On Gaussian rules for some modified Chebyshev weights.

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Abstract

In this paper, Gaussian rules for some modified Chebyshev weights introduced by Gautschi and Li in 1993 are considered. Our main concern is providing efficient estimations for the error of quadrature. Those estimations are checked by means of some numerical examples.

2010 Mathematics Subject Classifications : Primary 65D32, 65D30; Secondary 41A55

Keywords: Gauss quadrature formulae, Chebyshev weight functions, contour integral representation, remainder term for analytic functions, error bound.

1 Introduction

In [1], the authors considered a polynomial modification of a given positive measure $d\sigma$ supported on the real axis. Namely, if $n \in \mathbb{N}$ and π_n is the orthogonal polynomial of degree n with respect to $d\sigma$, they deal with the new sequence of polynomials $\{\widehat{\pi}_{m,n}\}$, being orthogonal with regard to the modified measure $d\widehat{\sigma}_n = \pi_n^2 \, d\sigma$. While in general is quite difficult getting explicit expressions for the induced orthogonal polynomials, it is not hard when dealing with the four Chebyshev weights, as pointed out by the authors in [1]. This new family of polynomials, hereafter referred to as "induced" orthogonal polynomials, has a number of applications in constructive approximation of functions, which justifies the interest in studying quadrature rules for approximating integrals with some kind of modified weights. In this note, we focus in estimating the error of Gauss rules for this modified weights in the case of the four Chebyshev weights, and the different bounds we obtain are tested by means of numerical examples.

The problem of estimating the quadrature error for Gauss–type rules has been thoroughly studied in the literature; to only cite a few, see the references [2]–[7].

2 Main Results

Throughout this note, we deal with integrals of the form

$$I_\sigma(f) = I(f;\sigma,n) = \int f(t)\, d\widehat{\sigma}_n(t)\,,$$

where $d\widehat{\sigma}_n = \pi_n^2\, d\sigma$, and $d\sigma$ is one of the four Chebyshev weights, namely

$$d\sigma^{[1]}(t) = \frac{dt}{\sqrt{1-t^2}}\,,\ d\sigma^{[2]}(t) = \sqrt{1-t^2}\, dt\,,$$
$$d\sigma^{[3]}(t) = \sqrt{\frac{1-t}{1+t}}\, dt\,,\ d\sigma^{[4]}(t) = \sqrt{\frac{1+t}{1-t}}\, dt\,.$$

by means of Gauss rules

$$I_m(f) = \sum_{j=1}^{m} A_{m,j}\, f(t_{m,j})\,,\ m = 1,2,\dots\,,$$

which means that the nodes $\{t_{m,j}\}$ are the zeros of the induced orthogonal polynomial $\{\widehat{\pi}_{m,n}\}$. While for the case where $i = 1$ and $n = 1$, whose related weight will be referred hereafter to as $d\sigma^{[I]}$, Gauss rules with an arbitrary number m are considered, otherwise we restrict ourselves to the case where $m = n$ for the sake of simplicity. In addition, since the orthogonal polynomials with respect to the measures $d\sigma^{[3]}$ and $d\sigma^{[4]}$ are easily connected to each other, only the results for $d\sigma^{[3]}$ are shown.

In this sense, our main concern is estimating the error of quadrature. It is well–known that in the usual case where the integrand f is analytic in a neighborhood Ω of a compact interval, say $[-1,1]$, this error admits the representation

$$R_m(f) = I_\sigma(f) - I_m(f) = \frac{1}{2\pi i}\oint_\Gamma K_m(z)\, f(z)\, dz\,,$$

where the kernel K_m is given by

$$K_m(z) = \frac{\varrho_{m,n}(z)}{\widehat{\pi}_{m,n}}\,,\ \ \varrho_{m,n}(z) = \int_{-1}^{1} \frac{\pi_m(t)}{z-t}\, w(t)\, dt\,,$$

Γ being any closed smooth contour contained in Ω and surrounding the real interval $[-1,1]$. As usual, elliptic contours with foci at ± 1 and sum of the semi–axes equal to $\rho > 1$, are considered. These level contours admit the expression

$$\mathcal{E}_\rho = \{z \in \mathbb{C} : |\phi(z)| = |z + \sqrt{z^2-1}| = \rho\}\,,$$

where the branch of $\sqrt{z^2-1}$ is taken so that $|\phi(z)| > 1$ for $|z| > 1$.

Next, we state our main results. For details about their proofs, as well as other possible error bounds, see [5]. On the sequel, we denote $\rho_f = \sup\{\rho > 1 : f \text{ is analytic on } D_\rho\}$.

Theorem 2.1 *The following L^∞–type bounds for the error, where $\|f\|_{\mathcal{E}_\rho} = \max_{z\in\mathcal{E}_\rho}$, hold.*

$$r_1^{[I]}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1 \left(\rho^2+1+(-1)^{m/2}\left(\rho^{m+2}+\rho^m\right)\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{\rho^{m+2}\left(\rho-\rho^{-1}\right)\left(\sum_{j=0}^{m/2}(-1)^j\rho^{m-2j}+\sum_{j=0}^{(m-1)/2}(-1)^j\rho^{2j-m}\right)},$$

if m is even, and

$$r_1^{[I]}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1 \left((m+2)\rho^2+m\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{m\rho^{m+2}\left(\rho-\rho^{-1}\right)\left(\sum_{j=0}^{(m-1)/2}(-1)^j\frac{m-2j}{m}\rho^{m-2j}+\sum_{j=0}^{(m-1)/2}(-1)^j\frac{m-2j}{m}\rho^{2j-m}\right)}$$

if m is odd. In the same way,

$$r_1^{(1)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(3\rho^{2n}+1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n-2}\rho^{3n}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)}. \quad (1)$$

$$r_1^{(2)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(2\rho^{2n+2}-\rho^{2n}-1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n}\rho^{3n+2}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)}, \quad n>1. \quad (2)$$

$$r_1^{(3)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(2\rho^{2n+1}+\rho^{2n}+1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n}\rho^{3n+1}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)}, \quad (3)$$

where $\rho^* > 1$ is a value obtained empirically (see [5] for details), and it was shown to be relatively closed to 1 in all the cases.

Theorem 2.2 *The following upper bounds for the error of quadrature, based on the Fourier–Chebyshev expansion of the error, hold.*

$$r_2^{(1)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n-2}}\frac{1}{\rho^{2n}-1}\|f\|_{\mathcal{E}_\rho},\ n\geq 1. \quad (4)$$

$$r_2^{(2)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n}}\left(\frac{1}{\rho^{2n}-1}+\frac{1}{2\rho^{2n+2}}\right)\|f\|_{\mathcal{E}_\rho}. \quad (5)$$

$$r_2^{(3)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n}}\left(\frac{1}{\rho^{2n}-1}\right)\|f\|_{\mathcal{E}_\rho}. \quad (6)$$

Theorem 2.3 *The following L^1–type bounds for the error of quadrature also hold.*

$$r_3^{(1)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n\cdot 2^{2n-1}}\sqrt{\frac{7\rho^{-2n}+9\rho^{2n}}{\rho^{4n}-1}}\|f\|_{\mathcal{E}_\rho}. \quad (7)$$

$$r_3^{(2)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n \cdot 2^{2n+1}} \sqrt{\frac{\rho^{2n-4} + 4\rho^{2n} + 3\rho^{-2n-4}}{\rho^{4n} - 1}} \, \|f\|_{\mathcal{E}_\rho}. \tag{8}$$

$$r_3^{(3)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n \cdot 2^{2n+1}} \sqrt{\frac{\rho^{2n-2} + 4\rho^{2n} + 3\rho^{-2n-2}}{\rho^{4n} - 1}} \, \|f\|_{\mathcal{E}_\rho}. \tag{9}$$

3 Numerical experiments and Conclusion

Now, we are concerned with checking the accuracy of the quadratures above, as well as of the bounds given in previous Theorems 2.1 and 2.2, when the characteristic example $f_1(z) = e^{\cos(\omega z)}$, $\omega > 0$; it is an entire function and, thus, $\rho_f = +\infty$. In the following Tables, the results obtained by applying our Gauss rules to the Chebyshev weights are displayed, along with the error bounds provided in the above theorems, as well as the actual values of the integrals and the errors. In Tables below the error bounds $r_j^{(i)}$, $i, j = 1, 2, 3$, given in (1)–(9), along with the actual values of the errors and the integrals, are displayed for $\omega = 1$ and some values of n. It is noteworthy that in general the estimates of the error are quite sharp, as well as the accuracy of the respective quadrature rules. More numerical results are displayed in [5].

n, ω	$r_1^{[1]}(f_1)$	$r_2^{[1]}(f_1)$	$r_3^{[1]}(f_1)$	Error$^{[1]}$	$I_\omega^{[1]}(f_1)$
$6, 1$	$4.856(-9)$	$3.095(-9)$	$4.643(-9)$	$5.596(-10)$	$3.3409...(-3)$
$10, 1$	$3.793(-17)$	$2.444(-17)$	$3.666(-17)$	$3.297(-18)$	$1.3050...(-5)$
$15, 1$	$8.548(-28)$	$5.545(-28)$	$8.317(-28)$	$5.915(-29)$	$1.2744...(-8)$
$20, 1$	$8.371(-39)$	$5.448(-39)$	$8.172(-39)$	$4.922(-40)$	$1.2446...(-11)$

n, ω	$r_1^{[2]}(f_1)$	$r_2^{[2]}(f_1)$	$r_3^{[2]}(f_1)$	Error$^{[2]}$	$I_\omega^{[2]}(f_1)$
$5, 1$	$6.844(-8)$	$6.995(-8)$	$6.668(-8)$	$9.110(-9)$	$3.3409...(-3)$
$10, 1$	$6.217(-18)$	$6.312(-18)$	$6.111(-18)$	$5.579(-19)$	$3.3626...(-6)$
$15, 1$	$1.406(-28)$	$1.423(-28)$	$1.386(-28)$	$9.984(-30)$	$3.1861...(-9)$
$20, 1$	$1.379(-39)$	$1.394(-39)$	$1.362(-39)$	$8.296(-41)$	$3.1115...(-12)$

n, ω	$r_1^{[3]}(f_1)$	$r_2^{[3]}(f_1)$	$r_3^{[3]}(f_1)$	Error$^{[3]}$	$I_\omega^{[3]}(f_1)$
$5, 1$	$7.797 - 8)$	$6.666(-8)$	$6.707(-8)$	$1.785(-8)$	$6.6819...(-3)$
$10, 1$	$6.896(-18)$	$6.110(-18)$	$6.136(-18)$	$1.099(-18)$	$6.5253...(-6)$
$15, 1$	$1.542(-28)$	$1.386(-28)$	$1.391(-28)$	$1.972(-29)$	$6.3723...(-9)$
$20, 1$	$1.502(-39)$	$1.362(-19)$	$1.366(-19)$	$1.641(-40)$	$6.2230...(-12)$

Acknowledgements

The research of R. Orive is supported in part by the Research Project of Ministerio de Ciencia e Innovación (Spain) under grant MTM2015-71352-P. The research of A.V. Pejčev and M.M. Spalević is supported in part by the Serbian Ministry of Education, Science and Technological Development (Research Project: "Methods of numerical and nonlinear analysis with applications" (#174002)).

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