9th International Congress of Serbian Society of Mechanics Vrnjačka Banja, Serbia, July 5-7, 2023

THE BRACHISTOCHRONIC MOTION OF CHAPLYGIN SLEIGH IN A VERTICAL PLANE WITH UNILATERAL NONHOLONOMIC CONSTRAINT

Aleksandar M. Obradović [1], Oleg Yu. Cherkasov[2], Luka N. Miličić[1]

[1] Faculty of Mechanical Engineering,
University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35, Serbia
e-mail: aobradovic@mas.bg.ac.rs, lmilicic@mas.bg.ac.rs
[2] Lomonosov Moscow State University,
Faculty of Mechanics and Mathematics, GSP-1, 1 Leninskye Gory, 119991, Moscow, Russia
e-mail: OYuChe@yandex.ru

Abstract:

The paper considers the procedure for determining the brachistochronic motion of the Chaplygin sleigh in a vertical plane, where the blade is such that it prevents the motion of the contact point in one direction only. The position of the sleigh mass center and orientation at the final positions is specified, as well as the initial value of mechanical energy. The simplest formulation of a corresponding optimal control problem is given and it is solved by applying Pontryagin's maximum principle. For some cases, analytical solutions of differential equations of the two-point boundary value problem (TPBVP) of the maximum principle, were found. Numerical integration was carried out for other cases using the shooting method, where the assessment of missing terminal conditions was given and it was shown that the solution obtained represents the global minimum time for the brachistochronic motion.

Key words: Chaplygin sleigh, maximum principle, unilateral nonholonomic constraint, analytical solutions.

1. Introduction

Classical Bernoulli's problem of determining the brachistochrone for the particle in a vertical plane [1] has experienced a large number of attempts at this problem generalizations for different, more complex, mechanical systems. A more detailed review of literature devoted to these generalizations can be found in a doctoral dissertation [2], and in papers [3,4]. This paper will deal with determining the brachistochronic motion of Chaplygin sleigh in a vertical plane, which was also the research subject in [5]. However, it differs basically from mentioned paper by a unilateral nonholonomic constraint, unlike classical bilateral constraint of the blade type, that is present in [5]. Details of this type of constraint can be seen in [6], which presents an appropriate blade profile that corresponds to a unilateral constraint.

Chaplygin sleigh (Fig. 1.) of mass m and the radius of inertia i with respect to the central axis is moving in a vertical plane. The sleigh position is defined by coordinates X and Y of the mass center C and angle φ between axis O_x and blade in the point C. The sleigh position is specified at the initial and final moment but the initial value of mechanical energy is also known, $E = mgL$, which is large enough for the sleigh to reach the final position, where g is acceleration of gravity. Let U and V be the projections of the velocity of point C onto a blade and perpendicular to a blade, respectively. If the constraint is bilateral [5], then $V = 0$, and this paper will also consider the cases of unilateral constraints, when $V \geq 0$ or $V \leq 0$,depending on two possible cases, related to the side to which the constraint restrains. The same figure also shows profiles of the corresponding blades for all mentioned cases.

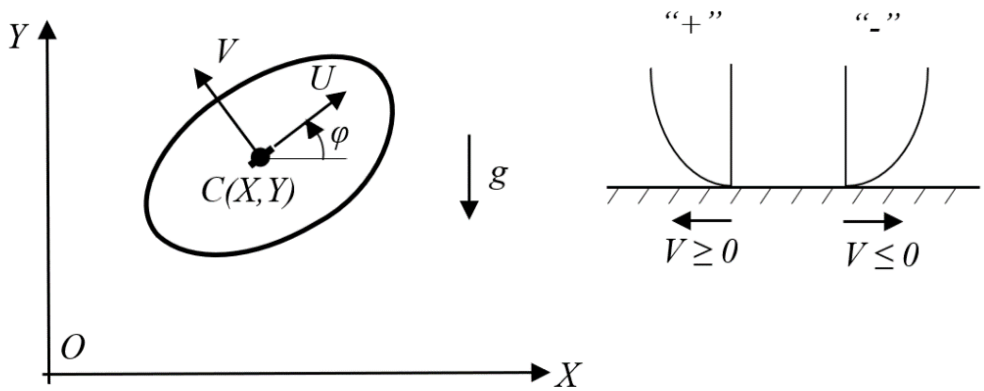

Fig. 1. Chaplygin sleigh in a vertical plane

During brachistochronic motion, which in the original understanding of this problem is realized by ideal mechanical constraints, without action of the active forces, the mechanical energy remains unchanged:

$$\frac{1}{2}m\left(U^2 + V^2 + i^2\Omega^2\right) + mgY = mgL \tag{1}$$

where L is a given constant, which has the dimension of length.

In this paper, the brachistochronic motion will be determined only using kinematic differential equations:

$$\begin{aligned} \dot{X} &= U\cos\varphi - V\sin\varphi \\ \dot{Y} &= U\sin\varphi + V\cos\varphi \\ \dot{\varphi} &= \Omega \end{aligned} \tag{2}$$

which significantly simplifies this problem, considering the determination of the brachistochronic motion itself. The variable Ω represents the angular velocity of the body. Although concrete realization of this motion by ideal mechanical constraints is not the subject of this paper, it is only remarked here in that when the reaction *Rc* is defined from dynamic equations, it must be checked whether its direction corresponds to the unilateral constraint orientation.

In order to determine the brachistochronic motion, which corresponds to a minimum time between two specified positions, Pontryagin's maximum principle will be applied [7]. Chapter 2 will present the simplest possible formulation of the optimal control problem. Chapter 3 will describe the procedure of solving this optimization task with special reference to the possibility of obtaining analytical solutions. Illustration of the procedure given by using concrete numerical examples will be reported in Chapter 4.

2. Formulation of the optimal control problem

Introducing dimensionless quantities:

$$X = ix, \quad Y = iy, \quad \Omega = \omega\sqrt{\frac{g}{i}}, \quad V = \pm v^2\sqrt{gi}, \quad U = u\sqrt{gi}, \quad t = \tau\sqrt{\frac{i}{g}}, \quad L = il \tag{3}$$

the equations of state are obtained:

$$\begin{aligned} x' &= u\cos\varphi \mp v^2\sin\varphi \\ y' &= u\sin\varphi \pm v^2\cos\varphi \\ \varphi' &= \omega \end{aligned} \tag{4}$$

where the appropriate sign, “**+**” or ”**–**“**,** corresponds to the direction for the case of a corresponding blade. The notation $(...)'$ represents differentiation with respect to dimensionless time τ . Conservation of mechanical energy in dimensionless variables is of the form:

$$u^2 + v^4 + \omega^2 + 2y - 2l = 0 \tag{5}$$

The initial conditions of motion are:

$$\tau_0 = 0 \quad x(\tau_0) = 0 \quad y(\tau_0) = 0 \quad \varphi(\tau_0) = 0 \tag{6}$$

whereas the final position is defined according to:

$$\tau_1 = ? \quad x(\tau_1) = x_1 \quad y(\tau_1) = y_1 \quad \varphi(\tau_1) = \varphi_1 \tag{7}$$

The problem of determining the brachistochronic motion consists of determining the optimal controls:

$$u = u(\tau) \quad v = v(\tau) \quad \omega = \omega(\tau) \tag{8}$$

and their corresponding final equations of motion of this system so that the system is converted from the initial state (6) in a minimum time $\tau_1$into the state (7) with a constraint (5). The functional being minimized in this problem has the form:

$$J = \int_0^{\tau_1} d\tau = \tau_1 \tag{9}$$

3. Solving the optimal control problem

In order to solve this problem, let us write the appropriate Pontryagin’s function H for the case of time minimization:

$$H = -1 + \lambda_x(u\cos\varphi \mp v^2\sin\varphi) + \lambda_y(u\sin\varphi \pm v^2\cos\varphi) + \lambda_\varphi\omega - \mu(u^2 + v^4 + \omega^2 + 2y - 2l) \tag{10}$$

where μ is the multiplier corresponding to the constraint of mechanical energy (5) and λ_x , λ_y , λ_φ are co-state variables. The co-state system of differential equations reads:

$$\lambda_x{}' = 0 \quad \lambda_y{}' = 2\mu \quad \lambda_\varphi{}' = -\left(\lambda_x\left(-u\sin\varphi \mp v^2\cos\varphi\right) + \lambda_y\left(u\cos\varphi \mp v^2\sin\varphi\right)\right) \tag{11}$$

Optimality conditions of the maximum principle:

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0, \quad \frac{\partial H}{\partial \omega} = 0 \tag{12}$$

yield expressions for optimal controls:

$$u=\frac{1}{2\mu}(\lambda_x\cos\varphi+\lambda_y\sin\varphi)$$
$$v^2=0\ \vee\ v^2=\frac{\pm1}{2\mu}(-\lambda_x\sin\varphi+\lambda_y\cos\varphi) \qquad (13)$$
$$\omega=\frac{1}{2\mu}\lambda_\varphi$$

where the multiplier μ, in cases when independent variable τ at the end of motion is not specified, is defined from conditions:

$$H(\tau)=0 \qquad (14)$$

and it has the value:

$$\mu(\tau)=\frac{1}{4(l-y(\tau))}>0 \qquad (15)$$

which is positive over the considered interval of motion.

In order that it would be the maximum of Pontryagin's function, the corresponding second-order derivatives must be negative:

$$\frac{\partial^2 H}{\partial u^2}<0,\quad \frac{\partial^2 H}{\partial v^2}<0,\quad \frac{\partial^2 H}{\partial \omega^2}<0 \qquad (16)$$

wherefrom the criterion is finally obtained for the choice of an appropriate control:

$$u=2(l-y)(\lambda_x\cos\varphi+\lambda_y\sin\varphi)$$
$$v^2=\begin{cases}0, & \pm(-\lambda_x\sin\varphi+\lambda_y\cos\varphi)\le 0\\ \pm2(l-y)(-\lambda_x\sin\varphi+\lambda_y\cos\varphi), & \pm(-\lambda_x\sin\varphi+\lambda_y\cos\varphi)>0\end{cases} \qquad (17)$$
$$\omega=2(l-y)\lambda_\varphi$$

Differential equations of the two-point boundary value problem (TPBVP) are obtained by substituting (17) in (4) and (11). It is necessary to supplement boundary conditions (6) and (7) with condition (5) at the beginning or end of the interval of motion in order to also define unknown dimensionless time of motion τ_1.

Here, it should be noted that, in a general case, different solutions of (17) can be combined over different intervals of motion depending on the sign of a corresponding expression. In a general case, there are not analytical solutions of differential equations of the TPBVP if $v^2=0$ over the entire interval of motion or over some subintervals of motion. In those cases, the problem must be solved numerically. If the shooting method is applied and backward numerical integration is done, by the choice of three parameters, λ_x , λ_{y1} , τ_1**,** three initial conditions are guessed **(**6**),** where it is, based on (5), (14) and (17):

$$\lambda_\varphi(\tau_1)=\pm\sqrt{\frac{1}{2(l-y_1)}-(\lambda_x\cos\varphi_1+\lambda_{y1}\sin\varphi_1)^2} \qquad (18)$$

and it should be checked which sign corresponds to the task concrete parameters.

Assessment of the interval of parameters' values is also obtained from (5):

$$\begin{aligned}&\left|\lambda_x\cos\varphi_1+\lambda_{y1}\sin\varphi_1\right|\le\frac{1}{\sqrt{2(l-y_1)}}\\&\left|\lambda_x\right|\le\frac{1}{\sqrt{2l}}\\&\left|\lambda_{y1}\right|<\frac{1}{\sqrt{2(l-y_1)}}\end{aligned}\tag{19}$$

and it significantly assists in shooting as well as in seeking the global minimum if there are multiple solutions for TPBVP.

If the constraint is side-oriented so that the extremal solution over the entire interval is on an open set, differential equations of TPBVP have a simpler form:

$$\begin{aligned}&x'=2(l-y)\lambda_x \qquad \lambda_x{}'=0\\&y'=2(l-y)\lambda_y \qquad \lambda_y{}'=\frac{1}{2(l-y)}\\&\varphi'=2(l-y)\lambda_\varphi \qquad \lambda_\varphi{}'=0\end{aligned}\tag{20}$$

and have general solutions in the analytical form:

$$\begin{aligned}&y=l-\frac{1+\cos(pt+\alpha)}{p^2}\\&x=\frac{2\lambda_x}{p^2}(t+\frac{1}{p}\sin(pt+\alpha))+C_1\\&\varphi=\frac{2\lambda_\varphi}{p^2}(t+\frac{1}{p}\sin(pt+\alpha))+C_2\end{aligned}\tag{21}$$

where $(p,\alpha,\lambda_x,\lambda_\varphi,C_1,C_2)$ are determined together with unknown moment τ_1 from (5), (6) and (7). It should be noted that these solutions also correspond to the case when the blade does not exist at all, consequently for the general case of the brachistochronic plane-parallel motion of a rigid body.

Form and structure of optimal controls will depend on whether the constraint is bilateral, unilateral and whether it exists at all, if unilateral it will also depend on corresponding direction, but on initial conditions (6), final conditions (7) and quantity l, as well.

4. Numerical examples

Let us show for the concrete parameters of the task:

$$l=2,\ x_1=\varphi_1=\frac{\pi+2}{2\sqrt{2}},\ y_1=1\tag{22}$$

the procedure described in previous chapters, particularly for the constraints $V\ge 0$ (Case "+") and $V\le 0$ (Case "-").

In the case "-", it is shown that over the entire interval of motion the solution is on an open set and that there are analytical solutions of the corresponding TPBVP:

$$\begin{aligned}&x=\frac{(\tau+\sin\tau)}{\sqrt{2}},\quad y=1-\cos\tau,\quad \varphi=\frac{(\tau+\sin\tau)}{\sqrt{2}}\\&\lambda_x=\frac{1}{2\sqrt{2}},\quad \lambda_y=\frac{\sin\tau}{2(1+\cos\tau)},\quad \lambda_\varphi=\frac{1}{2\sqrt{2}},\quad \tau_1^-=\frac{\pi}{2}\end{aligned}\tag{23}$$

It can be also noticed that the mass center trajectory is a deformed cycloid with the coefficient $\frac{1}{\sqrt{2}}$.

In the case "+", it is shown that over the entire interval of motion the extremal solution is on the boundary $V = 0$. Based on the assessments (19) of all possible values of the missing parameters, considering their positive signs, graphical representation can be given of the surfaces that correspond to obtaining the appropriate initial conditions. Figure 2 shows three surfaces of different colors, each of which corresponds to the fulfillment of one of the initial conditions (6).

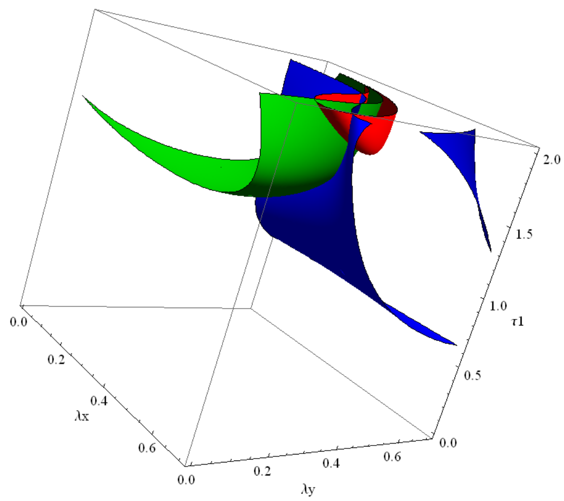

Fig. 2. Representation of the global minimum time of motion.

These surfaces have been obtained based on numerical calculations of the corresponding TPBVP in the program Wolfram Mathematica [8] using commands NDSolve[...], ContourPlot3D[...] and they represent numerical dependencies:

$$x_0(\lambda_x, \lambda_{y1}, \tau_1) = 0, \quad y_0(\lambda_x, \lambda_{y1}, \tau_1) = 0, \quad \varphi_0(\lambda_x, \lambda_{y1}, \tau_1) = 0 \tag{24}$$

The parametric values sought are at their intersection:

$$\lambda_x = 0.499037, \quad \lambda_{y1} = 0.369326, \quad \tau_1^+ = 1.76731 \tag{25}$$

and it can be also seen that the point of intersection is at the lowest position along the axis of dimensionless moment τ_1, whereby it is numerically shown that the obtained solution represents the global minimum time of motion. This method of graphical representation for any three-parameter shooting is very useful when TPBVP has multiple solutions, of which the one with the lowest time should be chosen [9].

Numerical solutions correspond to parametric values (25), as shown in Fig. 3, together with analytical solutions (23), which correspond to opposite orientation of a unilateral constraint.

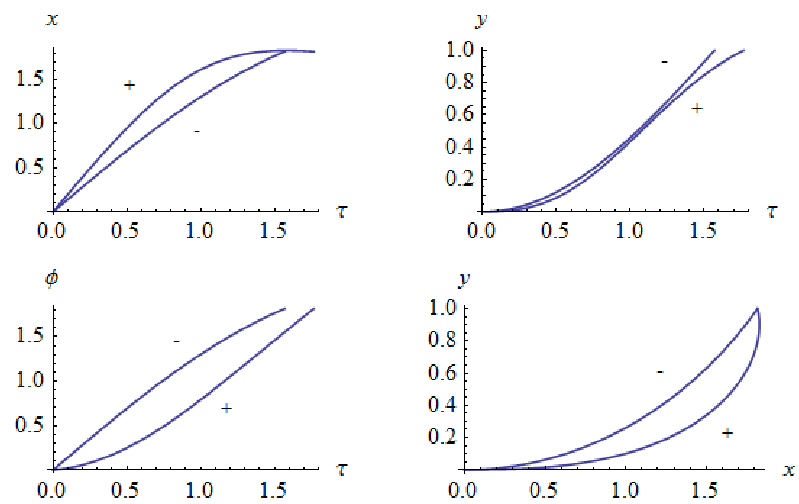

Fig. 3. Final equations of motion and mass center trajectory of the Chaplygin sleigh.

It can be noticed that the time of motion in Case “-”, τ_1^-, is lower compared to τ_1^+ in Case “+”. The first case corresponds to the brachistochronic plane-parallel motion of a rigid body in the vertical plane, while the second case corresponds completely to the case of a classical double-sided blade, as analyzed in [5].

The same figure depicts the trajectory of the mass center for both cases.

In addition, based on the numerical and analytical solution, it can be shown that the function $-\lambda_x sin\varphi + \lambda_y cos\varphi$ is negative on the entire interval of motion and that conditions (17) of the maximum principle are fully satisfied.

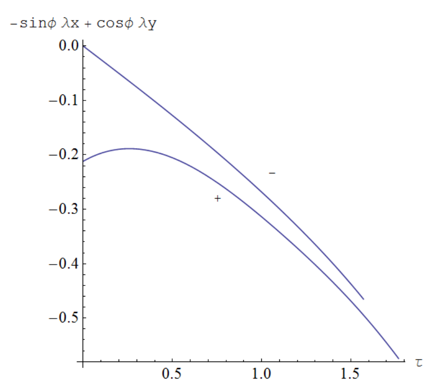

Fig. 4. Function $-\lambda_x sin\varphi + \lambda_y cos\varphi$.

5. Conclusions

The paper contributes to the unexplored area of optimal control of the motion of nonholonomic mechanical systems with unilateral restricted constraint reaction. The simplification of the optimal control task is also the originality of our work, based only on kinematic equations. A special contribution represents the analytical solution of differential equations for individual cases of unilateral constraints, when the solution is on an open set. It also entirely corresponds to the case of brachistochronic plane-parallel motion of a rigid body in the vertical plane, when the blade does not exist. The mass center trajectory is a deformed cycloid, unlike ordinary cycloid, in the case of classical brachistochrone. It is shown that in the concrete case, with numerical solution of TPBVP, the obtained solution represents, for specified task parameters, the global minimum time of motion.

Continuing research can take place in several directions. Such task parameters can be defined, contour conditions and initial mechanical energy, that the brachistochronic motion contains segments where optimal solution is from an open set and segments on which $V = 0$. It is also possible, as indicated in a classical work by Caratheodory [10], to impose maximum possible value of the unilateral constraint reaction of the blade. In that regard, the results of [11], which considered constrained classical nonholonomic bilateral constraint of the blade type, could be generalized to the motion in the vertical plane with unilateral constraint. Also, it would be necessary to show the method of realization by the help of ideal constraints, which would be also generalization of work [11], for the case of guides, or work [12], if motion is realized by rolling of the moving centroid along the stationary one. This additional research would require using of dynamical differential equations of the Chaplygin sleigh motion.

Acknowledgments

The results reported in this paper were obtained from the study supported by the Ministry of Science, Technological Development and Innovations of the Republic of Serbia. Contract No 451-03-47/2023-01/ 200105, date 03/02/2023

References

[1] Bernoulli, J.: *Problema novum ad cuius solutionem Mathematici invitantur (A new problem that mathematicians are invited to solve).* Acta Eruditorum. 15 (1696), 264–269.

[2] Zarodnyuk, A.V., *Optimization of controlled descent and generalized brachistochrone problems (in Russian)*, PhD, Moscow State University, Faculty of Mechanics and Mathematics, 2018.

[3] Sumbatov, A.S., *Problem on the brachistochronic motion of a heavy disk with dry friction*, International Journal of Non-Linear Mechanics, (2018), T. 99. C. 295-301.

[4] Sumbatov, A.S., *Brachistochrone with Coulomb friction as the solution of an isoperimetrical variational problem*, International Journal of Non-Linear Mechanics, (2017), T. 88. C. 135-141.

[5] Golubev, Y.F. *Brachistochrone for a rigid body sliding down a curve*, J. Comput. Syst. Sci. Int. 52, 571–587 (2013).

[6] Kozlov, V. V., *On the Dynamics of Systems with One-sided Non-integrable Constraints*, Theoretical and Applied Mechanics, Volume 46 (2019) Issue 1, 1–14.

[7] Pontryagin, L. S., Boltyansky, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., *The Mathematical Theory of Optimal Processes*, Interscience Publishers, John Wiley and Sons, New York, 1962.

[8] Wolfram, S. The Mathematica Book, 5th ed. Champaign, IL: Wolfram Media, 2003.

[9] Radulović, R., Šalinić, S., Obradović A., Rusov S., *A new approach for the determination of the global minimum time for the Chaplygin sleigh brachistochrone problem*, Mathematics and Mechanics of Solids, ISSN:1081-2865, (2017), vol. 22 br. 6, pp 1462-1482.

[10] Caratheodory, C., *Der Schlitten*, ZAMM-Z. Angew. Math. Me.13 (1933) 71–76.

[11] Šalinić, S., Obradović, A., Mitrović, Z. and Rusov, S., *On the brachistochronic motion of the Chaplygin sleigh*, Acta Mechanica 224 (2013) 2127-2141.

[12] Obradović A., Šalinić S., Jeremić O., Mitrović Z., *On the brachistochronic motion of a variable-mass mechanical system in general force fields*, Mathematics and Mechanics of Solids, ISSN: 1081-2865, 2014, Vol. 19(4) 398–410.