\xi_1^2 > 0$, according to Lemma 1, it can be deduced that

$$\frac{\left(\alpha_1 - \dfrac{1}{2}\xi_1^2\right) - \varepsilon}{\dfrac{\alpha_1}{K_1}} \le \langle T_1(t)\rangle \le \frac{\left(\alpha_1 - \dfrac{1}{2}\xi_1^2\right) + \varepsilon}{\dfrac{\alpha_1}{K_1}},$$

which implies that

$$\langle T_1(t) \rangle = \frac{\left(\alpha_1 - \frac{1}{2} \xi_1^2 \right)}{\frac{\alpha_1}{K_1}}. \tag{4.15}$$

substituting Eq (4.15) into Eq (4.13) and combining with the Eq (4.14), it yields

$$\lim_{t \to \infty} \frac{1}{t} \log T_1(t) \leq 0 \quad a.s. \tag{4.16}$$

Similarly, when $\delta + \frac{1}{4} \xi^2 - \alpha_2 - k > 0$, Lemma 1 allows us to deduce

$$\frac{\left(\alpha_2 - \delta_2 + k - \frac{1}{2} \xi_3^2 \right) - \varepsilon}{\frac{\alpha_2}{K_2}} \leq \langle R_1(t) \rangle \leq \frac{\left(\alpha_2 - \delta_2 + k - \frac{1}{2} \xi_3^2 \right) + \varepsilon}{\frac{\alpha_2}{K_2}},$$

$$\langle R_1(t) \rangle = \frac{\left(\alpha_2 - \delta_2 + k - \frac{1}{2} \xi_3^2 \right)}{\frac{\alpha_2}{K_2}}. \tag{4.17}$$

substituting Eq (4.17) into Eq (4.13) and combining with Eq (4.14), it yields

$$\lim_{t \to \infty} \frac{1}{t} \log R_1(t) \leq 0 \quad a.s. \tag{4.18}$$

Using the same method as in Theorem 4.2, the condition $\delta_2 + \frac{1}{2} \xi_3^2 - \alpha_2 - k > 0$ leads to $\lim_{t \to \infty} R_1(t) = 0$, then we get

$$\lim_{t \to \infty} \langle R_1(t) \rangle = 0 \; a.s. \tag{4.19}$$

substituting Eq (4.19) into Eq (4.13) and combining with Eq (4.14), it yields

$$\lim_{t \to \infty} \frac{1}{t} \log H_1(t) \leq 0 \quad a.s. \tag{4.20}$$

According to formula Eqs (4.16), (4.18), (4.20) and using the comparison theorem [26], we get

$$\lim_{t \to \infty} \frac{1}{t} \log T(t) \leq 0 \; a.s. \quad \lim_{t \to \infty} \frac{1}{t} \log H(t) \leq 0 \; a.s. \quad \lim_{t \to \infty} \frac{1}{t} \log R(t) \leq 0 \; a.s. \tag{4.21}$$

$\square$

Theorem 4.3. *When $\alpha_1 - \frac{1}{2} \xi_1^2 > 0$ and $\delta - \frac{\xi^2}{4} - \alpha_2 - k > 0$, the tumor cell population $T(t)$ is weakly persistently bounded, with $\limsup_{t \to \infty} \langle T(t) \rangle > 0$. Moreover, it can be deduced that $\sup_{t \to \infty} \langle T(t) \rangle \leq \frac{K_1 \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right)}{\alpha_1}$.*

Proof. According to the Itô's differential rule, we have

$$d \log T(t) = \left[\alpha_1 \left(1 - \frac{T}{K_1} \right) - \beta_1 H - \frac{1}{2} \xi_1^2 \right] dt + \xi_1 dB_1(t). \tag{4.22}$$

By integrating both sides from 0 to t and dividing by t, and taking the upper limit, we obtain

$$\limsup_{t \to \infty} \frac{1}{t} \log \frac{T(t)}{T(0)} = \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right) - \frac{\alpha_1}{K_1} \limsup_{t \to \infty} \langle T(t) \rangle - \beta_1 \limsup_{t \to \infty} \langle H(t) \rangle + \lim_{t \to \infty} \frac{1}{t} \int_0^t \xi_1 dB_1(t).$$

Applying Lemma 2, we have

$$\frac{\alpha_1}{K_1} \limsup_{t \to \infty} \langle T(t) \rangle \geq \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right) - \beta_1 \limsup_{t \to \infty} \langle H(t) \rangle + \lim_{t \to \infty} \frac{1}{t} \int_0^t \xi_1 dB_1(t). \tag{4.23}$$

By using the strong law of large numbers [26], we get $\lim_{t \to \infty} \frac{1}{t} \int_0^t \xi_1 dB_1(t) = 0$ a.s., and when $\delta - \frac{\xi^2}{4} - \alpha_2 - k > 0$, we have $\lim_{t \to \infty} H(t) = 0$. Thus, $\limsup_{t \to \infty} \langle H(t) \rangle = 0$ a.s.. Substituting it into Eq (4.23), we obtain

$$\frac{\alpha_1}{K_1} \limsup_{t \to \infty} \langle T(t) \rangle \geq \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right) > 0 \; a.s. \tag{4.24}$$

Therefore, $\limsup_{t \to \infty} \langle T(t) \rangle > 0$, which further indicates that the tumor cell population T is weakly persistent. In addition,

$$\frac{\log T(t) - \log T(0)}{t} \leq \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right) - \frac{1}{t} \int_0^t \frac{\alpha_1 T(s)}{K_1} ds + \frac{1}{t} \int_0^t \xi_1 dB_1(t).$$

Applying Lemma 1, when $\alpha_1 - \frac{1}{2} \xi_1^2 > 0$, we have

$$\limsup_{t \to \infty} \langle T(t) \rangle \leq \frac{K_1 \left(\alpha_1 - \frac{1}{2} \xi_1^2 \right)}{\alpha_1} \; a.s. \tag{4.25}$$

$\square$

We verify the weak persistence of the tumor cell population below. With noise intensities set to $\xi_1 = 0.3, \xi_2 = 2, \xi_3 = 2$, we calculate that

$$\alpha_1 - \frac{1}{2} \xi_1^2 = 0.135 > 0 \quad and \quad \delta + \frac{\xi^2}{4} - \alpha_2 - k = 0.853 > 0 \tag{4.26}$$

In accordance with Theorem 4.3, we conclude that $T(t)$ is weakly persistently average. As illustrated in Figure 4, the tumor cell population persists at a low level.
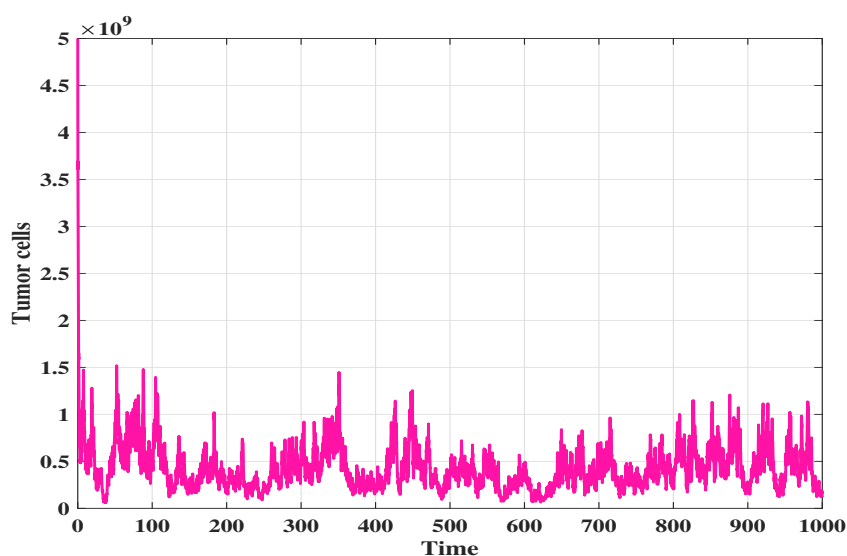
Figure 4. Simulation of the persistence of tumor cells $T(t)$ with values of $\xi_1 = 0.3$, $\xi_2 = 2$, $\xi_3 = 2$.

In order to further investigate the dynamical behavior of tumor cells in the persistent state, we control ξ_1 by selecting $\xi_1 = 0.1, 0.3, 0.5$ respectively to indicate the persistent state of tumor cells under different environmental noise.
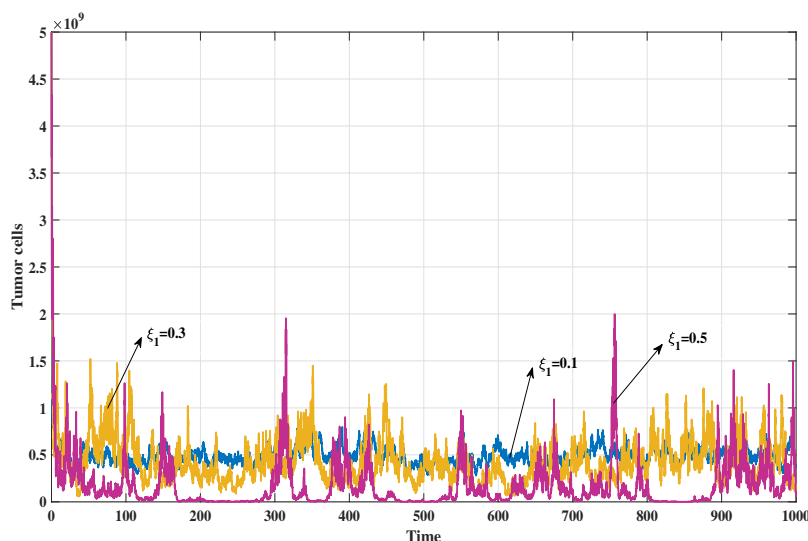

Figure 5. Simulation of the persistence of tumor cells $T(t)$ with values of $\xi_1 = 0.1, 0.3, 0.5, \xi_2 = 2, \xi_3 = 2$.

From Figure 5, it can be observed that as the intensity of noise increases, the fluctuation in the number of tumor cells becomes more pronounced, and the center of oscillation gradually shifts downwards. When $\xi_1 = 0.1$, the number of tumor cells fluctuates around 4.8×10^8, mainly within the range of $[3 \times 10^8, 7 \times 10^8]$. For $\xi_1 = 0.3$, the number of tumor cells is concentrated within the range of

$[1 \times 10^8, 1 \times 10^9]$ with the increase of fluctuation range and the decrease in the probability peak value. When ξ_1 is further increases to 0.5, the number of tumor cells start to fluctuate around zero, which indicating weaker persistence.
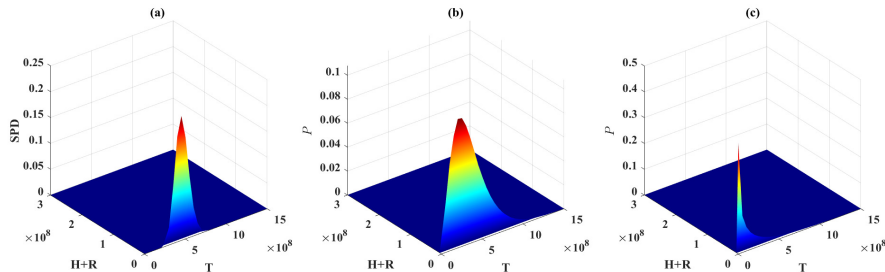

Figure 6. Stationary probability distributions of tumor cells and two types of T cells (a) Simulation of stationary probability distributions of tumor cells and two types of T cells with values of $\xi_1 = 0.1, \xi_2 = 2, \xi_3 = 2$. (b)Simulation of stationary probability distributions of tumor cells and two types of T cells with values of $\xi_1 = 0.3, \xi_2 = 2, \xi_3 = 2$. (c)Simulation of stationary probability distributions of tumor cells and two types of T cells with values of $\xi_1 = 0.5, \xi_2 = 2, \xi_3 = 2$.

Figure 6(a)–(c) depicts the distribution of persistent states of tumor cells under different noise intensity. We combine the populations of hunting T cells and resting T cells to obtain the two-dimensional stationary probability distribution of tumor cells and two types of T cells. It can be observed that the number of two types of T cells is primarily concentrated at position 0. In order to obtain the probability distribution function of tumor cells, we simplify system (2.1) by assuming $H(t) = R(t) = 0$. Thus, system (2.1) degenerates into system (4.27).

$$d\widetilde{T}(t) = \left(\alpha_1 \widetilde{T} \left(1 - \frac{\widetilde{T}}{K_1} \right) \right) \mathrm{dt} + \xi_1 \widetilde{T} dB_1(t). \tag{4.27}$$

By solving the Fokker-Planck equation of Eq (4.27), we obtain an analytical expression for the stationary probability density.

$$p(\widetilde{T}) = N \widetilde{T}^{\frac{2\alpha}{\xi_1^2} - 2} e^{-\frac{2\alpha}{K_1 \xi_1^2} \widetilde{T}}. \tag{4.28}$$

Here, N is the normalization coefficient. Then we validate the accuracy of the analytical expression for the probability density by utilizing the Milstein simulation method [28] to sample. By taking $\xi_1 = 0.1, 0.3, 0.5$ and using a time step of $\Delta t = 0.01$, we sample $N = 1 \times 10^7$ sample points, respectively, and obtain the quantities of tumor cells under different noise intensity.
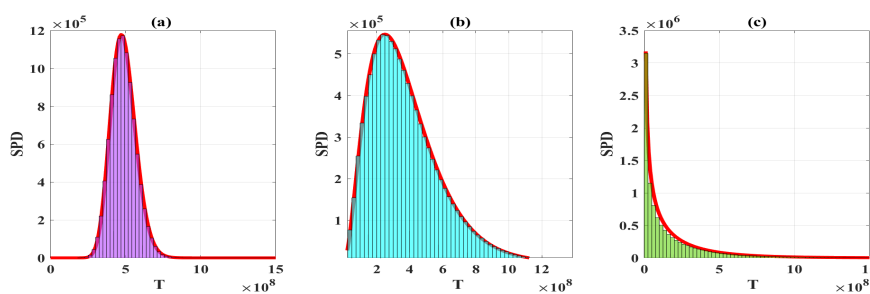
Figure 7. Steady-state probability density of tumor cells (a) Simulation with values of $\xi_1 = 0.1, \xi_2 = 2, \xi_3 = 2$. (b) Simulation with values of $\xi_1 = 0.3, \xi_2 = 2, \xi_3 = 2$. (c) Simulation with values of $\xi_1 = 0.5, \xi_2 = 2, \xi_3 = 2$.
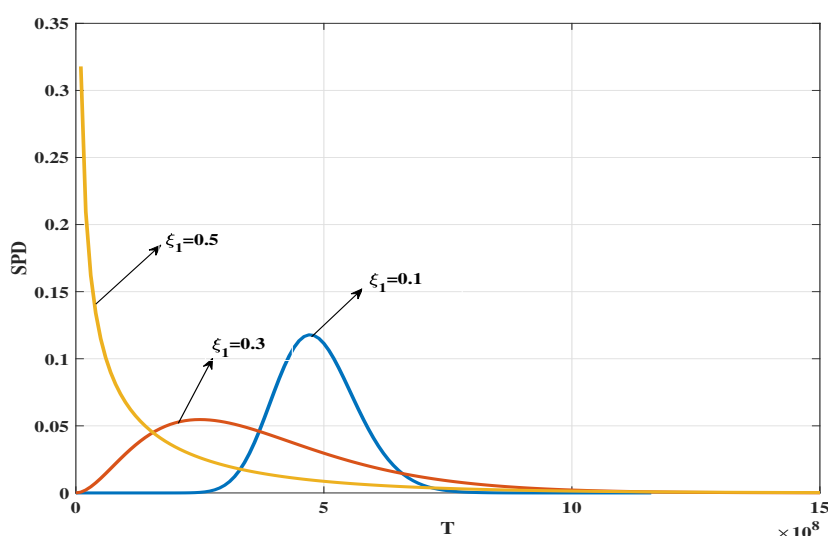

Figure 8. Stationary probability density function of tumor cells with values of $\xi_1 = 0.1, 0.3, 0.5, \xi_2 = 2, \xi_3 = 2$.

The histogram in Figure 7(a)–(c) represents the distribution of the number of sample points under different noise intensity, and the curve fitted on the histogram edges represents the probability distribution function based on Eq (4.28). It can be observed from Figure 7 that the function fits the distribution of sample points perfectly. By approximating the frequencies of tumor cells at different quantities in Figure 7 as probabilities, an accurate expression for $p(x)$ is obtained by substituting it into Eq (4.28). Figure 8 illustrates the steady-state probability distributions of tumor cells under different noise intensities. It can be observed that when the noise intensity is low, the probability density distribution exhibits a unimodal shape, indicating that the tumor maintains stable growth within a certain quantity range. As the noise intensity increases, the tumor cell population moves towards lower levels, and the peak value of the probability decreases, indicating that the increased noise intensity effectively suppresses the expansion of the tumor population. When ξ_1 continues to increase beyond a certain threshold, the probability density function transitions from a unimodal shape to a decreasing shape, and the tumor population concentrates around zero, indicating that tumor growth is significantly restricted, with lower invasiveness and metastatic potential.

5. Conclusions

We investigate the dynamic behavior and stationary response of tumor cells in a predator-prey like system with resting T cells. First, we prove the existence and uniqueness of global positive solutions for the stochastic system which guarantees the biological significance of the system model. Second, we establish the boundedness of moments for $T(t), H(t)$ and $R(t)$ by constructing appropriate auxiliary equations. Finally, we provide sufficient conditions for the extinction of resting T cells, as well as the threshold for persistence and extinction of tumor cells and validate the results through numerical simulations. When ξ_1 is small, the tumor remains in a persistent state. From Figure 6, it can be seen that in the persistent state, the quantities of two types of T cells tend to cluster around zero. By simplifying the system (2.1) through the assumption $H(t) + R(t) = 0$, we obtain the system (4.27) and derive an analytical expression for the stationary probability density of the tumor. The stochastic perturbations in the environment play a crucial role in eliminating tumor cells, increasing the noise intensity in the persistent state leads to stochastic bifurcation, and the stationary probability density of tumors transitions from unimodal to decreasing. When the noise intensity reaches a certain threshold, tumor cells eventually transition from a persistent state to an extinct state, and further increasing the noise intensity accelerates tumor extinction.

In fact, due to the influence of factors such as radiation and viruses, parameters in the tumor immune system may undergo mutations. Therefore, it is crucial to investigate the parameter changes and corresponding responses of the tumor immune system in different environments. The further efforts can be made to study the dynamic behavior of tumor immune system under stochastic switching.

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