University of East Sarajevo
Mathematical Society of the Republic of Srpska

Third Mathematical Conference of the Republic of Srpska

PROCEEDINGS

Trebinje, 07-08 June 2013

VOLUME I
Trebinje, 2014

ISBN 978-99976-600-0-8

Redakcija

Izdavač

Univerzitet u Istočnom Sarajevu

Fakultet za proizvodnju i menadžment Trebinje

Stepe Stepanovića bb, 89 101 Trebinje, BiH

Telefon: +387 (0)59 490 654

E-mail: fpmtrebinje@gmail.com

University of East Sarajevo
Mathematical Society of the Republic of Srpska

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA

PROCEEDINGS
Trebinje, 07-08 June 2013

VOLUME I
Trebinje, 2014

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE

ZBORNIK RADOVA
TOM I

Izdavač:
Fakultet za proizvodnju i menadžment Trebinje, Univerzitet u Istočnom Sarajevu

Za izdavača:
Doc. dr Dušan Jokanović

Glavni urednik:
Prof. dr Milenko Pikula

Tehnički urednik i kompjuterski slog:
Marina Zirojević

Lektura i korektura:
Maja Kovačević

Štampa:
"Grafokomerc" a. d. Trebinje

Tiraž:
300

Trebinje, 2014.

Sadržaj

SEKCIJA ZA ALGEBRU, GEOMETRIJU I DISKRETNU MATEMATIKU

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Algebras of Deductions in Category Theory*

Kosta Došen
Faculty of Philosophy, University of Belgrade, and
Mathematical Institute, SANU, Knez Mihailova 36, p.f. 367, 11001 Belgrade, Serbia
kosta@turing.mi.sanu.ac.rs

Original scientific paper

Abstract

In proof theory we use category theory to obtain identity criteria for deductions. Inference rules, by which we pass from one deduction to another, correspond there to partial algebraic operations. This is an algebra built on deductions and not on formulae. Logic is thereby tied to algebras of another kind than Boolean algebras, Heyting algebras, and similar lattice algebras investigated in universal algebra, induced by equivalence between formulae. Algebras of deductions are based on categories with structure given by so general and so important mathematical notions like product, coproduct, exponent, tensor product,... Here logic does not formalize other areas of mathematics to investigate them by its own means, but the subject matter of logic itself coincides with something investigated in contemporary algebra and other related areas of mathematics.

1 Logical models from universal algebra

Algebras associated with logic are first of all Boolean algebras. The semantics of classical propositional logic is built upon the two-element Boolean algebra, the two elements being interpreted as *truth* and *falsity*. The models of this logic are valuations into the two-element Boolean algebra, i.e. homomorphisms from the absolutely free algebra of formulae into the two-element Boolean algebra such that Boolean functions correspond to connectives. When the algebra of propositional formulae is factored through the equivalence relation induced by equivalence of formulae, i.e. identity of truth-value for every valuation, one obtains the Lindenbaum algebra of classical propositional logic, which is a freely generated Boolean algebra, with as much free generators as there are propositional letters.

The connection between Boolean algebras and classical logic is exhibited on a less technical level by the explanation that is given of the classical connectives

*Acknowledgement. I would like to express my heart-felt thanks to the organizers of the Third Mathematical Conference of the Republic of Srpska, who invited me to deliver a plenary lecture that this text records. I am grateful in particular to Dušan Jokanović for his hospitality. My work was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

of conjunction, disjunction and negation via intersection, union and complement. These are the operations of a set Boolean algebra, i.e. a Boolean algebra whose elements are sets. One encounters this explanation nowadays at the earliest stages of schooling, usually accompanied by diagrams of Euler and Venn. It is however questionable how much the notion of intersection, which is defined in terms of conjunction, can serve to explain conjunction. Conjunction is the more basic notion, and not intersection, and analogously for the notion of disjunction versus union, and the notion of negation versus complement.

With intuitionistic logic the role of Boolean algebras is taken over to a great extent by Heyting algebras, which make a more general class of algebras. These are distributive lattices with a least element and a binary operation of *residual* (also called *relative pseudo-complement*) with respect to the binary meet operation. This means that $c \cap a \leq b$ iff $c \leq a \Rightarrow b$, where $\cap$ is meet and $\Rightarrow$ is residual. Boolean algebras are complemented distributive lattices, where the residual $a \Rightarrow b$ is $-a \cup b$, with $-$ being the unary operation of complement and $\cup$ being join. The operation of residual in Heyting algebras, as well as in Boolean algebras, which are particular Heyting algebras, corresponds to the connective of implication.

The Lindenbaum algebra of intuitionistic propositional logic is a freely generated Heyting algebra, and the Kripke models of this logic, the most standard models one encounters for it, are obtained through a representation of Heyting algebras in preorderings. Heyting algebras are closely tied to topological Boolean algebras, i.e. Boolean algebras with unary operations corresponding to interior or closure. Topological Boolean algebras give models of the modal logic S4.

Various other logics have algebraic models of a similar kind, models out of the realm of universal algebra that resemble Boolean and Heyting algebras. In particular, in all these models the elements of the algebra are related to formulae, which are of the grammatical category of *propositions*. These elements are the values formulae may take. In the case of classical propositional logic and the two-element Boolean algebra, the values are the two truth-values. In other cases the values are more unusual, but they are again assigned to formulae, i.e. propositions.

2 Deductions make a category

Matters become different if in the algebraic modelling of logics the elements of the algebra are not expected any more to be assigned to formulae, i.e. propositions. For that one has to make a shift in the semantic conception of logic and language. One may cease to consider propositions as the main unit of language. One may look for a wider context in which propositions partake, and when logic is our inspiration that wider context is naturally found in *deductions*. The term *deduction*, which will later become for us a technical term, is intuitively synonymous with *inference*, or *proof from hypotheses*.

For the time being, to simplify matters we shall speak of deductions where we have not more than a single premise. This is not an essential restriction if we assume as usual that in deductions we have only finitely many premises, and we assume moreover that we have a connective like conjunction to connect these finitely many premises. The empty conjunction, i.e. a propositional constant like $\top$, replaces the empty set of premises. So we assume that in deductions we have a single premise and a single conclusion.

Then one may ask whether the deduction from premise A to conclusion B is completely determined by giving A and B. Is this deduction just the ordered pair (A, B)? Many logicians, if they wouldn't give to this question an explicitly affirmative answer, behave as if they would. They believe that consequence *relations* fully explain deductions. We may deduce B from A if and only if B is a consequence of A, i.e. if and only if the ordered pair (A, B) belongs to the consequence relation tied to our logic. Either there is a deduction from A to B, or there is none. There cannot be more than one deduction with the same premise and the same conclusion.

An alternative is that there may be more than one deduction with the same premise and the same conclusion. Suppose A is the conjunction $B \wedge B$. Then the deduction from $B \wedge B$ to B obtained by applying the first-projection rule *from* $C \wedge D$ *deduce* C and the deduction from $B \wedge B$ to B obtained by applying the second-projection rule *from* $C \wedge D$ *deduce* D would not be the same deductions. In another example, one may deduce $B \wedge B$ from $B \wedge B$ either by applying the identity rule *from* E *deduce* E or the commutativity rule *from* $C \wedge D$ *deduce* $D \wedge C$. This gives two different deductions. Still another example is given by the deductions from $B \wedge (B \Rightarrow B)$ to B obtained by applying either the first-projection rule or the rule *from* $C \wedge (C \Rightarrow D)$ *deduce* D, which is based on modus ponens.

In this alternative perspective deductions do not make a binary relation, i.e. a set of ordered pairs, but a graph where for the same ordered pair (A, B) there may be more than one arrow joining A and B. Deductions should be closed under composition and for every pair (A, A) there should be an identity arrow, i.e. identity deduction, from A to A, based on the identity rule. If we assume that composition is associative and that composing a deduction with the identity deduction is equal to this deduction, we obtain a category. In this category the formulae, i.e. propositions, are the objects and the deductions are the arrows. From a categorial perspective these arrows are more important.

3 Conjunction and product

In the categorial perspective we do not have only operations on objects, but also operations on arrows. These operations may be only partial, as composition is. The algebraic structure brought by these operations on objects and on arrows tied to connectives and other logical constants is of a different kind than what we had with Boolean and Heyting algebras. Before the binary connective of conjunction corresponded to lattice meet, and now it will correspond to the biendofunctor of product. Meet corresponds set-theoretically to intersection, while product corresponds to another, even more important, set-theoretic operation—namely, Cartesian product. Product on objects in the category **Set** of sets with functions as arrows is Cartesian product.

Tied to product, we have for every pair of formulae A_1 and A_2, the projection arrows $p^1_{A_1,A_2}\colon A_1 \wedge A_2 \to A_1$ and $p^2_{A_1,A_2}\colon A_1 \wedge A_2 \to A_2$, which correspond to the first-projection and second-projection rules of deduction mentioned above. We have moreover the partial operation of pairing of arrows, which applied to the arrows $f_1\colon C \to A_1$ and $f_2\colon C \to A_2$ gives the arrow $\langle f_1, f_2\rangle\colon C \to A_1 \wedge A_2$. Pairing corresponds to the natural deduction rule of conjunction introduction. For the arrow $h\colon C \to A_1 \wedge A_2$ and for $\circ$ being composition, we have the equations

$$p^1_{A_1,A_2} \circ \langle f_1, f_2\rangle = f_1, \quad p^2_{A_1,A_2} \circ \langle f_1, f_2\rangle = f_2, \quad \langle p^1_{A_1,A_2} \circ h, p^1_{A_1,A_2} \circ h\rangle = h.$$

These equations categorially define product. Replace $\wedge$ by $\times$ and, for $g_1 : A_1 \to B_1$ and $g_2\colon A_2 \to B_2$, define the total operation $\times$ on arrows by

$$g_1 \times g_2 =_{df} \langle g_1 \circ p^1_{A_1,A_2}, g_2 \circ p^2_{A_1,A_2}\rangle\colon A_1 \times A_2 \to B_1 \times B_2.$$

The operation $\times$ on objects and the operation $\times$ on arrows give the biendofunctor of product.

The equations assumed above make however perfect logical sense. They correspond to reductions involved in normalization in natural deduction (the first two equations correspond to β-reduction and the third to η-reduction). So it is natural to assume these equations as equations between deductions.

A category is a preorder when for every pair of objects A and B we have at most one arrow $f : A \to B$. Product becomes meet when our category is not only a preorder, but also a partial order, which means that if we have a pair of arrows $f\colon A \to B$ and $g\colon B \to A$, then the objects A and B must be the same. In a preorder, the equations assumed for product are trivially satisfied.

This categorial characterization of conjunction applies both to classical conjunction and intuitionistic conjunction. The connective of conjunction is the same in both logics. In the old universal-algebraic characterization too, conjunction was characterized in both logics as a meet operation.

4 Conjunction and disjunction

Analogously, the binary connective of disjunction, which before corresponded to join and union, now corresponds to binary coproduct, which set-theoretically is disjoint union on objects. This applies again both to classical and intuitionistic disjunction. It is often said that intuitionists understand disjunction not as it is understood in classical logic, and that because of that excluded middle fails for them. This should be taken carefully, because in a context where disjunction is alone—this is a context in which we do not have theorems, but we do have deductions—classical and intuitionistic disjunction do not differ. The essential novelty of intuitionistic logic is a new connective of implication, and not disjunction. This implication underlies negation, which we find in excluded middle (see below, the second paragraph of the next section).

When the connectives of conjunction and disjunction are together, then the universal-algebraic characterization is given by distributive lattices, and again classical and intuitionistic logic do not differ. Matters may however become different in terms of algebras of deductions.

The categories for intuitionistic conjunctive-disjunctive deductions are categories with binary (and hence all nonempty finite) products and coproducts, with product isomorphically distributive over coproduct; namely, there would be a natural isomorphism of the type $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$. There are also natural transformations of the dual type $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$, and of its converse type $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$, but they need not be isomorphisms. This is an asymmetry we have also in the category **Set**, where Cartesian product is isomorphically distributive over disjoint union, but not vice versa. When we restrict ourselves to finite sets, this reduces to the following equation for natural numbers $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, without the dual equation $a + (b \cdot c) = (a + b) \cdot (a + c)$ being true.

We may add to these categories also the empty product and coproduct, which are the terminal object $\top$ and initial object $\bot$. (An object is terminal when for every other object to it there is a single arrow; dually, it is initial when from it to every other object there is a single arrow.) These two objects correspond to the propositional constants $\top$ and $\bot$, which in the universal-algebraic approach correspond to the least element and greatest element of our lattices—an assumption made both for classical logic and intuitionistic logic. The assumptions about the terminality of $\top$ and the initiality of $\bot$ are parallel to the assumptions of binary product for $\wedge$ and binary coproduct for $\vee$. They are the same in classical and intuitionistic logic.

The isomorphic distributivity of conjunction over disjunction in intuitionistic logic is a consequence of the assumption that the monoendofunctor $A\wedge$ has as a right-adjoint $A \Rightarrow$, which involves intuitionistic implication $\Rightarrow$. This means that we have a natural bijection between the hom sets $Hom(A \wedge B, C)$ and $Hom(B, A \Rightarrow C)$. This means that there is a one-to-one correspondence between deductions of the type $A \wedge B \rightarrow C$ and deductions of the type $B \rightarrow A \Rightarrow C$. From left to right this correspondence is tied to the deduction theorem, and from right to left it has to do with modus ponens; from the identity deduction of type $A \Rightarrow C \rightarrow A \Rightarrow C$ one passes to the modus ponens deduction of the type $A \wedge (A \Rightarrow C) \rightarrow C$. (The naturality involved in this correspondence is given by transformations natural in B and C between the functors into the category **Set** involved in $Hom(A \wedge B, C)$ and $Hom(B, A \Rightarrow C)$; see [11], Sections II.2 and IV.1.)

With this adjunction involving $A\wedge$ and $A \Rightarrow$, together with the assumption that we have all finite products (including the empty one), we obtain *cartesian closed categories*; if we assume moreover that we have all finite coproducts including the empty one), we obtain *bicartesian closed categories* (see [10], Sections I.3

and I.8). The category **Set** is a bicartesian closed category.

5 Categories of classical deductions

In classical algebras of deductions there are arrows of the types $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ and $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$, as well as of the converse types, but Boolean duality would make us expect that either both of these kinds of arrows are isomorphisms or neither is. It is not in the Boolean spirit that $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ is an isomorphism while $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$ is not. The asymmetry of the bicartesian closed category **Set** that we have with respect to distribution of Cartesian product over disjoint union is foreign to the Boolean spirit. Since this asymmetry is a consequence of the adjunction involving $A\wedge$ and $A \Rightarrow$, we should not expect this adjunction for classical logic.

If $\neg A$ is defined as $A \Rightarrow \bot$, it may seem natural to assume in classical logic that A is isomorphic to $\neg\neg A$, or that there is at least a natural transformation whose components are of the type $\neg\neg A \to A$. There is an argument that shows that every bicartesian closed category together with one of these assumptions, or a similar assumption, is a preorder (see [2], Section 5, or [4], Section 14.3). It was concluded from that there cannot be any interesting categorial proof theory for classical logic, i.e. that there is no interesting categorification of Boolean algebras. But why must we assume that we have to start with a bicartesian closed category?

The book [4] is a detailed attempt to build a plausible categorial theory of classical deductions. The centre piece of that book is a notion of category with finite products and coproducts, which are distribute one over the other, but neither of these distributions need be an isomorphism. This way, in the Boolean spirit, symmetry is kept. For this notion of category it is established that it is coherent in the following sense. There is faithful functor from such a category freely generated by a set of objects to the category of relations between finite ordinals. This coherence result is of a semantical kind. It permits us to decide whether two arrows are equal by checking whether two simple graphs that come from the relations associated with the arrows are equal. This is a result of a semantical kind, analogous to the usual completeness theorems of logic, which enable us to decide provability by looking into a manageable model.

For this result there is given in [4] a long and rather involved proof based on a categorial version of Gentzen's cut elimination theorem for conjunctive-disjunctive classical logic, which is based on plural, i.e. multiple-conclusion, sequents. It is not established only that for every deduction we have a cut-free deduction with the same premise and conclusion, but it is established moreover that the two deductions are identical: the arrows standing for them are equal as arrows in our categories. It is remarkable that this equality of arrows has both a semantical justification via coherence, and a syntactical one via the cut-free normal from, and that the two justifications agree. Both justifications respect the Boolean duality between conjunction and disjunction. In the semantical justification product and coproduct are modeled by the same biproduct, while in the syntactical justification plural sequents treat conjunction and disjunction in the same way. All this makes quite plausible, and mathematically interesting, the proposed notion of identity of deduction for classical propositional logic.

This categorification which covers the conjunctive-disjunctive core of classical logic, together with the propositional constants $\top$ and $\bot$, is then extended with negation, and hence also implication, so that we do not have the adjunction involving $A \wedge$ and $A \Rightarrow$. We relinquish this adjunction of intuitionistic logic, and hence classical implication will not mirror deductions, as intuitionistic implication does. The resulting nontrivial categories are related to linear algebra.

The relationship with linear algebra is brought by having an operation of union, or addition, of deductions, and a zero deduction. In terms of relations, union of deductions is union of relations, the zero deduction is the empty relation, and union with the empty relation leaves a relation unchanged. Composition of relations corresponds to multiplication of zero-one matrices and union of relations corresponds to matrix addition. In the presence of zero deductions, we obtain a unique normal form for deductions like in linear algebra: every matrix is the sum of matrices with a single 1 entry, the others being zero. This normal form is related to cut elimination, i.e. composition elimination (see [4], Sections 1.6 and 14.4, Chapters 13 and 14).

6 Categories of deductions in other logics

The book [5] deals with categories of deductions in classical linear logic, presumably the main substructural logic, and an important alternative logic. In linear logic we have a binary connective of so-called *multiplicative* conjunction, which in terms of the algebra of deductions corresponds to tensor product. For this connective we assume associativity and commutativity, but not what corresponds to the structural rules of contraction and thinning. The algebras of deductions here come from categories to which belongs the category of vector spaces over a fixed field with linear transformations as arrows. Categories of deductions in intuitionistic linear logic are given by symmetric monoidal closed categories (see [11], Section VII.7), for which a coherence theorem is established in [9]. (This coherence theorem should be rephrased as a result taking account of syntax, as in [4] and [5].) The coherence proved in [5] is of the same kind, and it is established with respect to relations that one finds in Brauer algebras (for references see [5]). Brauer algebras, which come from representation theory, are related to Temperley-Lieb algebras, which came to play an important role in knot theory after Jones' seminal results in this field. Temperley-Lieb algebras are also related to a coherence result for adjunction (see [1], [3] and references therein).

The relations one finds in Brauer algebras are a particular case of relations one finds in a category called **Gen**, which has relations involved in the generality of deductions, i.e. relations connecting letters in a deduction that must remain the same in all generalizations of this deduction based on the same rules (see Section 2 above). The category **Gen** and the category of *split preorders*, of which it is a subcategory, are investigated in detail in [7]. These categories have properties relating them to Frobenius algebras, which have recently played a prominent role in two-dimensional topological quantum field theories—an area that seems far removed from logic and deduction theory. These matters (see [8], and references mentioned therein) are related to the coherence result for adjunction, and to Temperley-Lieb algebras, which we mentioned above. Frobenius algebras are also related to a coherence result involving deductions in the modal logic S5 (see [6]).

References

[1] K. Došen, Cut Elimination in Categories, Kluwer, Dordrecht, 1999 (*Addenda and Corrigenda* available at: www.mi.sanu.ac.rs/~kosta/AddCorrCutElimCat.pdf)

[2] ——, *Identity of proofs based on normalization and generality*, The Bulletin of Symbolic Logic, vol. 9 (2003), pp. 477-503 (version with corrected remark on difunctionality available at: arXiv.org)

[3] ——, *Simplicial endomorphisms*, Communications in Algebra, vol. 6 (2003), pp. 2681-2709 (available at: arXiv.org)

[4] K. Došen and Z. Petrić, Proof-Theoretical Coherence, KCL Publications (College Publications), London, 2004 (revised version of 2007 available at: www.mi.sanu.ac.rs/~kosta/coh.pdf)

[5] ——, Proof-Net Categories, Polimetrica, Monza, 2007 (preprint of 2005 available at: www.mi.sanu.ac.rs/~kosta/pn.pdf)

[6] ——, *Coherence for modalities*, Journal of Pure and Applied Algebra, vol. 215 (2013), pp. 1606-1628 (available at: arXiv.org)

[7] ——, *Syntax for split preorders*, Annals of Pure and Applied Logic, vol. 164 (2013), pp. 443-481 (available at: arXiv.org)

[8] ——, *Ordinals in Frobenius monads*, Journal of Pure and Applied Algebra, vol. 217 (2013), pp. 763-778 (available at: arXiv.org)

[9] G.M. Kelly and S. Mac Lane, *Coherence in closed categories*, Journal of Pure and Applied Algebra, vol. 1 (1971), pp. 97-140, 219

[10] J. Lambek and P.J. Scott, Introduction to Higher Order Categorical Logic, Cambridge University Press, Cambridge, 1986

[11] S. Mac Lane, Categories for the Working Mathematician, Springer, Berlin, 1971, second edition, 1998

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

On the Logic of the Ontological Argument

Slobodan Vujošević
University of Montenegro, Faculty of Natural Sciences and Mathematics
vslobo@t-com.me

Original scientific paper

Abstract

In his ontological argument Gödel has told nothing about its underlying logic. His argument is modal and at least of second-order and since he has used the S5 axiom, it was widely accepted that the logic of the argument is the S5 second-order modal logic. However, there is a step in his proof in which Gödel has applied the necessitation rule on the assumptions of the argument (see [3]), and this was repeated by all of his followers (see [1] and [4]). This application of the necessitation rule can seriously harm the consequence relation of the logic of the ontological argument. It seems that the only way to preserve the modal logic S5 for the ontological argument is to assume some of its axioms in the necessitated form.

1 Consequence and proof in modal logic

The notions of consequence and proof in modal logic are different from those in classical logic. The relation that some sentence A is a consequence of certain assumptions Σ can have two meanings in modal logic: A is true at each world at which the members of Σ are true and A is true in every model in which the members of Σ are true. The two notions are not equivalent and to distinguish between them some authors (see [2]) are using the terms *local* assumption and *local* consequence in the first, and the terms *global* assumption and *global* consequence in the second case. We shall show how this semantical distinction is reflected in the syntax of modal logic.

Assume some sentence A globally; if A is true at some arbitrary world w in some model, then A will be true at every world accessible to w (since A is true at every world), so $\Box A$ will be true at w. Since w was arbitrary, $\Box A$ must be true at every world of a model. This means that if A is a global assumption, the necessitation rule can be applied to A. Assuming A globally, we also assume $\Box A$, $\Box\Box A$ etc.

On the other hand, if we assume A locally, so that A is known to be true at a world w of some model, there is no reason to expect that $\Box A$ is also true at w. If A is a local assumption, the necessitation rule cannot apply to it.

The distinction between global and local assumptions in formal deductions comes down to the applicability or nonapplicability of the necessitation rule. A formal proof or derivation in modal logic do not allow the use of the necessitation rule to local premises and their consequences. To insure this, some authors define modal derivations as finite sequences divided in two separate parts, global and local (see [2]). The global part comes first, contains only global premises and the necessitation rule is allowed, while the local part comes second containing local premises, but the necessitation rule is not allowed.

2 Necessitation in Gödel's argument

It is well known that Gödel was involved in the foundation of the modern approach to modal logic. He was among the first logicians who introduced the necesitation rule that made possible the simple and elegant modal axiom systems that are in use today. But in the early 1970s, at the time Gödel wrote his note about the ontological argument, the idea of possible world semantics was new and perhaps not well appreciated. His argument was modal and was presented in at least second-order logic, but nothing was told about the exact logic he had in mind.

At some point in his note Gödel has formulated the theorem

$$G(x) \to \Box\exists y G(y),$$

where $G(x)$ means that x is godlike being (see page 403 in [3]), and without any comments he has proceeded in the following three steps:

$$\begin{array}{l} \exists x G(x) \to \Box\exists y G(y), \\ \Diamond\exists x G(x) \to \Diamond\Box\exists y G(y), \\ \Diamond\exists x G(x) \to \Box\exists y G(y). \end{array}$$

In the first step the existential quantifier was introduced, the second step havs come from the necessitation rule, and the third was the use of the S5 axiom. Since he was able to prove $\Diamond\exists x G(x)$, Gödel finally concluded $\Box\exists y G(y)$.

There is no doubt that the propositional skeleton of the logic of Gödel's argument is the modal logic S5, or something close to it. Besides the axioms of classical propositional logic, the modal axioms of the logic S5 are

$$\begin{array}{c} \Box(A \to B) \to (\Box A \to \Box B), \\ \Box A \to A, \\ \Box A \to \Box\Box A, \\ \Diamond\Box A \to \Box A, \end{array}$$

where $\Box$ is the necessity operator and, $\Diamond$ is the pssibility operator defined by $\Diamond A \leftrightarrow \neg\Box\neg A$, and the inference rules are modus ponens and necessitation: from A infer $\Box A$. The last axiom is usually called the S5 axiom.

According to what we have told about consequence in modal logic, to allow the unrestricted use of the necessitation rule in the logic S5, we have to assume the axioms of our theories globally. In the modal logic S5, where $\Box A \leftrightarrow \Box\Box A$, assuming A globally we assume $\Box A$. Formally, this means that all axioms of the theories in the S5 logic must come in the necessitated form, i.e. with $\Box$ prefixed.

Gödel, as well as his followers and commentators in this matter, have told nothing about the local or global character of the ontological argument axioms. They have presented these axioms in the unnecessitated form (see [2], [4] and [5]), and have used the necessitation rule on them and on their cosequences. Perhaps they had in mind global axioms?

Gödel's argument is a particular version of the general ontological argument that usually means two things: to prove that God's existence is possible and to prove that God exists necessarily if it exists. If Q is the statement that God exists, this means that in the general ontological argument we have to prove $\Diamond Q$ and $Q \to \Box Q$ (Anselm's principle). It is generally accepted that with these assumptions within S5 logic one can prove $\Box Q$: the necessitation of Anselm's principle gives $\Diamond Q \to \Diamond\Box Q$, the S5 axiom gives $\Diamond Q \to \Box Q$ and the first assumption finally gives $\Box Q$ (see [1], [3], [4] and [5]). But the use of necessitation in this proof was not correct. It seems that the only way to overcome this incorrectness is to formulate Anselm's principle in the form $\Box(Q \to \Box Q)$: it is necessary that God exists necessary if it exists.

Acknowledgement. I would like to express my thanks to Kosta Došen who warned me about the problem concerning the correctness of the necessitation rule in the ontological argument.

References

[1] Fitting, M., 2002, *Types, Tableaux and Gödel's God*, Dordrecht, Kluwer.

[2] Fitting, M. and Mendelsohn, R. L., 1990, *First-Order Modal Logic*, Dordrecht, Kluwer.

[3] Gödel, K., 1995, *Ontological Proof*, In *Collected Works, Volume III, Unpublished essays and lectures*, New York, Oxford University Press, pp. 403-404.

[4] Háyek, P. 2011, *Gödel's Ontological Proof and Its Variants*, In *Kurt Godel and the Foundations of Mathematics - Horizons of Truth*, Cambridge, Cambridge University Press, 2011, pp. 307-324.

[5] Sobel, J. H., 1987, *Gödel's Ontological Proof*, In *On Being and Saying: Essays in Honor of Richard Cartwright*, Cambridge, MA, MIT Press, pp. 24-61.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

O klasifikacijskim prostorima monoidalnih kategorija

Sonja Lj. Čukić
Matematički institut SANU
sonja@mi.sanu.ac.rs

Zoran Petrić
Matematički institut SANU
zpetric@mi.sanu.ac.rs

Originalni naučni rad

Apstrakt

U ovom kratkom radu dajemo pregled algebarskih struktura na kategorijama koje omogućavaju Segal-Tomasonovu bar konstrukciju. To je osnovni korak procedure koja kategoriji $\mathcal{M}$ pridružuje njeno raspetljavanje: prostor X takav da je prostor petlji nad X homotopski ekvivalentan klasifikacijskom prostoru kategorije $\mathcal{M}$. Još je od šezdesetih godina prošlog veka poznato da monoidalna struktura na kategoriji odgovara jednostrukom prostoru petlji. Segal-Tomasonova konstrukcija omogućava da se neke kategorije sa bogatijom strukturom višestruko raspetljaju. Najnoviji rezultati ovog tipa najavljeni su na konferenciji u Trebinju.

1 Uvod

Prve rezultate koji su omogućili da se povežu monoidalne kategorije s prostorima petlji dali su Stašev, [11], i Meklejn, [6]. Nakon toga se pojavila potreba da se odrede uslovi na algebarskoj strukturi kategorije koji omogućavaju njeno povezivanje s prostorima petlji raznih višestrukosti. Sedamdesetih godina prošlog veka su Segal, [10], i Tomason, [13], razvili tehniku koja je odredila jedan pravac u toj oblasti istraživanja.

Segal-Tomasonova ili redukovana, kako ćemo je ovde zvati, bar konstrukcija je postupak nalaženja simplicijalnog objekta u kategoriji čija je monoidalna struktura data konačnim proizvodima. U ovom radu će nam to biti 2-kategorija *Cat* čiji su objekti kategorije, morfizmi su funktori, a 2-morfizmi su prirodne transformacije. Monoid u *Cat* je striktna monoidalna kategorija. Jedna takva kategorija $\mathcal{M}$ je osnova za simplicijalni objekat u *Cat*, to jest funktor $\overline{W}\mathcal{M}$ iz Δ_+^{op} u *Cat*, gde je Δ_+ topološka simplicijalna kategorija. Ovo znači da je $\overline{W}\mathcal{M}(1) = \mathcal{M}$ i, uopštenije, $\overline{W}\mathcal{M}(n) = \mathcal{M}^n$.

Kada se $\overline{W}\mathcal{M}$ komponuje s funktorom nerv, a zatim s geometrijskom realizacijom, dobija se funktor iz Δ_+^{op} u *Top*, to jest simplicijalni prostor. Po rezultatima Segala i Mekdafa, [10], [8], prostor petlji geometrijske realizacije tog simplicijalnog prostora je homotopski ekvivalentan klasifikacijskom prostoru kategorije $\mathcal{M}$ ili preciznije, njenom grupnom kompletiranju. Dakle, taj simplicijalni prostor je raspetljavanje nerva od $\mathcal{M}$.

Uz malu modifikaciju, ova mašina za raspetljavanje se može prilagoditi nekim složenijim algebarskim strukturama na kategoriji tako da kao rezultat dobijemo višestruko raspetljavanje nerva polazne kategorije. Po rezultatima Segala, [10], i Tomasona, [13], ova tehnika dovodi u vezu simetrične monoidalne kategorije s beskonačnim prostorima petlji, dok se monoidalne kategorije pletenica dovode u vezu s dvostrukim prostorima petlji uz pomoć rezultata Žoajala i Strita, [5]. U ovim rezulatima sama redukovana bar konstrukcija nije dovoljna pošto njen rezultat nije funktor jer ne prolazi kroz kompoziciju morfizama. Međutim, pošto se svaki put može pokazati da je dobijen relaksiran funktor u smislu [12], onda se rezultati tog rada mogu iskoristiti da bi se dobio pravi funktor koji predstavlja traženi simplicijalni objekat. Da bi se pokazalo da je dobijen relaksiran funktor, svaki put se koristi koherencijski rezultat vezan za dati tip kategorija.

Balteanu, Fjodorovič, Švancl i Fogt, [2], su postavili pitanje dovoljnih uslova za to da kategorija s n monoidalnih struktura može da posluži kao osnova za redukovanu bar konstrukciju koja proizvodi relaksiran funktor odgovarajućeg tipa. Na taj način bismo dobili n-tostruko raspetljavanje nerva polazne kategorije. U tom radu su dati traženi uslovi, ali su oni suviše restriktivni u odnosu na jedinice—zahteva se da sve monoidalne strukture imaju zajedničku jedinicu. Došen i drugi autor ovog teksta, [4], su dali znatno oslabljenje tih uslova za slučaj $n = 2$, što verovatno predstavlja maksimum oslabljenja ukoliko i dalje hoćemo da koherencija bude sredstvo za dokazivanje toga da je rezultat redukovane bar konstrukcije jedan relaksiran funktor.

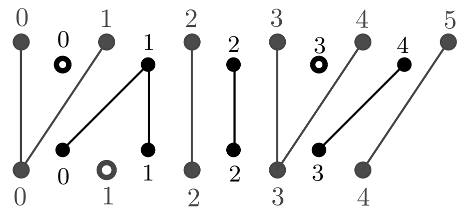
Autori ovog članka, [3], su dali željene uslove direktno proveravajući, bez koherencije, komutativnost dijagrama koji govore da je kao rezultat redukovane bar konstrukcije dobijen relaksiran funktor. Ti uslovi uopštavaju sve gorenavedene. Rezultati iz [3] su najavljeni tokom predavanja koje je drugi autor održao na konferenciji u Trebinju.

2 Redukovana bar konstrukcija

Neki dokazi koji su izostavljeni iz ovog odeljka mogu se naći u [9, Section 6]. Pod *simplicijalnim objektom* kategorije $\mathcal{C}$ podrazumevamo funktor iz Δ_+^{op} u $\mathcal{C}$, gde je Δ_+ standardna topološka *simplicijalna kategorija* (videti [7, VII.5]). U ovom odeljku ćemo pojasniti kako monoidalni objekat u monoidalnoj kategoriji čija je monoidalna struktura data konačnim proizvodima određuje jedan simplicijalni objekat te kategorije.

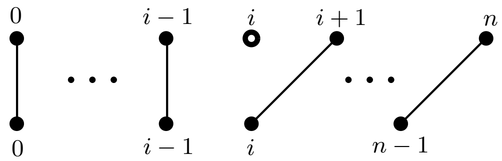
Ono što zbunjuje u ovoj konstrukciji je to što Δ_+^{op} sadrži *univerzalni komonoid*, a na raspolaganju imamo monoidalni objekat neke kategorije. Ovaj problem prevazilazimo tako što Δ_+^{op} utopimo u Δ, odnosno izjednačimo je s potkategorijom Δ_{Int} kategorije Δ.

Objekti kategorije Δ_{Int} su konačni ordinali veći ili jednaki 2, a strelice su monotone funkcije koje čuvaju prvi i poslednji element. Ta kategorija je slika kategorije Δ_+^{op} u kategoriji Δ pomoću funktora koji objekat n slika u $n+2$ (sa 0 označavamo prazan skup, dok je $n+1$ oznaka za skup $\{0,1,\ldots,n\}$), a morfizme slika „senčenjem” kao u sledećem primeru, gde unutrašnji graf predstavlja morfizam iz Δ_+^{op}, a spoljašnji je njegova „senka” iz Δ:



Ovaj funktor je veran i injektivan na objektima, pa stoga Δ_+^{op} možemo smatrati potkategorijom od Δ.

Za naše potrebe uvešćemo još jednu kategoriju srodnu simplicijalnoj koju označavamo sa Δ_{par}. Njeni objekti su takođe konačni ordinali, a morfizmi su monotone *parcijalne* funkcije. Za morfizme koji generišu tu kategoriju možemo uzeti one koji generišu Δ zajedno sa parcijalnim funkcijama $\rho_i^n \colon n{+}1 \to n$ za $n \geq 0$ i $0 \leq i \leq n$, koje se grafički predstavljaju kao



Posmatrajmo funktor koji preslikava Δ_{Int} u Δ_{par}, koji pridružuje objektu $n{+}2$ prve kategorije objekat n druge kategorije, a svakom morfizmu $f\colon m+2 \to n+2$ prve kategorije pridružuje morfizam $g\colon m \to n$ druge kategorije takav da je

$$g(x) = \begin{cases} f(x+1)-1, & \text{kada je } f(x+1)-1 \in n, \\ \text{nedefinisano}, & \text{inače.} \end{cases}$$

Na osnovu identifikacije kategorija Δ_+^{op} i Δ_{Int} on postaje funktor iz Δ_+^{op} u Δ_{par} koji je identitet na objektima.

Neka je $\mathcal{K}$ kategorija čija je monoidalna struktura data konačnim proizvodima. Na osnovu striktifikacije pokazane u [7, XI.3, Theorem 1], možemo slobodno smatrati da je ta monoidalna struktura striktna, to jest da je binarni proizvod asocijativan i da je terminalni objekat neutral za taj proizvod. Neka je $\langle C, \mu, \eta \rangle$ monoidalni objekat kategorije $\mathcal{K}$. Tada postoji jedinstven funktor iz Δ_{par} u $\mathcal{K}$ takav da se objekat n iz Δ_{par} slika u $C^n = C \times \ldots \times C$, zajednički generatori za Δ_{par} i Δ se slikaju u morfizme dobijene pomoću μ i η na osnovu toga što Δ sadrži *univerzalni monoid* (videti [7, VII.5, Proposition 1]), dok se generator ρ_i^n slika u morfizam iz C^{n+1} u C^n dobijen kao

$$\underbrace{\mathbf{1}_C \times \ldots \times \mathbf{1}_C}_{i} \times \kappa_C \times \underbrace{\mathbf{1}_C \times \ldots \times \mathbf{1}_C}_{n-i},$$

gde je κ_C jedinstven morfizam iz C u terminalni objekat kategorije $\mathcal{K}$.

Za nas je posebno interesantan slučaj kada umesto kategorije $\mathcal{K}$ posmatramo 2-kategoriju *Cat*, a umesto monoida C posmatramo monoid $\mathcal{M}$ u *Cat*, što znači da je $\mathcal{M}$ jedna striktna monoidalna kategorija. Na osnovu gorenavedenog postojaće funktor $\overline{W}\mathcal{M} \colon \Delta_+^{op} \to Cat$ koji je zadat sa

$$\overline{W}\mathcal{M}(n) = \mathcal{M}^n,$$

$$\overline{W}\mathcal{M}(d_0^n)(A_1, A_2, \ldots, A_n) = (A_2, \ldots, A_n),$$

$$\overline{W}\mathcal{M}(d_n^n)(A_1, \ldots, A_{n-1}, A_n) = (A_1, \ldots, A_{n-1}),$$

a za $1 \leq i \leq n-1$ i $0 \leq j \leq n-1$,

$$\overline{W}\mathcal{M}(d_i^n)(A_1, \ldots, A_i, A_{i+1}, \ldots, A_n) = (A_1, \ldots, A_i \otimes A_{i+1}, \ldots, A_n),$$

$$\overline{W}\mathcal{M}(s_j^n)(A_1, \ldots, A_j, A_{j+1}, \ldots, A_{n-1}) = (A_1, \ldots, A_j, I, A_{j+1}, \ldots, A_{n-1}),$$

gde su $d_i^n : n \to n-1$, za $n \geq 1$ i $0 \leq i \leq n$, i $s_j^n : n-1 \to n$, za $n \geq 1$ i $0 \leq j \leq n-1$, standardni generatori kategorije Δ_+^{op} (videti [3, Section 3]), dok je $\otimes$ *tenzor*, a I je *jedinica* striktne monoidalne kategorije $\mathcal{M}$.

Funktor $\overline{W}\mathcal{M}$ nazivamo *redukovana bar konstrukcija* bazirana na $\mathcal{M}$. S obzirom da je i $\mathcal{M}^k$ striktna monoidalna kategorija čija je struktura dobijena po komponentama od striktne monoidalne strukture na $\mathcal{M}$, onda postoji i redukovana bar konstrukcija bazirana na $\mathcal{M}^k$ i mi je označavamo sa $\overline{W}\mathcal{M}^k$. Ako kategorija $\mathcal{M}$ ima na sebi n monoidalnih struktura, onda redukovanu bar konstrukciju baziranu na i-toj monoidalnoj strukturi označavamo sa $\overline{W}\mathcal{M}_i$.

3 Višestruka redukovana bar konstrukcija

Osnovna ideja pomoću koje je u [10] i [13] pokazano da simetrične monoidalne kategorije odgovaraju beskonačnim prostorima petlji je da se za proizvoljno $n \geq 1$ one posmatraju kao kategorije s n monoidalnih struktura koje komuniciraju pomoću simetrije. Ta ideja je iskorišćena u [2] da bi se dao dovoljan uslov da kategorija s n striktnih monoidalnih struktura odgovara n-tostrukom prostoru petlji. Naša ideja u [3] je bila da, polazeći od kategorije s n striktnih monoidalnih struktura, bez ikakvih pretpostavki unapred, vidimo kakva komunikacija između tih struktura obezbeđuje da uopštenje redukovane bar konstrukcije proizvede jedan relaksiran funktor iz $(\Delta_+^{op})^n$ u *Cat* (videti definiciju niže).

Neka je $\mathcal{M}$ kategorija s n striktnih monoidalnih struktura. Uopštenje redukovane bar konstrukcije bazirano na $\mathcal{M}$ treba da nam proizvede dve funkcije—prva preslikava n-torke konačnih ordinala, što predstavlja objekte od $(\Delta_+^{op})^n$ u kategorije, to jest objekte od *Cat*, dok druga preslikava n-torke morfizama od Δ_+^{op} u funktore. Obe funkcije označavamo sa $\overline{W}\mathcal{M}$. Prva funkcija je sasvim jednostavno definisana i od nje zahtevamo da je

$$\overline{W}\mathcal{M}(m_1, \ldots, m_n) = \mathcal{M}^{m_1 \cdot \ldots \cdot m_n}.$$

Druga funkcija je nešto komplikovanije definisana i tu nam pomaže sledeća notacija. Neka je $\vec{f} = (f_1, \ldots, f_n)$ jedna n-torka morfizama od Δ_+^{op}. Za svako $1 \leq k \leq n$, neka je

$$i(k) = \prod_{k<l\leq n} s_l \quad \text{i} \quad o(k) = \prod_{1\leq l<k} t_l,$$

gde prazan proizvod računamo kao jedinicu i s_l je domen, a t_l je kodomen od f_l. Tada je druga funkcija definisana kao

$$\overline{W}\mathcal{M}(\vec{f}) = (\overline{W}\mathcal{M}_n^{i(n)}(f_n))^{o(n)} \circ \cdots \circ (\overline{W}\mathcal{M}_2^{i(2)}(f_2))^{o(2)} \circ (\overline{W}\mathcal{M}_1^{i(1)}(f_1))^{o(1)}.$$

Na primer, za $n = 3$ i $\vec{f} = (d_1^3, d_1^2, s_1^2)$ imamo da je $\overline{W}\mathcal{M}(\vec{f})$ funktor iz $\mathcal{M}^6$ u $\mathcal{M}^4$ zadat sa

$$\overline{W}\mathcal{M}(\vec{f})(A, B, C, D, E, F) = ((A \otimes_1 C) \otimes_2 (B \otimes_1 D), I_3, E \otimes_2 F, I_3),$$

gde su $\otimes_1$ i $\otimes_2$ tenzori prve, odnosno druge, monoidalne strukture, a I_3 je jedinica treće monoidalne strukture na $\mathcal{M}$.

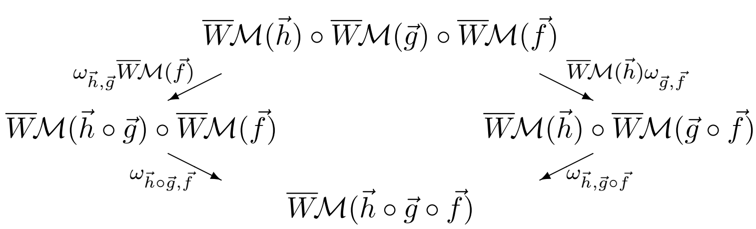
Ovaj par funkcija ne zadaje funktor iz $(\Delta_+^{op})^n$ u *Cat* zato što u opštem slučaju ne važi

$$\overline{W}\mathcal{M}(\vec{g}) \circ \overline{W}\mathcal{M}(\vec{f}) = \overline{W}\mathcal{M}(\vec{g} \circ \vec{f}).$$

Da bi $\overline{W}\mathcal{M}$ bio *relaksiran funktor* potrebno je da za svaki par kompozabilnih strelica $\vec{f}$ i $\vec{g}$ iz $(\Delta_+^{op})^n$ postoji prirodna transformacija

$$\omega_{\vec{g},\vec{f}} \colon \overline{W}\mathcal{M}(\vec{g}) \circ \overline{W}\mathcal{M}(\vec{f}) \xrightarrow{\cdot} \overline{W}\mathcal{M}(\vec{g} \circ \vec{f})$$

takva da sledeći dijagram komutira:



U [3] je pokazano da je neophodan uslov za postojanje ovakvih prirodnih transformacija ω to da za svako $1 \leq k < l \leq n$ postoje strelice $\kappa_{k,l} : I_k \to I_l$, $\beta_{k,l} \colon I_k \to I_k \otimes_l I_k$, $\tau_{k,l} \colon I_l \otimes_k I_l \to I_l$, kao i familija strelica $\iota_{k,l}$

$$(A \otimes_l B) \otimes_k (C \otimes_l D) \to (A \otimes_k C) \otimes_l (B \otimes_k D),$$

indeksirana četvorkama (A,B,C,D) objekata iz $\mathcal{M}$. Glavni rezultat tog rada je da su, pored prirodnosti transformacija $\iota_{k,l}$, sledeće jednakosti na strukturi kategorije $\mathcal{M}$ dovoljne za postojanje željenih prirodnih transformacija ω.

Za sve $1 \leq k < l \leq n$,

(1) $\iota_{k,l} \circ (\mathbf{1} \otimes_k \iota_{k,l}) = \iota_{k,l} \circ (\iota_{k,l} \otimes_k \mathbf{1})$,

(2) $\iota_{k,l} \circ (\mathbf{1} \otimes_k \beta_{k,l}) = \mathbf{1}$,

(3) $\iota_{k,l} \circ (\beta_{k,l} \otimes_k \mathbf{1}) = \mathbf{1}$,

(4) $\tau_{k,l} \circ (\mathbf{1} \otimes_k \tau_{k,l}) = \tau_{k,l} \circ (\tau_{k,l} \otimes_k \mathbf{1})$,

(5) $\tau_{k,l} \circ (\mathbf{1} \otimes_k \kappa_{k,l}) = \mathbf{1}$,

(6) $\tau_{k,l} \circ (\kappa_{k,l} \otimes_k \mathbf{1}) = \mathbf{1}$,

(7) $(\mathbf{1} \otimes_l \iota_{k,l}) \circ \iota_{k,l} = (\iota_{k,l} \otimes_l \mathbf{1}) \circ \iota_{k,l}$,

(8) $(\mathbf{1} \otimes_l \tau_{k,l}) \circ \iota_{k,l} = \mathbf{1}$,

(9) $(\tau_{k,l} \otimes_l \mathbf{1}) \circ \iota_{k,l} = \mathbf{1}$,

(10) $(\mathbf{1} \otimes_l \beta_{k,l}) \circ \beta_{k,l} = (\beta_{k,l} \otimes_l \mathbf{1}) \circ \beta_{k,l}$,

(11) $(\mathbf{1} \otimes_l \kappa_{k,l}) \circ \beta_{k,l} = \mathbf{1}$,

(12) $(\kappa_{k,l} \otimes_l \mathbf{1}) \circ \beta_{k,l} = \mathbf{1}$

i za sve $1 \leq k < l < m \leq n$,

(13) $\kappa_{l,m} \circ \kappa_{k,l} = \kappa_{k,m}$,

(14) $\beta_{l,m} \circ \kappa_{k,l} = (\kappa_{k,l} \otimes_m \kappa_{k,l}) \circ \beta_{k,m}$,

(15) $\tau_{l,m} \circ (\kappa_{k,m} \otimes_l \kappa_{k,m}) \circ \beta_{k,l} = \kappa_{k,m}$,

(16) $\iota_{l,m} \circ (\beta_{k,m} \otimes_l \beta_{k,m}) \circ \beta_{k,l} = (\beta_{k,l} \otimes_m \beta_{k,l}) \circ \beta_{k,m}$,

(17) $\kappa_{l,m} \circ \tau_{k,l} = \tau_{k,m} \circ (\kappa_{l,m} \otimes_k \kappa_{l,m})$,

(18) $\beta_{l,m} \circ \tau_{k,l} = (\tau_{k,l} \otimes_m \tau_{k,l}) \circ \iota_{k,m} \circ (\beta_{l,m} \otimes_k \beta_{l,m})$,

(19) $\tau_{l,m} \circ (\tau_{k,m} \otimes_l \tau_{k,m}) \circ \iota_{k,l} = \tau_{k,m} \circ (\tau_{l,m} \otimes_k \tau_{l,m})$,

(20) $\iota_{l,m} \circ (\iota_{k,m} \otimes_l \iota_{k,m}) \circ \iota_{k,l} = (\iota_{k,l} \otimes_m \iota_{k,l}) \circ \iota_{k,m} \circ (\iota_{l,m} \otimes_k \iota_{l,m})$.

Pojam kategorije koja zadovoljava ove uslove uveden je u [1, Section 7.6] pod imenom n-monoidalna kategorija. Ono što je važno za nas je da taj pojam uopštava pojmove simetrične monoidalne kategorije, monoidalne kategorije pletenica, bimonoidalne kategorije s intermutacijom (videti [4, Section 12]), simetrične bimonoidalne kategorije s intermutacijom (videti [4, Section 16] i [9, Section 2]) kao i pojam n-tostruke monoidalne kategorije uveden u [2, Section 1]. To znači da korektnost redukovane bar konstrukcije koju smo pokazali povlači korektnost svih redukovanih bar konstrukcija baziranih na kategorijama datih tipova. Naš rezultat, takođe, omogućuje da mnoge kategorije s više prirodno definisanih monoidalnih struktura na sebi dobiju priliku da se raspetljaju u smislu kako smo tu reč upotrebili u uvodu. Na taj način, svaka kategorija s konačnim koproizvodima i proizvodima, kakva je na primer kategorija konačnih skupova, može da se dvostruko raspetlja u odnosu na te dve monoidalne strukture.

Literatura

[1] M. Aguiar and S. Mahajan, Monoidal functors, species and Hopf algebras, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010

[2] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt, *Iterated monoidal categories*, Advances in Mathematics, vol. 176 (2003), pp. 277-349

[3] S.Lj. Čukić and Z. Petrić, *n-fold monoidal categories*, (arXiv:1309.6209)

[4] K. Došen and Z. Petrić, *Intermutation*, Applied Categorical Structures, vol. 20 (2012), pp. 43-95 (arXiv:math/0701325)

[5] A. Joyal and R. Street, *Braided tensor categories*, Advances in Mathematics, vol. 102 (1993), pp. 20-78

[6] S. Mac Lane, *Natural associativity and commutativity*, Rice University Studies, Papers in Mathematics, vol. 49 (1963), pp. 28-46

[7] ——, Categories for the Working Mathematician, Springer, Berlin, 1971 (expanded second edition, 1998)

[8] D. McDuff and G. Segal, *Homology fibrations and the "group-completion" theorem*, Inventiones mathematicae, vol. 31 (1976), pp. 279-284

[9] Z. Petrić and T. Trimble, *Symmetric bimonoidal intermuting categories and $\omega\times\omega$ reduced bar constructions*, to appear in Applied Categorical Structures, (arXiv:0906.2954)

[10] G. Segal, *Categories and cohomology theories*, Topology, vol. 13 (1974), pp. 293-312

[11] J.D. Stasheff, *Homotopy associativity of H-spaces*, I, II, Transactions of the American Mathematical Society, vol. 108 (1963), pp. 275-292, 293-312

[12] R. Street, *Two constructions on lax functors*, Cahiers de topologie et géométrie différentielle, vol. 13 (1972), pp. 217-264

[13] R.W. Thomason, *Homotopy colimits in the category of small categories*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 85, 91 (1979), pp. 91-109

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Manifolds over Polygons and Characteristic Classes

Đorđe Baralić
Mathematical Institute SASA Belgrade
djbaralic@mi.sanu.ac.rs

Vladimir Grujić
Faculty of Mathematics Belgrade
vgrujic@matf.bg.ac.rs

Original scientfic paper

Abstract

Quasitoric manifolds and their real analogues 'small covers' are important class of manifolds studied in toric topology. In this paper we study the characteristic classes of corresponding manifolds over polygons and its applications to classical questions in algebraic topology.

1 Introduction

Davis and Januszkiewicz in [3] introduced *quasitoric manifolds* as topological generalization of toric varieties in algebraic geometry. Quasitoric manifolds are $2n$-dimensional which poses locally standard action of group T^n and the orbit space of this action is a simple polytope P^n. Real analogue of quasitoric manifolds are *small covers*, i. e. n-dimensional manifolds with $\mathbb{Z}_2^n$ action instead of T^n. A nice exposition about quasitoric manifolds is given in the classical monograph [2].

In their remarkable paper Davis and Januszkiewicz described these manifolds and explained their cohomology ring and characteristic classes. Combinatorics of underlying polytope P^n plays important role in understanding of topological properties. Stanley-Reisner ideal strongly influence cohomology of this manifolds, but the group action also determines very special ideal which is necessary for full description. Thus, there exist nonhomeomorphic manifolds over the same polytopes.

The construction of a quasitoric manifold (small covers) from the characteristic pair (P^n, l) is described in [2, Construction 5.12]. Recall that P^n is a simple polytope with m facets and $\Lambda = (\lambda_1, \ldots, \lambda_m)$ is an integer $n \times m$ matrix, where $\lambda_j \in \mathbb{Z}^n$ $j = 1, \ldots, m$ corresponds to the generator of the Lie algebra isotropy subgroup of the characteristic submanifold M_j over the facet F_j. For every vertex $v = F_{i_1} \cap \cdots \cap F_{i_n} \in P^n$ the matrix has the property $\det \Lambda_{I_{(v)}} = \pm 1$ where $\Lambda_{I_{(v)}}$ is a square submatrix formed by the column vectors $\lambda_{i_1}, \ldots, \lambda_{i_n}$ corresponding to the facets $F_{i_1}, \ldots, F_{i_n}$. The matrix Λ is called *the characteristic matrix* of M.

Let $\lambda_j = (\lambda_{1j}, \ldots, \lambda_{nj})^t \in \mathbb{Z}^n$. Then we have

$$\theta_i := \sum_{j=1}^{m} \lambda_{ij} v_j$$

and let $\mathcal{J}$ be the ideal in $\mathbb{Z}[v_1, \ldots, v_m]$ generated by θ_i for all $i = 1, \ldots, n$. Let $\mathcal{I}$ denote the Stanley-Reisner ideal of P. The ordinary cohomology of quasitoric manifolds has the following ring structure:

$$H^*(M) \simeq \mathbb{Z}[v_1, \ldots, v_m]/(\mathcal{I} + \mathcal{J}).$$

The total Stiefel-Whitney class can be described by the following *Davis-Januszkiewicz formula*:

$$w(M^{2n}) = \prod_{i=1}^{m}(1 + v_i) \in H^*(M^{2n}; \mathbb{Z}_2),$$

where v_i is the $\mathbb{Z}_2$-reduction of the corresponding class over $\mathbb{Z}$ coefficients. An analogous formulas hold for small covers [3, Corollary 6.7]. In the same paper the total Chern class of quasitoric manifold M^{2n} is given by formula

$$c(M^{2n}) = \prod_{i=1}^{m}(1 + v_i) \in H^*(M^{2n}; \mathbb{Z}),$$

while Pontriagin class is given by

$$p(M^{2n}) = \prod_{i=1}^{m}(1 - v_i^2) \in H^*(M^{2n}; \mathbb{Z}).$$

Acknowledgements

This research was supported by the Grants 174020 and 174034 of the Ministry for Education and Science of the Republic of Serbia. Special thanks to Duško Jokanović and mathematical community of Trebinje for the invitation to present the results of this research at the Third Mathematical Conference of the Republic of Srpska.

2 Manifolds over polygons

Combinatorics of the underlying polytope strongly affects topology of quasitoric manifolds. Here we are particulary interested in a special case $n = 2$. In this case, small covers are 2-manifolds, the objects we know a lot about. However, quasitoric 4-manifolds are quiet interesting. Basic example is complex projective plane $\mathbb{C}P^2$ which orbit space of toric action is triangle. Quasitoric manifolds over square are known as Hirzebruch surfaces H_k.

We prove the main theorem of this contribution:

Theorem 2.1. *Let M^4 be a quasitoric manifold over the $2m$-gon. Then the classes $w_4(M^4)$ and $w_2^2(M^4)$ vanish.*

Proof: We observe that the total Stiefel-Whithney class is easily reduced on

$$w(M^4) = \prod_{i=1}^{m}(1 + v_i + v_{i+m}) \in H^*(M^4; \mathbb{Z}_2).$$

Ideal $\mathcal{J}$ in Davis-Januszkiewicz formula is given by two relations (we can consider that all coefficients are 0 or 1)

$$a_1v_1 + a_2v_2 + \cdots + a_{2m}v_{2m} = 0 \quad (1)$$
$$b_1v_1 + b_2v_2 + \cdots + b_{2m}v_{2m} = 0. \quad (2)$$

For every i modulo $2m$ we have

$$\det \begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix} = \pm 1. \quad (3)$$

Thus, from (1) and (2) we have $v_iv_{i-1} = v_iv_{i+1}$ for every i. Finally, in cohomology ring $H^*(M^4; \mathbb{Z}_2)$ we have

$$v_1v_2 = v_2v_3 = \cdots = v_{2m}v_1 \neq 0.$$

Now, the class $w_4(M^4)$ is determined by

$$w_4(M^{2n}) = v_1v_2 + v_2v_3 + \cdots + v_{2m}v_1 = 2m \cdot v_1v_2 = 0.$$

If one of the relations is $v_1 + v_2 + \cdots + v_{2m} = 0$ then $w_2(M^4)$ clearly vanish. So we assume this is not the case. Due to the relations (3) we have that $a_i = b_i = 1$ or exactly one of a_i and b_i is equal to 1. In the case $a_i = b_i = 1$ and $a_{i-1} = a_{i+1}$ we have $v_i^2 = 0$. In another case we deduce $v_i^2 = v_{i-1}v_i = v_iv_{i+1}$. Notice that in the case $a_i = b_i = 1$ and $a_{i-1} \neq a_{i+1}$ if collapse the edge i of the $2m$-gon into the vertex, we get $(2m-1)$-gon with regular sequence of vectors (we just 'deleted' $[a_ib_i]^t$). Then if we collapse every edge i for i such that $a_i = b_i = 1$ and $a_{i-1} \neq a_{i+1}$ we get new polygon with regular sequence of vectors. Now we consider the sides i such that $a_i = b_i = 1$ and $a_{i-1} = a_{i+1}$ and we collapse both sides i and $i+1$. Again we get the polygon with regular sequence. After collapsing all sides with vector $[1\,1]^t$ we get the polygon with regular sequence of alternating vectors $[1\,0]^t$ and $[0\,1]^t$ which must have even number of sides. This means that number k of the collapsed sides i such that $a_i = b_i = 1$ and $a_{i-1} \neq a_{i+1}$ is even. By previous observations, for the class $w_2^2(M^{2n})$ we have

$$w_2^2(M^{2n}) = (v_1 + v_2 + \cdots + v_{2m})^2 = k \cdot v_1v_2 = 0.$$

□

We have the following corollary:

Corollary 2.1. *The quasitoric manifold M^4 over the $2m$-gon is the boundary of a 5-dimensional manifold.*

In general, for manifolds over $(2m+1)$-gons we have various options. For example, complex projective plane $\mathbb{C}P^2$ is not boundary of a 5-manifold. Comparing the Stiefel-Whitney numbers one can prove that manifolds $\mathbb{C}P^2$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$ belong to the same unoriented cobordism class, see [4, Corrolary 4.11, p.53]. However, since for any quasitoric manifold M^4 the first Stiefel-Whitney class vanish, there is no quasitoric manifold cobordant to $\mathbb{R}P^4$. According to well-known result [5, Theorem 13.4, p.26], the unoriented cobordism classes of 4 manifolds are represented by $\mathbb{R}P^4$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.

In the same manner as in Theorem 2.1 we prove the following theorem:

Theorem 2.2. *Let M^4 be a quasitoric manifold over the $(2m+1)$-gon. Then the classes $w_4(M^4)$ and $w_2^2(M^4)$ are nontrivial.*

As corollary, we get:

Corollary 2.2. *The quasitoric manifold M^4 over the $(2m+1)$-gon has the same unoriented cobordism class as $\mathbb{C}P^2$.*

Theorems 2.1 and 2.2 shows that the parity of the side number of underlying polygon determines unoriented cobordism classes of quasitoric 4 manifolds. Oriented and complex bordisms are studied [2, Section 5.3, p.69-74].

3 Immersions and Embeddings

The results from previous sections could be used for studying the immersions and embeddings of quasitoric manifolds. As in [1], we use the dual Stiefel-Whitney classes as obstruction.

It is known that the top dual Stiefel-Whitney class of a oriented closed $2n$-dimensional is always zero. Using the argument from previous section we can prove this elementary for quasitoric 4-manifolds. Thus only interesting class is $\overline{w}_2(M^4)$ which is easily calculated

$$\overline{w}_2(M^4) = v_1 + v_2 + \cdot + v_m.$$

As corollary of 2.2 and [4], we have that:

Corollary 3.1. *For a quasitoric manifold M^4 over $(2m+1)$-gon we have*

$$\operatorname{imm}(M^4) \geq 6 \quad \textit{and} \quad \operatorname{em}(M^4) = 7. \tag{4}$$

Choosing the alternating sequence of columns $[0\,1]^t$ and $[1\,0]^t$ for the characteristic matrix of manifold M^4 over $2m$-gon, allows us to construct quasitoric manifold with trivial Chern, Pontriagin and Stiefel-Whitney characteristic classes. However, if we change one of the columns $[0\,1]^t$ with $[1\,1]^t$, we get manifold over $2m$-gon with non-trivial second Stiefel-Whitney class, for which the same relations as in Corollary 3.1 hold.

References

[1] Dj . Baralić : *Immersions and Embeddings of Quasitoric Manifolds over Cube*, accepted for publication in Publications de l'Institut Mathematique.

[2] V. Buchstaber and T. Panov, *Torus Actions and their applications in topology and combinatorics*, AMS University Lecture Series, volume 24, (2002).

[3] M. Davis and T. Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.* 62 (1991), no. 2, 417–451.

[4] J.W. Milnor, J.D. Stasheff. *Characteristic Classes.* Princeton Univ. Press 1974.

[5] T. Weston. *AN INTRODUCTION TO COBORDISM THEORY.* Lecture Notes avaible at https://www.math.umass.edu/ weston/oldpapers/cobord.pdf.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Some Properties of Skew Polynomial Rings of Laurent Type

Dusan Jokanović
University of East Sarajevo, Production and Management Faculty Trebinje
dusanjok@yahoo.com

Ana Marija Alilović
University of Mostar, Faculty of Science and Education

Original scientific paper

Abstract

In this paper we consider polynomials in the skew polynomial ring $R[x, \sigma]$, for some ring endomorphism σ, which is generalization of polynomial ring $R[x]$. Basic properties of rings of Laurent type are observed and conections with Armendariz property. We also consider some examples of semmicomutative and rigid matrix rings.

1 Preliminaries

Throughout this note each ring R is associative with identity, σ denotes an endomorphism of R and $R[x;\sigma]$ denotes skew polynomial ring with the ordinary addition and the multiplication subject to the relation $xr = \sigma(r)x$. Due to Rege and Chhawchharia [1], a ring R is called Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$, for all polynomials $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ from $R[x]$.There are large classes of rings which are Armendariz. It is well known that subrings of Armendariz rings are also Armendariz,and that class of Armendariz rings is not closed for factoring. There is very nice characterization of Armendariz rings through a bijection between the sets of annihilators of subsets of R and subsets of $R[x]$ (see [4]). Recently, several types of generalizations of Armendariz rings have been introduced. Armendariz rings can be obtained through typical ring constructions. It is well known that subrings of Armendariz rings are Armendariz. Rege and Chhawchharia studied conditions for which trivial extension $T(R, R/I)$ of reduced rings are Armendariz. Hong also generalized the notions of Armendariz and rigid ring to σ-skew Armendariz ring. Ring R is called σ-skew Armendariz if $f(x)g(x) = 0$ implies $a_i\sigma^i(b_j) = 0$, for all $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ from $R[x;\sigma]$ (see [5]). As a generalization of σ-skew Armendariz rings, Ouyang (see [2]) introduced a notion of weak σ-skew Armendariz ring R as a ring in which $f(x)g(x) = 0$ implies $a_i\sigma^i(b_j)$ is the nilpotent element of R for all $f(x) = \sum_{i=0}^{n} a_ix^i$

and $g(x) = \sum_{j=0}^{m} b_j x^j$ from $R[x;\sigma]$. Liu and Zhao have studied a generalization of Armendariy rings, which they called weak Armendariz ring. A ring is called a weak-Armendariz if $f(x)g(x) = 0$ implies $a_i b_j \in nil(R)$. $g(x) = \sum_{j=0}^{m} b_j x^j$ from $R[x;\sigma]$. Each semmicomutative ring is weak Armendariz which means that weak Armendariz rings are a common generalization of semmicomutative rings and Armendariz rings. There is example of semicomutative ring which is not weak Armendariz. Let $R = R_1 \oplus R_2$, where R_1 and R_2 are any reduced rings. It easy to see thar R is semmicomutative ring . Let $\sigma : R \to R$ be an endomophism defined by $\sigma(a,b) = (b,a)$. For the polynomials $f(x) = (0,1) - (0,1)x$ and $g(x) = (1,0) + (0,1)x$ we obtain $f(x)g(x) = 0$ but $(0,1)(0,1) \notin nil(R)$. So R is not $\alpha-$ weak Armendariz ring. A ring R satisfies it insertion of factor-property(simply IFP) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. In this paper we consider conditions which Chen and Tong (see [3]) have proved that if R and S are rings and σ is an isomorphism of rings R and S and R is α-skew Armendariz ring, then S is $\sigma\alpha\sigma^{-1}$-skew Armendariz ring. In this paper we prove a variant of this theorem for weak skew Armendariz rings. We also prove that if α is endomorphism of ring R, and the factor ring $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz then $V_n(R)$ is weak $\widetilde{\alpha}$-skew Armendariz.

2 Triangular ring $T(R,n)$

For a ring R consider a following set of triangular matrices

$$T_n(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \mid a_{ij} \in R \right\}.$$

We also consider the following set of triangular matrices

$$T(R,n) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \mid a_{ij} \in R \right\}.$$

It is well known that $T_n(R)$ and $T(R,n)$are subrings of the triangular matrix rings with matrix addition and multiplication. Each endomorphism α of a ring R can be naturally extended to a endomorphism

$$\overline{\alpha} : T_n(R) \to T_n(R),$$

and

$$\overline{\alpha} : T(R,n) \to T(R,n),$$

with:

$$\overline{\alpha}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}\right) =$$

$$\begin{bmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \alpha(a_{13}) & \dots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \alpha(a_{23}) & \dots & \alpha(a_{2n}) \\ 0 & 0 & \alpha(a_{33}) & \dots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(a_{nn}) \end{bmatrix},$$

and

$$\overline{\alpha}\left(\begin{bmatrix} a_0 & a_1 & a_{13} & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} \alpha(a_0) & \alpha(a_1) & \alpha(a_2) & \dots & \alpha(a_{n-1}) \\ 0 & \alpha(a_0) & \alpha(a_1) & \dots & \alpha(a_{n-2}) \\ 0 & 0 & \alpha(a_{33}) & \dots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(a_0) \end{bmatrix}$$

Let $E_{ij} = (e_{st} : 1 \leq s,t \leq n)$ denotes $n \times n$ unit matrices over ring R, in which $e_{ij} = 1$ and $e_{st} = 0$ when $s \neq i$ or $t \neq j$, $0 \leq i,j \leq n$, for all $n \geq 2$. If $V = \sum_{i=1}^{n-1} E_{i,i+1}$, then $V_n(R) = RI_n + RV + ... + RV^{n-1}$ is the subring of upper triangular skew matrices.
In the next section we will show that under the assumption that $R[x]/(x^n)$ is weak Armendariz we obtain the weak Armendariz property of ring $T(R,n)$.

Theorem 2.1. *Suppose that α is an endomorphism of ring R. If the factor ring $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz, then $T(R,n)$ is weak $\widetilde{\alpha}$-skew Armendariz.*

Proof. Suppose that $R\,[x]\,/(x^n)$ is weak $\widetilde{\alpha}-$skew Armendariz and define the ring isomorphism $\theta : V_n(R) \to R\,[x]\,/\,(x^n)$ by

$$\theta(r_0 I_n + r_1 V + .. + r_{n-1}V^{n-1}) = r_0 + r_1 x + .. + r_{n-1}x^{n-1} + (x^n).$$

Now we have that $V_n(R)$ is weak $\theta^{-1}\widetilde{\alpha}\theta$-skew Armendariz and
$\theta^{-1}\widetilde{\alpha}\theta(r_0 I_n + r_1 V + .. + r_{n-1}V^{n-1}) = \theta^{-1}\widetilde{\alpha}(r_0 + r_1 x + ... + r_{n-1}x^{n-1} + (x^n))$

$$= \theta^{-1}(\alpha(r_0) + \alpha(r_1)x + ...\alpha(r_{n-1})x^{n-1} + (x^n))$$
$$= \alpha(r_0)I_n + \alpha(r_1)V + ... + \alpha(r_{n-1})V^{n-1}$$
$$= \widetilde{\alpha}(r_0 I_n + r_1 V + .. + r_{n-1}V^{n-1}),$$

which means that $V_n(R)$ is weak $\widetilde{\alpha}$-skew Armendariz ring. On the other hand there is a ring isomorphism $f : R[x]\backslash(x^n) \to T(R, n)$ given by

$$f(a_0 + a_1x + ... + a_{n-1}x^{n-1}) = (a_0, a_1, ..., a_{n-1})$$

,. where (x^n) is ideal in $R[x]$ generated with x^n. Now by a theorem of isomorphism of rings we have that $Kerf = (x^n)$. ⊣

3 Extensions of weak Armendariz rings

In this section we generalize some results from [3], which are related to σ-skew Armendariz rings, to the weak σ-skew Armendariz case.

A ring R is weak Armendariz if $f(x)g(x) = 0$ implies $a_ib_j \in nil(R)$ for every two polynomials $f(x) = a_0 + a_1x + ... + a_nx^n$, $g(x) = b_0 + b_1x + ... + b_mx^m$ from the ring $R[x]$. This definition is equivalent with the fact that ideal 0 is weak Armendariz ideal. We will prove that the class of weak Armendariz rings is closed for direct products. Also, if the factor ring R/I is weak Armendariz ring, for some nilpotent ideal I, then the ring R is weak Armendariz.

Theorem 3.1. *Finite direct product of weak Armendariz rings is weak Armendariz ring.*

Proof. Suppose that $R_1, R_2, ..., R_n$ are weak Armendariz rings and $R = \prod_{i=1}^{n} R_i$. If $f(x)g(x) = 0$ for some polynomials

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n, \;\; g(x) = b_0 + b_1x + ... + b_mx^m \in R[x],$$

where $a_i = (a_{i1}, a_{i2}, ..., a_{in})$, $b_i = (b_{i1}, b_{i2}, ..., b_{in})$ are elements of the product ring R, define

$$f_k(x) = a_{0k} + a_{1k}x + ... + a_{nk}x^n, \;\; g_k(x) = b_{0k} + b_{1k}x + ... + b_{mk}x^m.$$

From $f(x)g(x) = 0$, we have

$$a_0b_0 = 0, \; a_0b_1 + a_1b_0 = 0, \; ..., \; a_nb_m = 0,$$

and this implies

$$a_{01}b_{01} = a_{02}b_{02} = ... = a_{0n}b_{0n} = 0$$

$$a_{01}b_{11} + a_{11}b_{01} = ... = a_{0n}b_{1n} + a_{1n}b_{0n} = 0$$

$$a_{n1}b_{m1} = a_{n2}b_{m2} = ... = a_{nn}b_{mn} = 0.$$

This means that $f_k(x)g_k(x) = 0$ in $R_k[x]$, $1 \le k \le n$, and since R_k are weak Armendariz rings, we have $a_{ik}b_{jk} \in nil(R_k)$. Now, for each i, j, there exists positive integers m_{ijk} such that $(a_{ik}b_{jk})^{m_{ijk}} = 0$ in the ring R_k, $1 \le k \le n$. If we take $m_{ij} = \max\{m_{ijk} : 1 \le k \le n\}$, than it is clear that $(a_ib_j)^{m_{ij}} = 0$ and this means that R is weak Armendariz ring. ⊣

Theorem 3.2. *If I is nilpotent ideal of ring R such that R/I is weak Armendariz ring, then R is weak Armendariz ring.*

Proof. Let $f(x) = a_0 + a_1x + ... + a_nx^n$ and $g(x) = b_0 + b_1x + ... + b_mx^m$ are polynomials from $R[x]$ such that $f(x)g(x) = 0$. This implies

$$(\overline{a_0} + \overline{a_1}x + ... + \overline{a_n}x^n)(\overline{b_0} + \overline{b_1}x + ... + \overline{b_m}x^m) = 0,$$

and since R/I is weak Armendariz, we have that $\overline{a_i}\overline{b_j} \in nil(R|I)$. From the fact that the ideal I is nilpotent, we obtain that $a_ib_j \in nil(R)$. ⊣

Recall that a ring R is weak σ-rigid if $a\sigma(a) \in nil(R) \Leftrightarrow a \in nil(R)$. It is easy to see that the notion of weak σ-rigid ring generalizes the notion of a σ-rigid ring. Every homomorphism σ of rings R and S can be extended to the homomorphism of rings $R[x]$ and $S[x]$ by $\sum_{i=0}^{m} a_ix^i \mapsto \sum_{i=0}^{m} \sigma(a_i)x^i$, which we also denote by σ. Chen and Tong in [3] prove that if σ is ring isomorphism of rings R and S and R is α-skew Armendariz, then S is $\sigma\alpha\sigma^{-1}$ skew Armendariz ring. We prove the weak skew Armendariz variant of this theorem.

Theorem 3.3. *Let R and S be rings with a ring isomorphism $\sigma : R \to S$. If R is weak α-skew Armendariz then S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ are polynomials from the ring $S[x;\sigma\alpha\sigma^{-1}]$. We have to prove that $f(x)g(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S)$, for all i and j.

As we noted, σ extends to the isomorphism of corresponding polynomial rings, so that there exists polynomials $f_1(x) = \sum_{i=0}^{m} a'_ix^i$ and $g_1(x) = \sum_{j=0}^{m} b'_jx^j$ from $R[x]$ such that $f(x) = \sigma(f_1(x)) = \sum_{i=0}^{m} \sigma(a'_i)x^i$ and $g(x) = \sigma(g_1(x)) = \sum_{j=0}^{m} \sigma(b'_j)x^j$.

First, we shall show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. If $f(x)g(x) = 0$, we have

$$a_0b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + ... + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0,$$

for any $0 \le k \le m$. From the definition of $f_1(x)$ and $g_1(x)$, we have,

$$\sigma(a'_0)\sigma(b'_k) + \sigma(a'_1)(\sigma\alpha\sigma^{-1})\sigma(b'_{k-1}) + ... + \sigma(a'_k)(\sigma\alpha\sigma^{-1})^k\sigma(b'_0) = 0,$$

so that $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain

$$a'_0b'_k + a'_1\alpha(b'_{k-1}) + ... + a'_k\alpha^k(b'_0) = 0,$$

which means that $f_1(x)g_1(x) = 0$ in the ring $R[x;\alpha]$.

It remains to prove that $f_1(x)g_1(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S)$. From the fact that R is weak α-skew Armendariz we have $a'_i\alpha^i(b'_j) \in nil(R)$

and since $a'_i = \sigma^{-1}(a_i), b'_j = \sigma^{-1}(b_j)$, we have $\sigma^{-1}(a_i)\alpha^i\sigma^{-1}(b_j) \in nil(R)$. This implies

$$\sigma^{-1}(a_i)\sigma^{-1}\sigma\alpha^i\sigma^{-1}(b_j) = \sigma^{-1}(a_i(\sigma\alpha\sigma^{-1})^i(b_j)) \in nil(R)$$

and finally we obtain

$$a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S),\ 0 \leq i, j \leq m.$$

Hence S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz. ⊣

4 Skew Polynomial Laurent series Rings

In this section we introduce Laurent σ-Armendariz rings and Laurent σ-skew power series rings and we give their useful characterization in terms of σ-skew Armendariz rings. Throughout this section σ is a ring automorphism.

A ring R is σ-skew Armendariz ring of Laurent type if for every two polynomials

$$f(x) = \sum_{i=-p}^{q} a_i x^i, \quad g(x) = \sum_{j=-t}^{s} b_j x^j$$

from $R\left[x, x^{-1}; \sigma\right]$,

$$f(x)g(x) = 0 \ \text{ implies } \ a_i\sigma^i(b_j) = 0, -p \leq i \leq q, -t \leq j \leq s.$$

We say that R is σ-skew power series Armendariz ring of Laurent type if for every

$$f(x) = \sum_{i=-p}^{\infty} a_i x^i, \quad g(x) = \sum_{j=-t}^{\infty} b_j x^j$$

from the power series ring $R[[x, x^{-1}; \sigma]]$,

$$f(x)g(x) = 0 \ \text{ implies } \ a_i\sigma^i(b_j) = 0, -p \leq i \leq \infty, -t \leq j \leq \infty.$$

In the following two theorems we give a useful characterization of Laurent σ-skew Armendariz rings and Laurent σ-skew power series rings.

Theorem 4.1. *The following conditions are equivalent:*

1. *R is σ-skew Armendariz ring,*

2. *R is σ-skew Armendariz ring of Laurent type.*

Proof. Suppose that $f(x) = \sum_{i=-p}^{q} a_i x^i$ and $g(x) = \sum_{j=-t}^{s} b_j x^j$ are polynomials from the ring $R[x, x^{-1}; \sigma]$ such that $f(x)g(x) = 0$. Since $x^p f(x)$ and $x^t g(x)$ are polynomials from the ring $R[x; \sigma]$ we have that $x^p f(x) g(x) x^t = 0$ which gives $\sigma^p(a_i)\sigma^{i+p}(b_j) = 0, -p \leq i \leq q, -t \leq j \leq s$. Since σ is an automorphism,

$$\sigma^p(a_i \sigma^i(b_j)) = 0,$$

so that we have $a_i \sigma^i(b_j) = 0$. The converse is evident since $R[x; \sigma] \subset R\left[x, x^{-1}; \sigma\right]$. ⊣

Theorem 4.2. *The following conditions are equivalent:*

1. *R is σ-skew power series Armendariz ring,*
2. *R is σ-skew power series Armendariz ring of Laurent type.*

Proof. The same as the proof of the previous theorem. ⊣

References

[1] M. R. Rege, S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A. Math. Sci. **73**(1997), 14–17

[2] L. Ouyang, *Extensions of generalized α-rigid rings*, International Journal of Algebra, **3**(2008), 105–116

[3] W. Chen, W. Tong, *On skew Armendariz and rigid rings*, Houston Journal of Mathematics, **22**(2)(2007)

[4] Y. Hirano, *On annihilator ideals of polynomial ring over a noncommutative ring*, J. Pure and Appl. Algebra **151(3)** (2000), 105–122

[5] C. Y. Hong, N. K. Kim, T. K. Kwak, *On skew Armendariz rings, Comm. Algebra*, **(31)**(2)(2003), 105–122

[6] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. **(3)(4)** (1996), 289–330

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

On a Fixed Point Theorem of Chatterjea

Vesna Mišić
University of East Sarajevo - Faculty of Transport and Traffic Engineering
vesnasmisic@yahoo.com
Ivan D. Aranđelović
University of Belgrade - Faculty of Mechanical Engineering
iarandjelovic@mas.bg.ac.rs

Original scientific paper

Abstract

I. D. Aranđelović and V. Mišić [4] (see also [5]) introduced the notion of a contractive linear operator on metric linear spaces. In [3] authors consider contractive linear operators on locally convex topological vector spaces. General theory of contractive bounded linear operators on partial ordered (non-necessarily locally convex) Haussdorff topological vector spaces and theirs basic properties was presented in [6].

In this talk (paper) we present one common fixed point theorem with operator contractive condition which generalize some earlier results obtained by S. Chatterjea [10], M. Abbas and G. Jungck [1] - Theorem 2.4, L.-G. Huang and X. Zhang [4] - Theorem 4, Sh. Rezapour and R. Hamlbarani [18] - Theorem 2.7.

1 Introduction

There have been a number of generalizations of metric space. One such generalization is the notion of a TVS-cone metric space initiated by I. Beg, A. Azam and M. Arshad [9] which include cone metric spaces in Huang - Zhang sense [4]. They have proved some fixed point theorems for this class of spaces which generalize Banach's and Kannan's contraction mappings principles in a cone metric space with normal cone. This result's were extended to cone metric space with solid cone by Sh. Rezapour and R. Hamlbarani [18]. I. D. Aranđelović and V. Mišić [4],[5] introduced the F - cone metric spaces and the notion of a contractive linear operator and present some fixed point results with operator contractive condition which generalize some results from [4] and [18]. In [3] authors consider contractive linear operators on locally convex topological vector spaces. General theory of contractive bounded linear operators on partial ordered (non-necessarily locally convex) Haussdorff topological vector spaces and theirs basic properties was presented in [6].

G. Jungck [13] initiated the investigation of common fixed point theorems for commuting mappings. S. Sessa [19] introduced the notion of weak commutativity of mappings, which further generalizes concept of R-weak commutativity of R. Pant [17]. Further G. Jungck [14] defined compatible and weakly compatible [15] pairs of self-mappings. Common fixed point results in cone metric spaces were presented in papers [12], [1], [8], [16],... In this talk (paper) we present one common fixed point theorem with operator contractive condition which generalize some earlier results obtained by S. Chatterjea [10], M. Abbas and G. Jungck [1] - Theorem 2.4, L.-G. Huang and X. Zhang [4] - Theorem 4, Sh. Rezapour and R. Hamlbarani [18] - Theorem 2.7.

2 Preliminary Notes

Let E be a linear topological space. Let E be a linear topological space. A subset P of E is called a cone if:

1) P is closed, nonempty and $P \neq \{0\}$;

2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

3) $P \cap (-P) = \{0\}$.

Given a cone, $P \subseteq E$ we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

Let E be a linear topological space and let $P \subseteq E$ be a cone. We say that P is a solid cone if and only if $\text{int}P \neq \emptyset$. Then c is an interior point of P if and only if $[-c, c]$ is a neighborhood of Θ in E.

Ordered topological vector space (E, P) is order-convex if its base of neighborhoods of zero consists of order-convex subsets. In this case the cone P is said to be normal, or P-saturated.

Let E be a topological vector space and $P \subseteq E$ be a cone. P is a solid cone if and only if $\text{int}P \neq \emptyset$. There exists solid cone which is non-normal [18]. In paper [6] we introduced the notion of a contractive operator by the following way.

Definition 2.1. ([6]) If $A : E \to E$ is a one to one function such that $A(P) = P$, (I-A) is one to one and $(I - A)(P) = P$ then A is contractive operator.

Basic properties of contractive bounded linear operator we present in next Lemma.

Lemma 2.1. *([6]) If $A : E \to E$ is the contractive bounded linear operator then*

1) there exists A^{-1} and it is bounded linear operator;

2) there exists $(I - A)^{-1}$ and it is bounded linear operator;

3) $A(x) \ll x$ for any $x \in \text{int}P$;

4) $x \leq y$ implies $A(x) \leq A(y)$ for any $x, y \in P$;

5) $x \ll y$ implies $A(x) \ll A(y)$ for any $x, y \in P$;

6) $(I-A)(x) \ll x$ *for any* $x \in \text{int}P$*;*
7) $I + A + \cdots + A^n = (I-A)^{-1} \circ (I - A^{n+1})$*;*
8) for each $x \in P$ *and any* $c \in \text{int}P$ *there exists a positive integer* n_0 *such that*

$$A^n(x) \ll c$$

for all $n > n_0$*;*
9) $\lim_{n\to\infty}(I + A + \cdots + A^n) = (I-A)^{-1}$.

Recently I. Beg, A. Azam and M. Arshad [9] introduced the notion of TVS-cone metric spaces, such that distance function take values Hausdorff (not necessarily locally convex) topological vector space.

In the following, we always suppose that E is a real (not necessarily locally convex) Hausdorff topological vector space, P is a solid cone in E such that $\leq$ is a partial ordering on E with respect to P. By I we denote the identity operator on E i.e. $I(x) = x$ for each $x \in E$.

Definition 2.2. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:
1) $\Theta \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \Theta$ if and only if $x = y$;
2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.
Then d is called a TVS-cone metric on X and (X,d) is called a TVS-cone metric space.

Definition 2.3. Let (X,d) be a solid TVS-cone metric space, $x \in X$ and (x_n) a sequence in X. Then
1) (x_n) TVS-cone converges to x if for every $c \in \text{int}P$ there exists a positive integer N such that for all $n \geq N$ $d(x_n, x) \ll c$. We denote this by $\lim x_n = x$ or $x_n \to x$;
2) (x_n) is a TVS-cone Cauchy sequences if for every $c \in \text{int}P$ there exists a positive integer N such that for all $m, n \geq N$ $d(x_m, x_n) \ll c$;
3) (X,d) is a TVS-cone complete cone metric space if every Cauchy sequence is convergent.

Lemma 2.2. *([7]) Let* (X,d) *be a TVS-cone metric space,* $(x_n) \subseteq X$ *and* $A : E \to E$ *a one to one bounded linear operator such that* $2A$ *is contractive operator and* $B = (I-A)^{-1} \circ A$*. If*

$$d(x_{n+1}, x_{n+2}) \leq B(d(x_n, x_{n+1})) \tag{1}$$

for any n*, then* (x_n) *is a Cauchy sequence.*

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. The element $x \in X$ is a fixed point for f if $x = f(x)$.

Let X, Y be a nonempty sets, $f, g : X \to Y$ and $f(X) \subseteq g(X)$. Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Containing this process, having chosen

$x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. $f(x_n)$ is called Jungck sequence with initial point x_0.

Let X, Y be a nonempty sets and $f, g : X \to Y$. If $f(x) = g(x) = y$ for some $y \in Y$ then x is called a coincidence point of f and g, and y is called a point of coincidence of f and g.

Let X be a nonempty set and $f, g : X \to X$. f and g is weakly compatible self mappings if they commute at their coincidence point.

Lemma 2.3. *([1]) Let X be a nonempty set and $f, g : X \to X$ be weakly compatible self mappings. If f and g have unique point of coincidence $y = f(x) = g(x)$, then y is the unique fixed point of f and g.*

3 Results

Next Theorem generalizes Theorem 2.4 of [1], Theorem 2.7 of [18] and Theorem 4 of [4]. It also include famous result of S. Chatterjea [10].

Theorem 3.1. *Let (X, d) be a TVS - cone metric space, $f, g : X \to X$ and $A : E \to E$ one to one bounded linear operator such that $2A$ is contractive operator. Suppose that the rang of g contains the rang of f, and $g(X)$ is a complete subspace of X. If for any $x, y \in X$*

$$d(f(x), f(y)) \leq A(d(g(x), f(y)) + d(g(y), f(x))), \tag{2}$$

then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a common unique fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$.

By (3.1) we get that

$$\begin{aligned}
&d(g(x_n), g(x_{n+1})) = d(f(x_{n-1}), f(x_n)) \leq \\
&\leq A(d(g(x_{n-1}), f(x_n)) + d(g(x_n), f(x_{n-1}))) = \\
&= A(d(g(x_{n-1}), g(x_{n+1})) + d(g(x_n), g(x_n))) = \\
&= A(d(g(x_{n-1}), g(x_{n+1})) \leq A(d(g(x_{n-1}), g(x_n)) + A(d(g(x_n), g(x_{n+1})),
\end{aligned}$$

which implies

$$(I - A)(d(g(x_n), g(x_{n+1}))) \leq A(g(d(x_{n-1}), g(x_n))).$$

Let $B = (I - A)^{-1} \circ A$. Now we have that $(g(x_n))$ is convergent because it satisfies the hypotheses of Lemma 2.2 and $g(X)$ is complete.

Let $\lim g(x_n) = q$. $q \in g(X)$, because $g(X)$ is complete, which implies that there exists $p \in X$ such that $q = g(p)$. Let $0 \ll c$. Then there exists positive integer n_0 such that $n \geq n_0$ implies $d(g(x_n), q) \ll \dfrac{c}{3}$ and $d(g(x_n), g(x_{n+1})) \ll \dfrac{c}{3}$. Such n_0 exists because $\lim g(x_n) = q$. Let $n > n_0$. Hence

$$\begin{aligned}
&d(g(p), f(p)) \leq d(f(p), g(x_{n+1})) + d(g(x_{n+1}), g(p)) = \\
&= d(f(p), f(x_n)) + d(g(x_{n+1}), q) \leq \\
&\leq A(d(f(p), g(x_n)) + d(g(p), f(x_n))) + d(g(x_{n+1}), q) \\
&\leq A(d(f(p), g(p)) + d(g(p), g(x_n)) + d(g(p), f(x_n))) + d(g(x_{n+1}), q) \\
&= A(d(f(p), g(p)) + d(q, g(x_n)) + d(q, g(x_{n+1}))) + d(g(x_{n+1}), q) \\
&\leq A(d(f(p), g(p))) + A(d(q, g(x_n))) + A(d(q, g(x_{n+1}))) + d(g(x_{n+1}), q),
\end{aligned}$$

which implies

$$\begin{aligned}
&d(g(y), f(y)) - A(d(g(y), f(y))) \leq \\
&A(d(q, g(x_n))) + A(d(q, g(x_{n+1}))) + d(g(x_{n+1}), q).
\end{aligned}$$

Hence,

$$(I - A)(d(g(p), f(p))) \ll c.$$

So we obtain $f(p) = g(p) = q$, because $I - A$ is one to one linear operator, which implies that $(I - A)(x) = 0$ if and only if $x = 0$.

Let $z \in g(X)$, $z \neq p$ and $g(z) = f(z)$. From (3.1) it follows

$$\begin{aligned}
d(g(z), g(p)) \quad &= d(f(z), f(p)) \leq A(d(g(z), f(p)) + d(g(p), f(z))) \leq \\
&\leq A(2d(g(z), g(p))).
\end{aligned}$$

So $0 \leq (2A - I)(d(z, p))$. It follows $d(z, p) = 0$ which is a contradiction. So f and g have a unique coincidence point in X. Assume now that f and g f and g are weakly compatible. From Lemma 2.3 it follows that f and g have a common unique fixed point. ♢

Acknowledgement. The first author was partially supported by Ministry of Education and Science Grant, No. 174002, Serbia.

References

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341, No. 1 (2008) 416-420.

[2] M. Abbas and B. Rhoades, Fixed and periodic points results in cone metric space, Applied Mathematics Letters 22 (2009) 511–515.

[3] I. Aranđelović, D. Kečkić, On nonlinear quasi-contractions on TVS-cone metric spaces, Applied Mathematics Letters 24 (2011) 1209–1213.

[4] I. D. Aranđelović and V. Mišić, Contractive linear operators and its applications in cone metric fixed point, International Journal of Mathematical Analaysis 4/41 (2010) 2005-2015.

[5] I. D. Aranđelović and V. Mišić, Common fixed point results with linear operator contractive condition in F - cone metric spaces, International Journal of Contemporary Mathematical Sciences 5/39 (2010), 1931-1942.

[6] I. D. Aranđelović and V. Mišić, Contractive operators on topological vector spaces, Bull. Int. Math. Virtual Institute 2 (2012), 167-171.

[7] I. D. Aranđelović and V. Mišić, On cone metric fixed point theory,

[8] C. Di Bari and P. Vetro, φ-pairs and common fixed point in cone metric spaces, Rend. Circ. Mat. Palermo, 57 (2008) 279–285.

[9] I. Beg, A. Azam and M. Arshad, Common fixed points for maps on topological vector space valued cone metric spaces, Internat. J. Math. Math. Sci, vol. 2009, Article ID 560264, 8 pages, 2009.

[10] S. Chatterjea, Fixed points theorems, *C. R. Acad. Bulgare Sci.* **25** (1972).

[11] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.

[12] D. Ilić and V. Rakočević, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2008) 876–882.

[13] C. Jungck, Commuting maps and fixed points, Am. Math. Monthly, 83 (1976).

[14] C. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.

[15] C. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci. 4 (1986) 199–215.

[16] G. Jungck, S. Radenović, S. Radojević and V. Rakočević, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, Fixed point theory and applications 2009, Article ID 643640, 12 pages, doi: 10.1155/2009/643640.

[17] R. P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994), 436–440.

[18] Sh. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl. 345 (2008) 719–724.

[19] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd) 32 (1982) 149–153.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Diofantove jednačine $x^2 - axy + y^2 = \pm a$

Vidan Govedarica
Elektrotehnički fakultet, Univerzitet u Istočnom Sarajevu
vidangov@yahoo.com

Jovana Janković
Filozofski fakultet, Univerzitet u Istočnom Sarajevu
jovanajankovic91@yahoo.com

Originalni naučni rad

Apstrakt

U ovom radu se razmatraju Diofantove jednačine $x^2 - axy + y^2 = a$ i $x^2 - axy + y^2 = -a$ sa dvije cjelobrojne promjenljive x, y i jednim prirodnim parametrom a. Ispituje se za koje vrijednosti parametra a ove jednačine imaju bar jedno rješenje, a kada imaju beskonačno mnogo cjelobrojnih rješenja. U slučajevima kada neka od ovih jednačina ima beskonačno mnogo rješenja nalazi se njeno opšte rješenje. Ovim ćemo uopštiti odgovarajuća tvrdjenja iz [1], odnosno [2].

Uvod

Poznato je da je teorija kvadratnih Diofantovih jednačina sa dvije promjenljive skoro u potpunosti ispitana. Naime, za svaku ovakvu jednačinu se mozhe utvrditi da li ona ima rješenja, a u slučaju kad ima rješenja, može se polazeći od jednog njenog rješenja naći rekurentni niz kojim su odredjena sva njena rješenja. S sruge strane, teorija kvadratnih Diofantovih jednačina sa dvije promjenljive i jednim ili više cjelobrojnih parametara je aktuelna i danas.

Neka je data familija (niz) kvadratnih formi

$$P_a(x, y) = x^2 - axy + y^2,$$

gdje je a prirodan broj, a x i y cjelobrojne promjenljive. Ispitajmo za koje vrijednosti parametra a i neke vrijednosti cjelobrojnog parametra b jednačina $P_a(x, y) = b$ ima bar jedno cjelobrojno rješenje.

1 Diofantova jednačina $x^2 - axy + y^2 = 1$

Razmotrimo prvo slučaj $b = 1$, tj. posmatrajmo Diofantovu jednačinu

$$x^2 - axy + y^2 = 1. \tag{1}$$

Ako je par (x, y) rješenje jednačine (1), onda je očito i par $(-x, -y)$ takodje njeno rješenje. Kako za $xy < 0$ imamo $x^2 - axy + y^2 > 1$, a za $xy = 0$ dobijamo dva rješenja $(1, 0)$ i $(0, 1)$, slijedi da se dalje možemo ograničiti na rješavanje jednačine (1) u skupu prirodnih brojeva.

Za $a = 1$ dobijamo jednačinu $x^2 - xy + y^2 = 1$, koja u skupu prirodnih brojeva ima samo jedno rješenje $(1, 1)$. Za $a = 2$ dobijamo jednačinu $(x - y)^2 = 1$, čija su sva rješenja u skupu $\mathbb{N}$ oblika $(n, n + 1)$, $(n + 1, n)$, $n \in \mathbb{N}$. Neka je, dalje, $a > 2$. Tada možemo uzeti da je $x \neq y$, pošto je za $x = y$ lijeva strana jednačine (1) negativna.

Par $(1, a)$ je jedno rješenje jednačine (1). Neka je par (x, y) jedno rješenje jednačine (1). Pokažimo da tada ona ima i rješenje oblika (y, z), koje je različito od (x, y) (i (y, x)). Zaista, iz jednakosti $P_a(x, y) = 1$ i $P_a(y, z) = 1$ dobijamo redom

$$x^2 - axy + y^2 = y^2 - ayz + z^2,$$
$$x^2 - z^2 = axy - ayz,$$
$$(x - z)(x + z) = ay(x - z),$$

odakle, zbog $x \neq z$, slijedi da je $x + z = ay$, tj. $z = ay - x$. Dakle, ako je (x, y) rješenje jednačine (1), onda je i $(y, ay - x)$ njeno rješenje. Na ovaj način, polazeći od rješenja $(1, a)$, dobijamo da jednačina (1) ima beskonačno mnogo rješenja i da su ta rješenja oblika $(x, y) = (x_n, x_{n+1})$, gdje su x_n i x_{n+1} uzastopni članovi niza (x_n) definisanog rekurentnom formulom

$$x_0 = 1,\ x_1 = a, \quad x_{n+1} = ax_n - x_{n-1} \ \ (n \in \mathbb{N}). \tag{2}$$

Pokažimo da su sa (2) data sva rješenja jednačine (1) u skupu prirodnih brojeva.

U suprotnom, postoji njeno minimalno (u odnosu na zbir koordinata) rješenje (p, q), $p < q$, takvo da p i q nisu uzastopni članovi niza (2). Ako je $p = 1$, onda je $q = a$, tj. $(p, q) = (x_0, x_1)$, što je suprotno pretpostavci.

Neka je, zbog toga, $p > 1$. Iz prethodno dokazanog slijedi da par cijelih brojeva $(r, p) = (ap - q, p)$ takodje zadovoljava jednačinu (1). Pri tome je $r^2 - arp + p^2 - 1 = 0$, odakle slijedi da je $r > 0$ (za $r \leq 0$ imali bismo da je $r^2 - arp + p^2 - 1 > 0$). Slično dobijamo da je $r < p$, jer bismo za $r \geq p$, zbog $p < q$, imali

$$-r^2 + arp = r(ap - r) = rq > rp \geq p^2 > p^2 - 1,$$

tj. $r^2 - arp + p^2 - 1 < 0$. Dakle, $0 < r < p$.

Kako par prirodnih brojeva (r, p) zadovoljava jednačinu (1) i pri tome je $r + p < p + q$, na osnovu pretpostavke postoji $k \in \mathbb{N}_0$ takav da je $r = x_k$ i $p = x_{k+1}$. Medjutim, tada je $q = ap - r = ax_{k+1} - x_k = x_{k+2}$, tj. (p, q) je takodje par uzastopnih članova niza (x_n). Kontradikcija.

Ovim smo dokazali sljedeće tvrdjenje

Teorema 1.1. Za svaki prirodan broj $a \geq 2$ jednačina (1) ima beskonačno mnogo rješenja u skupu prirodnih brojeva. Sva njena rješenja su oblika $(x, y) = (x_n, x_{n+1})$, gdje su x_n i x_{n+1} uzastopni članovi niza (2).

2 Diofantova jednačina $x^2 - axy + y^2 = a$

Neka je sada $b = a$, $a \in \mathbb{N}$, tj. posmatrajmo Diofantovu jednačinu

$$x^2 - axy + y^2 = a. \tag{3}$$

u skupu nenegativnih cijelih brojeva x i y. Ako je $x = 0$ onda je $y^2 = a$, tj. a je potpun kvadrat. Imamo sljedeća dva slučaja.

1° Ako je a potpun kvadrat, tada je $a = b^2$ za neko $b \in \mathbb{N}$, pa dobijamo jednačinu

$$x^2 - b^2xy + y^2 = b^2. \tag{4}$$

Pri tome možemo uzeti da je $b > 1$, pošto smo jednačinu $x^2 - xy + y^2 = 1$ već razmatrali.

Kako je (b, b^3) jedno rješenje jednačine (4), kao i ranije dobijamo da je svaki par $(x, y) = (x_k, x_{k+1})$ uzastopnih članova niza (x_n) definisanog sa

$$x_0 = 0,\ x_1 = b, \quad x_{n+1} = b^2x_n - x_{n-1} \ \ (n \in \mathbb{N}) \tag{5}$$

takodje njeno rješenje.

Pokažimo da su sa (5) data sva rješenja jednačine (4) u supu prirodnih brojeva.

U suprotnom, postoji njeno minimalno (u odnosu na zbir koordinata) rješenje (p, q), takvo da p i q nisu uzastopni članovi niza (5). Pri tome je $p \neq q$, jer bismo za $p = q$ imali $(2 - b^2)p^2 = b^2$, što je nemoguće, pošto je lijeva strana ove jednakosti negativna, a desna strana pozitivna. Takodje ne može biti $p = 1$, jer bismo u tom slučaju imali da je $1 - b^2q + q^2 = b^2$, tj. $b^2 = \dfrac{1+q^2}{1+q}$, a ovaj razlomak je cio broj samo za prirodan broj $q = 1$, što je u kontradikciji sa $p \neq q$. Dakle, možemo uzeti da je $1 < p < q$.

Iz prethodno dokazanog slijedi da par cijelih brojeva $(r, p) = (b^2p - q, p)$ takodje zadovoljava jednačinu (5). Posmatrajmo kvadratnu funkciju

$$f(x) = x^2 - b^2xp + p^2 - b^2.$$

Kako je $f(q) = 0$ i $f(p) = (2 - b^2)p^2 - b^2 < 0$ za nule r i q ove funkcije važi $r < p < q$. Imamo sljedeća tri slučaja.

1) $f(0) = 0$. Tada je $r = 0$, $p = b$, $q = b^3$, pa je $(p, q) = (x_1, x_2)$. Kontradikcija.

2) $f(0) > 0$. Tada je $p^2 - b^2 > 0$, tj. $p > b$, pa kako je $f(r) = 0$ i $f(p) < 0$, dobijamo da je $0 < r < p$. Kako par prirodnih brojeva (r, p) zadovoljava jednačinu (4) i pri tome je $r + p < p + q$, na osnovu pretpostavke postoji $k \in \mathbb{N}$ takav da je $r = x_k$ i $p = x_{k+1}$. Medjutim, tada je $q = b^2p - r = b^2x_{k+1} - x_k = x_{k+2}$, tj. (p, q) je takodje par uzastopnih članova niza (x_n). Kontradikcija.

3) $f(0) < 0$. Tada je $p^2 - b^2 < 0$, tj. $1 < p > b$ i $r < 0$, pa je

$$0 = f(r) = r^2 - b^2rp + p^2 - b^2 = r^2 + p^2 + b^2(-rp - 1) > 0,$$

jer je $-rp - 1 > 0$. Kontradikcija.

2° Neka a nije potpun kvadrat. Tada je $a \geq 2$. Pretpostavimo da jednačina (3) ima rješenje (x, y) u skupu nenegativnih cijelih brojeva. Kako za $x = 0$ (ili $y = 0$) dobijamo da je a potpun kvadrat, slijedi da su x i y prirodni brojevi. Neka je (p, q) njeno minimalno rješenje. Pri tome je $p \neq q$, jer bismo za $p = q$ imali $(2 - a)p^2 = a$, što je nemoguće, pošto je $(2 - a)p^2 \leq 0$. Dakle, možemo uzeti da je $0 < p < q$.

Par cijelih brojeva $(r, p) = (ap - q, p)$ je takodje rješenje jednačine (3), tj. važi $r^2 - arp + p^2 - a = 0$. Pokažimo da je $0 < r < p$.

Posmatrajmo kvadratnu funkciju

$$f(x) = x^2 - axp + p^2 - a.$$

Kako je $f(q) = 0$ i $f(p) = (2 - a)p^2 - a < 0$ za nule r i q funkcije f važi $r < p < q$. Zaista, za $x < 0$ je $f(x) = x^2 - axp + p^2 - a > -axp - a \geq 0$, tj. $f(x) > 0$, a ne može biti $r = 0$ (jer bi u tom slučaju a bio potpun kvadrat).

Dakle, par (r, p) zadovoljava jednačinu (3) i važi $0 < r < p < q$, što je u kontradikciji sa pretpostavkom da je (p, q) njeno minimalno rješenje.

Ovim smo dokazali sljedeće tvrdjenje, koje je uopštenje zadatka 6 sa Medjunarodne matematičke olimpijade 1988. godine.

Teorema 2.1. Jednačina (3) ima beskonačno mnogo rješenja u skupu prirodnih brojeva samo ako je a potpun kvadrat. Ako je $a = b^2$ potpun kvadrat, onda su sva njena rješenja oblika $(x, y) = (x_n, x_{n+1})$, gdje su x_n i x_{n+1} uzastopni članovi niza (5).

3 Diofantova jednačina $x^2 - axy + y^2 = -a$

Neka je $b = -a$, $a \in \mathbb{N}$, tj. posmatrajmo Diofantovu jednačinu

$$x^2 - axy + y^2 = -a. \tag{6}$$

u skupu (nenegativnih) cijelih brojeva x i y.

Pokažimo da ona ima cjelobrojnih rješenja samo za $a = 5$.

1° Pretpostavimo da jednačina (6) ima cjelobrojnih rješenja. Tada je $a > 2$, jer je

$$x^2 - xy + y^2 + 1 = \frac{1}{4}\left((2x - y)^2 + 3y^2\right) + 1 > 0,$$

$$x^2 - 2xy + y^2 + 2 = (x-y)^2 + 2 > 0.$$

Dalje, za $a > 2$ jednačina (6) nema cjelobrojnih rješenja takvih da je $x = y$, jer za $x = y$ dobijamo da je

$$(2-a)x^2 + a, \quad \text{tj.} \quad x^2 = \frac{a}{a-2}\,.$$

a za $a \geq 3$ važi $1 < \dfrac{a}{a-2} < 4$, odakle slijedi da $\dfrac{a}{a-2}$ ne može biti potpun kvadrat.

Primijetimo još da jednačina (6), posmatrana kao kvadratna jednačina po x pri fiksiranim $a > 2$ i $y \in \mathbb{N}$, nema cjelobrojnih rješenja. Zaista, u suprotnom bi njena diskriminanta jednaka nuli, tj. $a^2y^2 - 4(y^2 + a) = 0$, odnosno $y^2 = \dfrac{4a}{a^2-4}$. Kako za $a > 2$ važi $\dfrac{4a}{a^2-4} \leq \dfrac{12}{5}$ i ne može biti $\dfrac{4a}{a^2-4} = 1$, jer jednačina $a^2 - 4a - 4 = 0$ nema cjelobrojnih rješenja, to $\dfrac{4a}{a^2-4}$ ne može biti potpun kvadrat. Kontradikcija.

Pretpostavimo da je (p, q), $p < q$, minimalno rješenje jednačine (6) u skupu prirodnih brojeva, tj. rješenje sa najmanjim $y = p$. Za to $y = p$ kvadratna jednačina

$$x^2 - axp + p^2 + a = 0 \tag{7}$$

ima dva različita rješenja $x = q$ i $x = r$, $q < r$. Tada je $p < q < r$ pa je $q \geq p+1$ i $r \geq p+2$. Na osnovu Vijetovih formula za korijene q i q jednačine (7) važi

$$q + r = ap, \quad qr = p^2 + a.$$

Zbog toga je

$$p^2 + a - ap = qr - q - r = (q-1)(r-1) - 1 \geq p(p+1) - 1 = p^2 + p - 1,$$

tj. $p^2 + a - ap \geq p^2 + p - 1$, što je ekvivalentno sa $(1-p)(1+a) \geq 0$. Posljednja nejednakost je moguća samo za $p = 1$, a u ovom slučaju sve gornje nejednakosti postaju jednakosti pa dobijamo da je

$$q = p + 1 = 2, \ \ r = p + 2 = 3 \ \text{ i } \ a = qr - p^2 = 5.$$

2° Neka je $a = 5$. Tada dobijamo jednačinu

$$x^2 - 5xy + y^2 + 5 = 0. \tag{8}$$

Za $x = 1$ dobijamo jednačinu $y^2 - 5y + 6 = 0$, koja ima dva cjelobrojna rješenja $y = 2$ i $y = 3$, što znači da su $(1, 2)$ i $(1, 3)$ rješenja jednačine (8).

Ako je (x, y) rješenje jednačine (8), na osnovu ranije pokazanog, slijedi da je i $(y, 5y - x)$ njeno rješenje. Zbog toga će jednačinu (8) zadovoljavati svi parovi (x_k, x_{k+1}) i (x'_k, x'_{k+1}) susjednih članova sljedeća dva niza (x_n) i (x'_n) definisanih rekurentnim formulama

$$x_0 = 1, \quad x_1 = 2, \quad x_{n+1} = 5x_n - x_{n-1}, \quad n \in \mathbb{N}, \tag{9}$$

$$x'_0 = 1, \quad x'_1 = 2, \quad x'_{n+1} = 5x'_n - x'_{n-1}, \quad n \in \mathbb{N}. \tag{10}$$

Pokažimo da su na ovaj način odredjena sva rješenja jednačine (8). U suprotnom postoji njeno minimalno rješenje (p, q), $p < q$, takvo da p i q nisu uzastopni članovi jednog od nizova (9) ili (10). Ako je $p = 1$, onda je $q = 2$ ili $q = 3$ pa je $(p, q) = (x_0, x_1)$ ili $(p, q) = (x'_0, x'_1)$, što je suprotno pretpostavci.

Neka je, zbog toga, $p > 1$. Tada je par $(p, r) = (p, 5p - q)$ takodje rješenje jednačine (8) pa je $r^2 - 5rp + p^2 + 5 = 0$, odakle slijedi da je $r > 0$ (za $r \leq 0$ je lijeva strana ove jednačine pozitivna). Pokažimo da je $r < p$. Neka je $f(x) = x^2 - 5xp + p^2 + 5$. Tada je $f(q) = 0$ i $f(p) = -3p^2 + 5 < 0$, pa za nule r i q ove funkcije važi $r < p < q$, odnosno $0 < r < p < q$. Kako par (r, p) zadovoljava jednačinu (8), slijedi da postoji k takvo da je

$$r = x_k, \quad p = x_{k+1} \quad \text{ili} \quad r = x'_k, \quad p = x'_{k+1}.$$

Medjutim, tada je $q = 5p - r = 5x_{k+1} - x_k = x_{k+2}$ ili $q = 5p - r = 5x'_{k+1} - x'_k = x'_{k+2}$, tj. (p, q) je par uzastopnih članova jednog od nizova (x_n) i (x'_n). Kontradikcija.

Ovim smo dokazali sljedeće tvrdjenje, koje je uopštenje zadatka $M1225$ iz [2].

Teorema 3.1. Jednačina (6) ima rješenja u skupu prirodnih brojeva samo za $a = 5$. Za $a = 5$ su sva njena rješenja oblika $(x, y) = (x_n, x_{n+1})$ ili (x'_n, x'_{n+1}), gdje su x_n i x_{n+1}, odnosno x'_n i x'_{n+1}, uzastopni članovi nizova (9) i (10).

Literatura

[1] Djukić, D., Janković, V., Matić, I., Petrović, N., *The IMO Compendium,* Springer, 2009.

[2] Mamikonyan, S., Oganesyan, G., *Zadača M1225,* Naučno-populyarnyĭ fiziko-matematičeskiĭ žurnal Kvant, 10(1990), 25–26, Moskva.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Jedna klasa trinomnih jednačina i polupravilni poligoni

Nenad Stojanović
Visoka Medicinska škola,Prijedor
nsnest1@gmail.com

Originalni naučni rad

Apstrakt

U radu su analizirani konveksni jednakostrani polupravilni poligoni i to sa stanovišta geometrijske konstrukcije i geometrijskog prikaza. Analizirani su uslovi pod kojim se može geometrijski konstruisati polupravilni jednakostrani konveksni poligon,a zatim je dato rješenje problema vezanog za određe broja geometrijskih prikaza polupravilnih poligona $\mathcal{P}_N$ sa $N = n \cdot m$ stranica. Formulisana je veza geometrijskog prikaza polupravilnih poligona i trinomnih jednačina.Na osnovu te veze dat je analitički pristup konstrukciji polupravilnih jednakostranih poligona čiji vrhovi upisanog mu pravilnog poligona leže na jediničnoj kružnici.

1 Uvod

Navedimo nekoliko elementarnih pojmova iz algebre koji su nam potrebni za dalje izlaganje,a koji su povezani sa geometrijskom konstrukcijom pravilnih poligona[1].

Neka je $\xi = a + bi, \quad a, b \in \mathbb{R}$ proizvoljan kompleksan broj.Od posebnog interesa su kompleksni brojevi čiji je modul jednak 1.Neka je z takav broj. Tada je $z = \cos\varphi + i\sin\varphi$. Slike stepena broja z leže na kružnici poluprečnika $r = 1$. Faza broja Z^2 jednaka je 2φ,faza broja Z^3 je 3φ, faza broja z^n jednaka je $n\varphi$.

Lako se pokaže da dijeljenje kružnice na $n \in \mathbb{N}$ jednakih dijelova vodi ka defenisanju jednačine $z^n - 1 = 0$. Radi toga jednačina takvog oblika nosi još i naziv jednačina dijeljenja kružnice.

Stavimo li da je $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ tada je $\varepsilon^k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

Korijeni n-tog stepena iz jedinice; $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots, \varepsilon^{n-1}$ su međusobno različiti i njihove slike predstavljaju vrhove pravilnog n-tougla, upisanog u kružnicu poluprečnika $r = 1$, pri čemu tačka koja odgovara broju 1 odgovara jednom od vrhova pravilnog poligona. Primijetimo da, kako je $\varepsilon^n = 1$, vrijedi

$$\varepsilon^{n-k} = \varepsilon^n \varepsilon^{-k} = 1 \cdot \varepsilon^{-k} = \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} \tag{1}$$

Konstrukcija pravilnog n-tougla svodi se dakle, na određivanje n-tog korijena iz jedinice, tj.takvog broja ε koji zadovoljava jednačinu $z^n - 1 = 0$.

Kako je $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$ slijedi da je 1 korijen jednačine $z^n - 1 = 0$ i da svaki drugi korijen ε zadovoljava jednačinu

$$\varepsilon^{n-1} + \varepsilon^{n-2} + \cdots + \varepsilon = -1. \qquad (2)$$

Konstrukcija pravilnog n tougla može se ostvariti pomoću šestara i lenjira kada korijeni jednačine 2 mogu biti izraženi u kvadratnim radikalima (dokaz pogledati u [1]).

2 Geometrijske konstrukcije i prikazi polupravilnih poligona

Neka je $\mathcal{P}_N^{a,\delta}$ konveksan jednakostrani polupravilni poligon sa $N = n \cdot m$ stranica kome je pravilni poligon $\mathcal{P}_n^b, n \geq 3, \in \mathbb{N}$ "upisani" a $\mathcal{P}_k^a, k \geq 3 \in \mathbb{N}$ jednakokraki poligoni sa $m = k-1$ jednakih kraka konstruisani nad svakom stranicom pravilnog poligona,kao zajedničkom,i neka je sa δ označen ugao za koji vrijedi

$$\delta = \angle(d_{i-1}, d_i); i = 1, 2, \ldots, m-1; d_0 = a; d_{m-1} = d_{k-2} = b, \qquad (3)$$

i $\delta \in (0; \frac{\pi}{N-n}), \delta \neq \frac{\pi}{N}$ tada vrijedi teorema

Teorema 2.1. Konveksni jednakostrani polupravilni poligon $\mathcal{P}_N^{a,\delta}$, stranice a i ugla δ možemo geometriski konstruisati ako i samo ako se može geometrijski konstruisati "upisani" mu pravilni poligon i odgovarajući ivični poligoni sa $m = k-1$ jednakih kraka nad svakom stranicom tog upisanog pravilnog poligona tako da je $N = n \cdot m$.

Dokaz: Neka je dat polupravilan poligon $\mathcal{P}_N$ i neka su $A_1, A_2, \ldots, A_n$ vrhovi "upisanog" mu pravilnog poligona $\mathcal{P}_n$.

Sa $A_j = B_1, B_2, B_3, \ldots, B_m, B_{m+1} = A_{j+1}, j = 1, 2, \ldots n$ označimo vrhove jednakokrakog poligona $\mathcal{P}_k$ konstruisanog nad svakom stranicom A_jA_{j+1} "upisanog" mu pravilnog poligona.

Pokazano je [Gaus,1796.] da se pravilan poligon sa n stranica može geometrijski konstruisati ako i samo ako vrijedi

$$n = 2^{r_0} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots \cdot p_s^{\alpha_s}, r_0 \geq 0, r_0 \in \mathbb{Z}, \alpha_i \in \{0,1\}, i = 1, 2, \ldots, s. \qquad (4)$$

gdje su p_i različiti prosti brojevi Ferme.To je ne samo dovoljan nego i potreban uslov za konstrukciju pravilnog poligona [1]. Neka ti uslovi vrijede za "upisani" pravilni poligon.Tada njegova konstrukcija postoji.

Ostaje da se analizira konstrukcija jednakokrakih poligona $\mathcal{P}_k^{a,\delta}$ konstrusanih nad svakom stranicom,kao zajedničkom,upisanog pravilnog poligona ako je zadana dužina kraka a i ugao δ.

Primijetimo da dijagonale $d_i, i = 1, 2, \ldots, m-1$ povučene iz vrha A_j stranice $A_jA_{j+1} = b$, i $j = 1, 2, \ldots, n$ pravilnog poligona $\mathcal{P}_n^b$ dijele poligon $\mathcal{P}_k^a$ na trouglove:

$$\triangle A_jB_2B_3, \triangle A_jB_3B_4, \ldots, \triangle A_jB_{i-1}B_i, \triangle A_jB_iB_{i+1}, \cdots \triangle\ A_jB_mA_{j+1}.$$

Dakle,egzistencija konstrukcije poligona $\mathcal{P}_k^a$,zavisi od konstrukcije tih trouglova.

Ne gubeći ništa na uopštenosti uzmimo da je jednakokraki poligon $\mathcal{P}_k^a$ konstruisan nad stranicom $A_1A_2 = b$ pravilnog poligona $\mathcal{P}_n^b$.

Dijagonale d_i, $i = 1, 2, \ldots (m-1)$, povučene iz vrha A_1 dijele poligon $\mathcal{P}_k^a$ na trouglove

$$\triangle A_1B_2B_3, \triangle A_1B_3B_4, \ldots, \triangle A_1B_iB_{i+1}, \ldots, \triangle A_1B_mA_2.$$

Vrijedi:

1. Za jednakokraki trougao $\triangle A_1B_2B_3$ je zadato $A_1B_2 = B_2B_3 = a$, $\angle A_1 = \angle B_3 = \delta$ i možemo ga konstruisati,a $A_1B_3 = d_1$ je osnovica tog trougla.

2. Za trougao $\triangle A_1B_3B_4$ vrijedi $B_3B_4 = a, \angle A_1 = \delta A_1B_3 = d_1$ i $k(B_3, a) \cap A_1B_4 = B_4$.

3. Za trouglove $\triangle A_1B_iB_{i+1}, i = 2, 3, \ldots, m$,$B_{m+1} = A_2$, je zadato; $B_iB_{i+1} = a, \angle A_1 = \delta, A_1B_i = d_{i-2}$.Dakle, mogu se konstruisati.

Slijedi,postoji konstrukcija poligona $\mathcal{P}_k^a$ stranice a i osnovice $A_1A_2 = b$.

Ponovimo li konstrukciju jednakokrakih poligona $\mathcal{P}_k^a$ nad svakom stranicom upisanog pravilog poligona dobijamo polupravilni poligon $\mathcal{P}_N^a$ kome je zadana stranica a i ugao δ. $\Diamond$

Neka je $\mathcal{P}_N$ konveksni jednakostrani polupravilni poligon sa N stranica i $N = n \cdot m, n, k \geq 3$,$m = k-1, n, k \in \mathbb{N}$ tada ima smisla definicija

Definicija 2.1. Kažemo da paru prirodnih brojeva (n, m) odgovara geometrijska konstrukcija $\mathcal{K}_n^m$ konveksnog jednakostranog polupravilnog poligona $\mathcal{P}_N$ sa N stranica ako:

1. Za svaki $N \in \mathbb{N}$ postoje prirodni brojevi $n, k \geq 3$ i prirodan broj $m = k-1$ takvi da $N = n \cdot m$.

2. Za broj stranica n upisanog pravilnog poligona vrijedi prikaz

$$n = 2^{r_0} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots \cdot p_s^{\alpha_s}, r_0 \geq 0, r_0 \in \mathbb{Z}, \alpha_i \in \{0, 1\}, i = 1, 2, \ldots, s.$$

   gdje su p_i različiti prosti brojevi Ferme,

3. Nad svakom stranicom pravilnog poligona,kao zajedničkom, postoji konstrukcija jednakokrakog poligona sa $m = k-1$ jednakih kraka.

Vrijedi i obratno: Geometrijskom konstrukcijom $\mathcal{K}_n^m$ polupravilnog poligona $\mathcal{P}_N$ sa N stranica je generisan uređen par (n, m) prirodnih brojeva takvih da je $n \cdot m = N$ i $n, k \in \mathbb{N}, n, k \geq 3, m = k - 1$, i n je broj stranica upisanog mu pravilnog poligona,a m broj kraka jednakokrakih poligona konstruisanih nad svakom stranicom, kao zajedničkom, tog upisanog pravilnog poligona.

Kako prikaz broja $N = n \cdot m$ u obliku proizvoda dva prirodna broja nije jedinstven,jednakostrani polupravilni poligon nema jedinstvenu geometrijsku konstrukciju.

Ako je $\mathcal{K}_N$ skup svih geometrijskih konstrukcija polupravilnog poligona $\mathcal{P}_N$ tada je

$$\mathcal{K}_N = \{\mathcal{K}_n^m | \quad n, k \geq 3, n, k \in \mathbb{N}, m = k - 1 \wedge N = n \cdot m\} \tag{5}$$

i pri tome n ima prikaz oblika 4.

Ako ne zahtijevamo da postoji stroga geometrijska konstrukcija pravilnog poligona nego samo da postoji "upisani" pravilni poligon $\mathcal{P}_n$, odnosno da zahtjeve definicije oslabimo tako da vrijedi definicija

Definicija 2.2. Ako se broj stranica N jednakostranog polupravilnog poligona $\mathcal{P}_N$ može prikazati kao proizvod dva faktora $n \cdot m$ tako da je n broj stranica "upisanog" pravilnog poligona, a m broj jednakih kraka poligona konstruisanog nad svakom stranicom tog pravilnog poligona i pri tome je $n, k \geq 3, n, k \in \mathbb{N}, m = k - 1$ i$N = n \cdot m$,(nije obavezno da n ima prikaz oblika 4) kažemo da je parom (n, m) određen *geometrijski prikaz $\mathcal{P}_n^m$ polupravilnog poligona* .

Vrijedi i obratno:Geometrijskim prikazom $\mathcal{P}_n^m$ je generisan uređen par (n, m). Označimo li sa $\mathcal{G}_N$ skup svih geometrijskih prikaza polupravilnog poligona $\mathcal{P}_N$ sa $N = n \cdot m$ stranica tada je

$$\mathcal{G}_N = \left\{\mathcal{P}_n^m | \exists n, k \in \mathbb{N} n, k \geq 3, m = k - 1, N = n \cdot m\right\}. \tag{6}$$

Dakle,možemo govoriti o *skupu $\mathcal{K}_N$ svih geometrijskih konstrukcija*kada par prirodnih brojeva (n, m) zadovoljava uslove definicije 2.1 i o *skupu $\mathcal{G}_N$ svih geometrijskih prikaza* konveksnog polupravilnog poligona $\mathcal{P}_N$ kada par prirodnih brojeva (n, m) zadovoljava uslove definicije 2.2,odnosno kada se ne zahtijeva da n obavezno ima prikaz u obliku 4.

Na osnovu izloženog slijedi da je svaki konstruktivni prikaz ujedno i geometrijski, a da obratno ne vrijedi.

Sljedećim primjerima pokazujemo razliku između geometrijske konstrukcije i geometrijskog prikaza.

Primer 2.1. Par prirodnih brojeva $(6, 2)$ generiše gemetrijsku konstrukciju polupravilnog dvanestougla $\mathcal{P}_{12}$ jer postoji geometrijska konstrukcija pravilnog šestougla $\mathcal{P}_6$ i jednakokrakog trougla $\mathcal{P}_3$ nad svakom njegovom stranicom. Takođe geometrijsku konstrukciju jednakostranog polupravilnog $\mathcal{P}_{12}$ generišu i parovi

$(3,4),(4,3)$ jer se polupravilni dvanestougao može konstruisati ako se nad svakom stranicom jednakokrakog trougla konstruiše poligon sa 4 jednaka kraka,kao i kada se nad svakom stranicom kvadrata konstruiše poligon sa tri jednaka kraka.

Na osnovu toga skup geometrijskih konstrukcija konveksnog polupravilnog dvanaestougla čine parovi

$$\mathcal{K}_{12} = \{(6,2),(3,4),(4,3)\} = \{\mathcal{K}_6^2, \mathcal{K}_3^4, \mathcal{K}_4^3\}$$

Osim toga skup geometrijskih prikaza polupravilnog dvanaestougla je

$$\mathcal{G}_{12} = \{(6,2),(3,4),(4,3)\} = \{\mathcal{P}_6^2, P_3^4, \mathcal{P}_4^3\}$$

Parom $(2,4)$ nije generisana ni geometrijska konstrukcija niti geometrijski prikaz polupravilnog osmougla $\mathcal{P}_8$ jer nije ispunjen zahtjev da je $n \geq 3$,dok par $(4,2)$ generiše geometrijsku konstrukciju tog polupravilnog poligona.

Na osnovu toga je skup svih geometrijskih konstrukcija polupravilnog jednakostranog osmougla $\mathcal{K}_8 = \{(4,2)\} = \{\mathcal{K}_4^2\}$ i jednak je skupu geometrijskih prikaza $\mathcal{G}_8 = \{(4,2)\} = \{\mathcal{P}_4^2\}$.

Za razliku od toga geometrijska konstrukcija polupravilnog osamnaestougla parom $(9,2)$ nije generisana,jer se pravilni devetogao nemože konstruisati,odnosno nije ispunjen uslov definicije 2.1,ali je generisan jedan geometrijski prikaz,tako da je skup svih geometrijskih konstrukcija polupravilnog $\mathcal{P}_{18}$

$$\mathcal{K}_{18} = \{(3,6),(6,3)\} = \{\mathcal{K}_3^6, \mathcal{K}_6^3\}$$

dok je skup geometrijskih prikaza

$$\mathcal{G}_{18} = \{(9,2),(3,6),(6,3)\} = \{\mathcal{P}_9^2, \mathcal{P}_3^6, \mathcal{P}_6^3\}$$

.

Pregled geometrijskih prikaza i geometrijskih konstrukcija za neke polupravilne poligone prikazan je tabelom 1.

3 Određivanje broja geometriksih prikaza polupravilnog poligona

Označimo sa $d(\mathcal{G}_N)$ broj geometrijskih prikaza polupravilnog jednakostranog poligona $\mathcal{P}_N$ sa $N = n \cdot m$ stranica.Problem određivanja broja geometrijskih prikaza $d(\mathcal{G}_N)$ tog polupravilnog poligona razmatran je sljedećim teoremom

Teorema 3.1. Broj geometrijskih prikaza $d(\mathcal{G}_N)$ polupravilnog poligona $\mathcal{P}_N$ sa N stranica,kome je $\mathcal{P}_n$ odgovarajući "upisani" pravilni poligon izračunavamo po formuli

$$d(\mathcal{G}_N) = \begin{cases} d(N) - 3, & \text{ako je } N \text{ paran broj} \\ d(N) - 2 & \text{ako je } N \text{ stepen prostog broja ili nije oblika } 2^{2^n} + 1 \end{cases}$$

a $d(N)$ broj pozitivnih djelitelja broja N.

Tabela 1: Geometrijski prikazi i geometrijske konstrukcije,broj geometrijskih prikaza i geometrijskih konstrukcija polupravilnih jednakostranih poligona u rasponu od $N = 6$ do $N = 30$ stranica.

N	n	$\mathcal{G}_N$	$\mathcal{K}_N$	$d(\mathcal{G}_N)$	$d(\mathcal{K}_N)$
6	3	$\{(3,2)\}$	$\{(3,2)\}$	1	1
7	0	0	0	0	0
8	4	$\{(4,2)\}$	$\{(4,2)\}$	1	1
9	3	$\{(0,0)\}$	0	1	0
10	5	$\{(5,2)\}$	$\{(5,2)\}$	1	1
11	0	0	0	0	0
12	3,4,6	$\{(6,2),(3,4),(4,3)\}$	$\{(6,2),(3,4),(4,3)\}$	3	3
13	0	0	0	0	0
14	7	$\{(7,2)\}$	0	1	0
15	3,5	$\{(3,5),(5,3)\}$	$\{(3,5),(5,3)\}$	2	2
16	4,8	$\{(4,4),(8,2)\}$	$\{(4,4),(8,2)\}$	2	2
17	0	0	0	0	0
18	3,6	$\{(9,2),(3,6),(6,3)\}$	$\{(3,6),(6,3)\}$	3	2
19	0	0	0	0	0
20	4,5	$\{(4,5),(5,4),(10,2)\}$	$\{(4,5),(5,4),(10,2)\}$	3	3
21	3,7	$\{(3,7),(7,3)\}$	$\{(3,7)\}$	2	1
22	11	$\{(11,2)\}$	0	1	0
23	0	0	0	0	0
24	3,4,6,8,12	$\{(3,8),(4,6),(6,4),(8,3)(12,2)\}$	$\{(3,8),(4,6),(6,4),(8,3)(12,2)\}$	5	5
25	5	$\{(5,5)\}$	$\{(5,5)\}$	1	1
26	13	$\{(13,2)\}$	0	1	0
27	3,9	$\{(3,9),(9,3)\}$	$\{(3,9)\}$	2	1
28	4,7	$\{(4,7),(7,4),(14,2)\}$	$\{(4,7)\}$	2	1
29	0	0	0	0	0
30	3,5,6,10,15	$\{(3,10),(5,6),(6,5),(10,3),(15,2)\}$	$\{(3,10),(5,6),(6,5),(10,3),(15,2)\}$	5	5

Dokaz: Neka je $\mathcal{P}_N$ konveksni jednakostrani polupravilni poligon sa N stranica i $N = \prod_{i=1}^{s} p_i^{\alpha_i}$, a p_i različiti prosti brojevi $\alpha_i \in (0,1), i = 1, 2, \ldots, s$, kanonski prikaz broja N.Broj pozitivnih djelitelja broja N određen je relacijom

$$d(N) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) \tag{7}$$

Po pretpostavci je $N = n \cdot m$ pa je broj njegovih predstavljanja kao proizvoda dva faktora,gdje se vodi računa o poretku faktora,(vidjeti u [2] str. 65) jednak broju njegovih pozitivnih djelitelja $d(N)$ tj.

$$d_2(N) = d(N) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1). \tag{8}$$

Svakim prikazom broja N kao proizvoda dva faktora n, m gdje se vodi računa o poretku faktora određen je uređen par (n, m). Ovom uređenom paru prirodnih brojeva prema definiciji 2.2 odgovara jedno geometrijsko predstavljanje. Od ukupnog broja tako određenih uređenih parova moramo isključiti one koji ne ispunjavaju uslove te definicije.To su parovi$(n, 1), (1, m), (2, m)$.

Dakle,ukupan broj parova generisanih geometrijskim prikazima je

$$d(\mathcal{G}_N) = d(N) - 3$$

Ako je N stepen prostog neparnog broja, ili nije oblika $2^{2^n} + 1$, tada se od ukupnog broja oduzimaju parovi $(n, 1), (1, m)$,koji ne ispunjavaju uslove definicije 2.2 pa je ukupan broj geometrijskih prikaza

$$d(\mathcal{G}_N) = d(N) - 2.$$

Iz navedenih definicija i dokazanog teorema slijedi da broj geometrijskih konstrukcija nije jednak broju geometrijskih prikaza za svaki polupravilni poligon P_N sa $N = n \cdot m$ stranica,ali postoje polupravilni poligoni kod kojih su ti brojevi jednaki.

Primer 3.1. Polupravilnom $\mathcal{P}_{24}$ se mogu "upisati" sljedeći pravilni poligoni $\{3, 4, 6, 8, 12\}$, dakle ima ih ukupno $d = 5$. Na osnovu (7), odnosno (8) imamo da je $d(\mathcal{G}_{24}) = d(N) - 3 = 8 - 3 = 5$.

Navedimo nekoliko problema koji su vezani za broj geometrijske prikaza polupravilnih poligona

Problem 3.1. Dokazati da postoji beskonačno mnogo polupravilnih jednakostranih poligona kojima je broj geometrijkih prikaza i geometrijskih konstrukcija jednak broju 5 tj. da vrijedi jednakost

$$d(\mathcal{K}_N) = d(\mathcal{G}_N) = 5.$$

Na primjer kao što su to polupravilni poligoni $\mathcal{P}_{24}$i $\mathcal{P}_{30}$.

Problem 3.2. Dokazati da postoji beskonačno mnogo polupravilnih jednakostranih poligona sa $N = n \cdot m$ stranica za koje vrijedi

$$d(\mathcal{K}_N) = d(\mathcal{G}_N) = d(n)$$

gdje je $d(n) > 2$ broj svih odgovarajućih pravilnih poligona $\mathcal{P}_n$ za koje postoji geometrijska konstrukcija. Kao što su: $\mathcal{P}_{12}, \mathcal{P}_{24}, \mathcal{P}_{30}$.

4 Geometrijski prikazi polupravilnog poligona i trinomne jednačine

Paru prirodnih brojeva (n, m) kojim je generisana geometrijska konstrukcija $\mathcal{K}_n^m$ jednakostranog polupravilnog poligona $\mathcal{P}_N$ sa $N = n \cdot m$ stranica pridružimo funkciju $f_{n,m}(x)$ definisanu sa

$$f_{n,m}(x) = Ax^{n \cdot m} + Bx^n + C \tag{9}$$

čiji koeficijenti $A, B, C \in \mathbb{R}$ zadovoljavaju uslov $A + B + C = 0$ i pri tome su A i C suprotnog predznaka tj.$signA = -signC$.

Sljedećim teoremom je analiziran analitički pristup konstrukciji jednakostranih polupravilnih poligona, odnosno, i pokazano je da afiksi nula jednačine

$$Ax^N + Bx^n + C = 0 \tag{10}$$

čiji koeficijenti zadovoljavaju navedene uslove određuju vrhove polupravilnog jednakostranog poligona $\mathcal{P}_N$.

Teorema 4.1. Afiksi rješenja jednačine $Ax^N + Bx^n + C = 0$ predstavljaju vrhove jednakostranog polupravilnog poligona $\mathcal{P}_N$ sa $N = n \cdot m$ stranica kome vrhovi "upisanog" pravilnog poligona $\mathcal{P}_n$ leže na jediničnoj kružnici i nad svakom stranicom tog pravilnog poligona je konstruisan poligon $\mathcal{P}_k$ sa $m = k - 1$ jednakih kraka, ako i samo za njene koeficijente $A, B, C \in \mathbb{R}$ vrijedi $A + B + C = 0$ i $signA = -signC$

Dokaz: Pretpostavimo da uslov $A + B + C = 0$ vrijedi i da je $signA = -signC, A, B, C \neq 0$. Pokažimo da afiksi rješenja jednačine predstavljaju vrhove polupravilnog poligona P_N. Kako je $A = -(B + C)$ polaznu jednačinu transformišimo na sljedeći način

$$\begin{aligned} Ax^N + Bx^n + C &= 0 \\ &\Leftrightarrow \quad -(B+C)x^N + Bx^n + C = 0 \\ &\Leftrightarrow \quad -B(x^N - x^n) - C(x^N - 1) = 0 \\ &\Leftrightarrow \quad B(x^{n \cdot m} - x^n) + C(x^N - 1) = 0 \\ &\Leftrightarrow \quad Bx^n(x^{n \cdot m - n} - 1) + C(x^{n \cdot m - 1}) = 0. \end{aligned}$$

Primijetimo da vrijede rastavi

$$\begin{aligned} x^{nm-n} - 1 &= (x^n)^{m-1} - 1 \\ &= (x^n - 1)(x^{n(m-2)} + x^{n(m-3)} + \cdots + x^n + 1)x^{nm} - 1 \\ &= (x^n)^m - 1 = (x^n - 1)(x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n + 1). \end{aligned}$$

Na osnovu toga iz posljednje jednakosti imamo da je

$$\begin{aligned} Bx^n[(x^n - 1)(x^{n(m-2)} + x^{n(m-3)} + \cdots + x^n + 1)] \\ +C[(x^n - 1)(x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n + 1] = 0 \\ \Leftrightarrow (x^n - 1)[Bx^n(x^{n(m-2)} + x^{n(m-3)} \\ + \cdots + x^n + 1) + C(x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n + 1)] = 0 \\ \Leftrightarrow (x^n - 1)[(B+C)x^{n(m-1)} + (B+C)x^{n(m-2)} + \cdots + (B+C)x^n + C] = 0. \end{aligned}$$

Koristeći uslov $B + C = -A$ i dijeljenjem posljednje jednačine sa $-A$ dobijamo

$$(x^n - 1)\Big[x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n - \frac{C}{A}\Big] = 0 \tag{11}$$

Posljednja jednakost zamjenom $x^n = y$ prelazi u oblik

$$(y - 1)\Big[y^{m-1} + y^{m-2} + \cdots + y - \frac{C}{A}\Big] = 0 \tag{12}$$

Jedno rješenje ove jednačine je $y = 1$,a svako drugo rješenje $y = \varepsilon$ mora zadovoljiti jednačinu

$$\varepsilon^{m-1} + \varepsilon^{m-2} + \cdots + \varepsilon = \frac{C}{A} \tag{13}$$

Kako je $y = x^n$ za $y = 1$ imamo jednačinu $x^n - 1 = 0$ čija su rješenja određena relacijom

$$x_k = \sqrt[n]{1} = \cos \frac{2k\pi n}{+} \sin \frac{2k\pi n}{,} k = 1, 2, 3, \cdots, n$$

Označimo li rješenje sa $w = \cos \frac{2\pi}{n} + \sin \frac{2\pi}{n}$ tada je $w^k = \cos \frac{2k\pi}{n} + \sin \frac{2k\pi}{n}$, pa se prvoodređeni međusobno različiti korijeni n tog stepena iz jedinice mogu predstaviti u obliku $w, w^2, w^3, \cdots, w^{n-1}, w^n = 1$. Grafički prikaz tih korijena predstavlja vrhove pravilnog poligona $\mathcal{P}_n$ upisanog u jediničnu kružnicu pri čemu tačka koja odgovara broju 1 odgovara jednom vrhu. Ostaje da se pokaže da afiksi rješenja jednačine (13) predstavljeju vrhove jednakokrakih poligona $\mathcal{P}_k$ konstruisanih nad svakom stranicom pravilnog poligona $\mathcal{P}_n$.

Neka su $y = \varepsilon_k, k = 1, 2, 3, \cdots, m-1$ rješenja jednačine (13). Primijetimo da svakom tom rješenju odgovara jedna binomna jednačina

$$x^n - \varepsilon_k = 0, \quad k = 1, 2, 3, \cdots, m-1. \tag{14}$$

Svaka od tih jednačina ima n različitih prvoodređenih rješenja koja leže na kružnici poluprečnika $r_k = \sqrt[n]{|\varepsilon_k|}$. Dakle, na taj način se na $m-1$ koncentričnih kružnica nalazi ukupno $n(m-1) = nm - n$ tačaka. Ako se tome doda i n tačaka generisanih binomnom jednačinom $x^n = 1$ dobijamo da je ukupan broj tačaka konstruisanih rješenjima binomnih jednačina $mn - n + n = mn = N$.

Obratno:Ako afiksi rješenja jednačine $Ax^N + Bx^n + C = 0$, predstavljaju vrhove polupravilnog jednakostranog poligona $\mathcal{P}_N$, sa $N = mn, m = k-1$, $n, k \geq 3, n, k \in \mathbb{N}$ stranica, pokažimo da je tada $A + B + C = 0$.Zaista, ako u jednačini $Ax^{mn} + Bx^n + C = 0$, stavimo da je $x^n = \varepsilon$ dobijamo $A\varepsilon^m + B\varepsilon + c = 0$.

Uzmimo da je pravilni poligon $\mathcal{P}_n$ upisan u jediničnu kružnicu, čiji su vrhovi određeni afiksima rješenja jednačine $x^n - 1 = 0$. Kako su ti vrhovi i vrhovi polupravilnog poligona $\mathcal{P}_N$ jedno rješenje jednačine $Ax^N + Bx^n + C = 0$ je $x = 1$, na osnovu toga imamo da je $A + B + C = 0$. $\square$

Napomena 4.1. Ako jednačinu (13) transformišemo na sljedeći način

$$\begin{aligned}
\varepsilon^{m-1} + \varepsilon^{m-2} + \cdots + \varepsilon &= \frac{C}{A} \\
\Leftrightarrow \varepsilon^{m-1} + \varepsilon^{m-2} + \cdots + \varepsilon + 1 &= 1 + \frac{C}{A} \\
\Leftrightarrow \frac{\varepsilon^m - 1}{\varepsilon - 1} &= 1 + \frac{C}{A}, \varepsilon \neq 1 \\
\Leftrightarrow \frac{\varepsilon^m - 1}{\varepsilon - 1} &= \frac{A + C}{A} \\
\Leftrightarrow \varepsilon^m - 1 &= (\varepsilon - 1)(\frac{A + C}{A})
\end{aligned}$$

$\Leftrightarrow \varepsilon^m - (1 + \frac{C}{A})\varepsilon + \frac{C}{A} = 0$ dolazimo do oblika

$$A\varepsilon^m + B\varepsilon + C = 0, \varepsilon \neq 1. \tag{15}$$

Ova jednačina ima $m-1$ prvoodređenih različitih rješenja. U zavisnosti od toga da li je jednačina (15) rješiva zavisi i konstrukcija polupravilnog jednakostranog poligona.

Napomena 4.2. Jednačinu $Ax^{nm}+Bx^n+C=0$ napišimo u obliku $Ax^m+B+\frac{C}{x^n}=0$. Povećavamo li broj stranica "upisanog" pravilnog poligona odnosno, pretpostavimo da $n \rightarrowtail \infty$ tada bi jednačina težila obliku $Ax^m+B=0$. Afiksi tješenja te jednačine su vrhovi pravilnog poligona $\mathcal{P}_m$ upisanog u kružnicu poluprečnika $r=\sqrt[m]{|-\frac{B}{A}|}$.

5 Analitički pristup konstrukciji polupravilnih poligona $\mathcal{P}_{2n}$

Analiziramo klasu trinomnih jednačina koja generiše konstrukciju polupravilnih jednakostranih poligona $\mathcal{P}_N$ sa $N=2\cdot n$ stranica kod kojih vrhovi "upisanog" pravilnog poligona $\mathcal{P}_n, n\geq 3$ leže na jediničnoj kružnici,a poligoni $\mathcal{P}_k$ konstruisani nad svakom stranicom tog pravilnog poligona, kao zajedničkom, su jednakokraki trouglovi.

Teorema 5.1. Rješenja trinomne jednačine $Ax^{2n}+Bx^n+C=0$ za čije koeficijente $A,B,C\in\mathbb{R}$ vrijedi $A+B+C=0$ i A i C su suprotnog predznaka, predstavljaju vrhove jednakostranog polupravilnog poligona $\mathcal{P}_{2n}$ sa $N=2\cdot n$ stranica, kod koga vrhovi "upisanog" pravilnog poligona $\mathcal{P}_n$ leže na jediničnoj kružnici.

Dokaz: Riješimo li jednačinu $Ax^{2n}+Bx^n+C=0$, pod pretpostavkom da vrijede uslovi za koeficijente A,B,C,stavljanjem da je $x^n=y$ nalazimo da je jedno rješenje $y=1$, a drugo rješenje $y=\frac{C}{A}$. Na taj način smo problem određivanja vrhova polupravilnog poligona $\mathcal{P}_{2n}$ sveli na određivanje rješenja jednačina $x^n=1$ i $x^n=\frac{C}{A}$ čiji afiksi predstavljaju vrhove tog polupravilnog poligona.

Označimo sa $w=\cos\frac{2\pi}{n}+\sin\frac{2\pi}{n}$ rješenje binomne jednačine $x^n-1=0$ tada su međusobno različiti korijeni n tog stepena iz jedinice redom $w,w^2,w^3,\cdots,w^{n-1},w^n=1$, a grafički prikaz tih korijena predstavlja vrhove pravilnog poligona $\mathcal{P}_n$ upisanog u jediničnu kružnicu pri čemu tačka koja odgovara broju jedan odgovara jednom vrhu.

Da pokažemo da afiksi rješenja binomne jednačine $x^n=\frac{C}{A}$ predstavljaju vrhove jednakokrakih trouglova konstruisanih nad svakom stranicom pravilnog poligona $\mathcal{P}_n$, dovoljno je pokazati da:

1. svako rješenje ϵ_k $k=1,2,\cdots,n$ jednačine $x^n=\frac{C}{A}$ pripada simetrali stranice pravilnog poligona nad kojom je konstruisan odgovarajući jednakokraki trougao
2. sve stranice su međusobno jednake tj. $|\epsilon_j w_i|=|\epsilon_j w_{j+1}|$ za sve $i=1,2,\cdots n-1$ i $j=1,2,\cdots n$.

Dokažimo prvu osobinu. Neka je s_k simetrala ugla $\angle A_{k-1}OA_k$ gdje je $A_{k-1}A_k$ stranica "upisanog" pravilnog poligona $\mathcal{P}_n$, S njena središnja tačka i $O=(0,0)$ centar jedinične kružnice. Neka je tačka B_k afiks rješenja ϵ_k jednačine $x^n-\frac{C}{A}=0$. Koordinate tačaka A_{k-1} i A_k su redom:

$$A_{k-1}=\Big(\cos\frac{2\pi(k-1)}{n},\sin\frac{2\pi(k-1)}{n}\Big), A_k=\Big(\cos\frac{2k\pi}{n},\sin\frac{2k\pi}{n}\Big),$$

a tačka B_k ima koordinate

$$B_k=r\Big(\cos\frac{\pi(2k+1)}{n},\sin\frac{\pi(2k+1)}{n}\Big)$$

gdje je $r=\sqrt[n]{|\frac{C}{A}|}$ i $k=0,1,2,\cdots n-1$

Elementarnim transforamcijama lako se pokaže da koordinate tačke B_k zadovoljavaju jednačinu simetrale s_k

$$y=\frac{\sin\frac{2\pi(k-1)}{n}+\sin\frac{2k\pi}{n}}{\cos\frac{2\pi(k-1)}{n}+\cos\frac{2k\pi}{n}}\cdot x.$$

Na osnovu toga je pokazano da vrijedi prvi zahtjev. Kako je svaka tačka na simetrali duži jednako udaljena od krajeva vrijedi i drugi zahjev$|\epsilon_j w_i|=|\epsilon_j w_{j+1}|$.

U narednom primjeru je razmotrena veza trinomnih jednačina i konstrukcije polupravilnih jednakostranih poligona $\mathcal{P}_{2n}$ ako je $n=3$.

5.1 Konstrukcija polupravilnih jednakostranih šestouglova

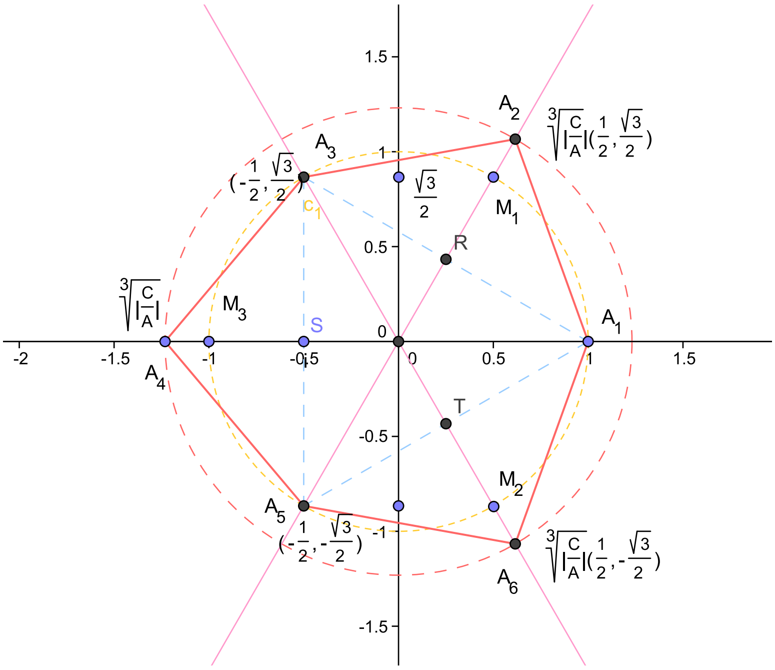
Pokažimo da trinomna jednačina $Ax^6+Bx^3+C=0$, sa realnim koeficijentima za koje je $A+B+C=0$ i $signA=-signC$ generiše jednakostrane polupravilne šestouglove, kojima je odgovarajući pravilni poligon "upisan" u jediničnu kružnicu sa centrom u koordinantnom početku. Zamijenimo li $x^3=y$ polazna jednačina prelazi u jednačinu $Ay^2+By+C=0$. Obzirom na uslov jedno rješenje jednačine je $y=1$, a drugo $y=\frac{C}{A}$. Kako je $x^3=y$ imamo dvije binomne jednačine; $x^3=1$ i $x^3=\frac{C}{A}$.

Iz prve jednačine nalazimo da je $x_0=1, x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}, x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}$,pa su odgovarajući vrhovi pravilnog jednakostraničnog trougla $A_1(1,0), A_2(-\frac{1}{2},\frac{\sqrt{3}}{2})$, $A_3(-\frac{1}{2},-\frac{\sqrt{3}}{2})$.

Iz druge jednačine nalazimo da su rješenja data relacijom

$$x'_k=\sqrt[3]{|\frac{C}{A}|}(\cos\frac{(2k+1)\pi}{3}+i\sin\frac{(2k+1)\pi}{3}), k=0,1,2. \qquad (16)$$

Uvrstimo li vrijednosti za k, nalazimo da je $x'_0=\sqrt[3]{|\frac{C}{A}|}(\frac{1}{2}+\frac{\sqrt{3}}{2}), x'_1=-\sqrt[3]{|\frac{C}{A}|}, x'_2=\sqrt[3]{|\frac{C}{A}|}(\frac{1}{2}-\frac{\sqrt{3}}{2})$ (Slika 1).

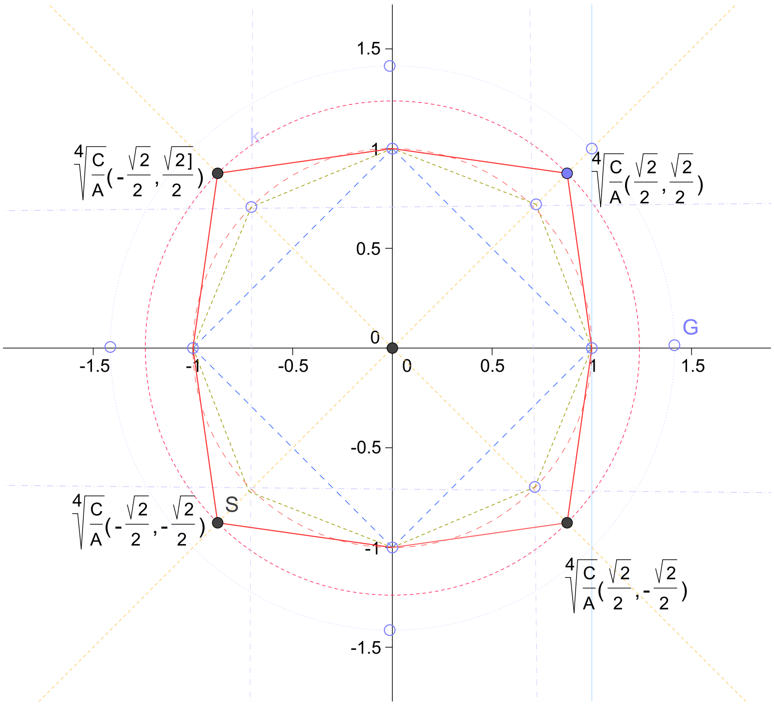


Slika 1: Konstrukcija polupravilnog šestougla

Primijetimo da konveksnost jednakostranog polupravilnog šestougla zavisi od vrijednosti konstante $\lambda = \sqrt[3]{|\frac{C}{A}|}$ kao i egzistencija same konstrukcije.

Razlikujemo sljedeće slučajeve:

1. Ako je $1 < \lambda < 2$ tada je $1 < \frac{C}{A} < 8$, polupravilan šestougao postoj i konveksan je. (Slika 1)
2. Ako je $\frac{1}{2} < \lambda < 1$ tada je Za $\lambda = 1$ je $C = A$ i polupravilan jednakostrani šestougao postaje pravilan
3. Za $\lambda = \frac{1}{2}$ tada je $\frac{C}{A} = \frac{1}{8}$ i vrhovi se poklapaju sa središtima stranica pravilnog poligona, pa jednakostrani polupravilan šestougao ne postoji.
4. Ako je $\frac{1}{8} < \frac{C}{A} < 1$, polupravilan šestougao postoji i konveksan je.
5. Ako je $0 < \lambda < \frac{1}{2}$ tada je $0 < \frac{C}{A} < \frac{1}{8}$,polupravilan šestougao postoji i nekonsveksan je.
6. Ako je $\lambda = 2$ tada jednakostrani polupravilan šestougao ne postoji.
7. Ako je $\lambda > 2$ tada je $\frac{C}{A} > 8$, polupravilan šestougao je nekonveksan ako postoji.



Slika 2: Konstrukcija jednakostranog polupravilnog osmougla

5.2 Konstrukcija polupravilnog jednakostranog osmougla

Ako u trinomnu jednačinu koja generiše $2n$-tougaone jednakostrane polupravilne poligone stavimo da je $n = 4$ dobijamo jednačinu $Ax^8 + Bx^4 + C = 0$ sa realnim koeficijentima za koje vrijedi $A + B + C = 0, singA = -singC$ koja generiše polupravilne jednakostrane osmouglove. Stavimo li da je $x^4 = y$ dobijamo kvadratnu jednačinu čija su rješenja $y = 1$ i $y = \frac{C}{A}$. Na osnovu toga imamo dvije binomne jednačine $x^4 = 1$ i $x^4 = \frac{C}{A}$. Rješenja tih binomnih jednačina određena su relacijama

$$x_k = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}, k = 0, 1, 2, 3$$

i

$$x'_k = \sqrt[4]{|\frac{C}{A}|}\left(\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right), k = 0, 1, 2, 3$$

Iz tih relacija nalazio da su vrhovi polupravilnog jednakostranog osmougla $A_1(1, 0), A_2(0, 1), A_3(-1, 0), A_4(0, -1), A_5 = \lambda(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), A_6 = \lambda(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), A_7 = -\lambda(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), A_8 = \lambda(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ gdje je $\lambda = \sqrt[4]{\frac{C}{A}}$(Slika 2.) Očito za različite vrijednosti parametra $\lambda = \sqrt[4]{\frac{C}{A}}$ imamo različite slučajeve, slično prethodnom razmatranju.

Literatura

[1] A. August,*Teorija geometrijskih konstrukcija*, Lenjingrad,(1940).

[2] A. Ilić, *Uvod u analitičku teoriju brojeva*, Novi Sad, 1996.

[3] N. Stojanović, *Inscribed circle of general semi-regular polygon and some of its features*, International Journal of Geometry, Vol.2., (2013), No. 1,5 - 22.

[4] N. Stojanović, *Some metric properties of general semi-regular polygons*, Global Journal of Advanced Research on Classical and Modern Geometries, Vol. 1, Issue 2, (2012) pp. 39-56.

[5] N. Stojanović, V. Govedarica, *Jedan pristup analizi konveksnosti i računanju površine jednakostranih polupravillnih poligona*, II Matematička konferencija Republike Srpske, Zbornik radova, Trebinje, (2013), str.87-105.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

A Construction Weighted Projective Plane of Order 11 and (2,11-1) - quasigroup

Alija Mandak
University of Pristina-Kosovska Mitrovica
Faculty of Teacher Training Prizren-Leposavic, Serbia
alija.mandak@pr.ac.rs

Original scientific paper

Abstract

We introduce a notion of weighted projective planes which is a generalization of usual projective planes. We prove that a Frobenius group G of order 24 operates on a projective plane P of order 11 as a colineation group. Using this operation the plane P may be constructed. A weighted projective plane P of order 11 is equivalent to a totally simmetric (2,11-1) - quasigroup.

Key words: Projective plane, quasigroup, group, colineation, orbit.

1 Introduction

An incidence structure is a triple $D = (V, B, I)$, where V and B are disjoint sets and $I \subseteq V \times B$. The elements of V are called *points*, and the elements of B are called *blocks*. If A is a point of V, the set of all blocks incident with A is denoted by (A). Thus

$$(A) = \{b : b \in B,\ AIb\}.$$

Moreover, for $A_1, A_2, \dots A_n$, the set of all the blocks incident with all the points A is denoted by $(A_1, A_2, \dots A_n)$. Thus

$$(A_1, A_2, \dots A_n) = \{b : b \in B,\ A_i\, I\, b \text{ for all } i \in N_n\},$$

where N is the set of all positive integers and $N_n = \{1, 2, \dots n\}$. Dually, for $b, b_1, b_2, \dots, b_n \in B$,

$$(b) = \{A : A \in V,\ AIb\},$$
$$(b_1, b_2, \dots b_n) = \{A : A \in V,\ AIb \text{ for all } i \in N_n\}.$$

We consider only the incidence structures where distinct blocks have distinct sets of points. We identify each block b with the set (b) and identify the incidence relation with the membership relation $\in$.

1.1 Some definition and results

Definition 1.1. A incidence structure $P = (V, B, I)$ is called *projective plane* if and only if it satisfies the following axioms:

1. $(P.1)$ Any two distinct points are joined by exactly one line.
2. $(P.2)$ Any distinct lines intersect in a unique point.
3. $(P.3)$ There exists a *quadrangle*, i.e 4 points no three of which are on a common line.

The following theorem is proved in [1].

Theorem 1.1. *Let $P = (V, B, I)$ be a finite projective planes. Then there exists a natural number n, called the* order *of P, satisfying:*

$$a)\ |(A)| = |(g)| = n + 1, \quad \text{for all } A \in V \quad \text{and } g \in B;$$
$$b)\ |V| = |B| = n^2 + n + 1.$$

The finite projective plane of order n will be denoted by $S(2, n+1, n^2+n+1)$. The following definition generalizes the notion of finite projective planes of order n.

Definition 1.2. A finite incidence structure $P = (V, B, I)$ is called *weighted projective plane* with parameters $n^2 + n + 1, n + 1, 1 \in N$, if for any $b \in B$ there is a mapping $f_b : (b) \to N$, if and only if it satisfies the following axioms:

$(WD.1)$ $|V| = n^2 + n + 1;$
$(WD.2)$ $|(A, B)| = 1,$ for any two distinct points $A, B \in V$;
$(WD.3)$ $k_b = n + 1,$ for any block $b \in B$, where :

$a)$ the image $\mathrm{f_b(A)}$ is denoted by $\mathrm{t_{Ab}}$, and is called the *weight* of the point A in the block b,

$b)$ for $\mathrm{A \in V}$, its *weight* is $\mathrm{t_A} = \sum_{\mathrm{A \in b_i}} \mathrm{t_{Abi}}$, and

$c)$ for $\mathrm{b \in B}$, the number $\mathrm{k_b} = \sum_{\mathrm{Ai \in b}} \mathrm{t_{Aib}}$ is called the *size* of the block b.

Definition 1.3. A weighted projective plane $S' = (V', B', \in)$ is an *extension* of a weighted projective plane $S = (V, B, \in)$, if $V \subseteq V'$ and for each $b \in B$ there is $b' \in B'$ such that $(b) \subseteq (b')$, and for each $A \in (b)$, $t_{Ab'} = t_{Ab}$.

Definition 1.4. An extension $(V', B', \in)$ of a weighted projective plane with parameters $n^2 + n + 1, n + 1, 1$ defined by

$a) V' = V;$

$b) B^{'} = B \cup B^{''}$ where $B^{''} = \{\{A^{n+1}\} : A \in V\}$, and

$c)$ for each $\mathrm{A} \in \mathrm{V},\ \mathrm{t_A} = \mathrm{r} + \mathrm{n} + 1,$ where r is the number of block in B containing A, is called a *complete weighted projective plane* with parameters $n^2 + n + 1, n + 1, 1,$ and is denoted by $S^{'}(2, n + 1, n^2 + n + 1)$.

Next we compare complete weighted projective plane $S^{'}(2, n + 1, n^2 + n + 1)$ with the notion of totally symmetric (2,n-1)– quasigroup given below.

Definition 1.5. Let Q be nonempty set, n and m positive integers, and

$$f : (x_1, x_2, \ldots x_n) \to f(x_1, x_2, \ldots x_n)$$

a mapping from Q^n into Q^m. Then we say that $Q(f)$ is an (n, m)– groupoid.

A (n, m) – groupoid $Q(f)$ is said to be a (n, m)– quasigroup if and only if the following statement is satisfied:
(A). For each "vector" $(a_1, a_2, \ldots a_n) \in Q^n$ and each injection φ from $N_n = \{1, 2, \ldots, n\}$ into N_{n+m} there exists unique "vector" $(b_1, b_2, \ldots, b_{n+m}) \in Q^{n+m}$ such that $b_{\varphi(1)} = a_1, \ldots, b_{\varphi(n)} = a_n$ and

$$f(b_1, b_2, \ldots, b_n) = (b_{n+1}, b_{n+2}, \ldots, b_{n+m})$$

In the paper [3] an (n, m)– quasigroup is interpreted as a (n, m)– quasigroup relation.

Definition 1.6. A $(n + m)$- ary relation $\rho \subseteq Q^{n+m}$ is called (n, m)- quasigroup relation if and only if the following statement is satisfied:
(A). For each "vector" $(a_1, a_2, \ldots, a_n) \in Q^n$ and each injection φ from $N_n = \{1, 2, \ldots, n\}$ into N_{n+m} there exists unique "vector" $(b_1, b_2, \ldots, b_{n+m}) \in Q^{n+m}$ such that $b_{\varphi(1)} = a_1, \ldots, b_{\varphi(n)} = a_n$ and

$$(b_1, b_2, \ldots, b_{n+m}) \in \rho.$$

The following theorem is proved in [3].

Theorem 1.2. *A (n, m)– groupoid (Q, f) is a (n, m) – quasigroup if and only if the $(n + m)$ – ary relation $\rho \subseteq Q^{n+m}$ defined by*

$$(x_1, x_2, \ldots x_{n+1}) \in \rho \Leftrightarrow f(x_1, x_2, \ldots x_{n+1}) = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})$$

is an (n, m)– quasigroup relation.

Definition 1.7. A (n, m)– quasigroup is called *totally symmetric*, if and only if

$$f(x_1, x_2, \ldots x_n) = (x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \Leftrightarrow f(y_1, y_2, \ldots y_n) = (y_{n+1}, y_{n+2}, \ldots, y_{n+m})$$

for any $(x_1, x_2, \ldots, x_{n+m}) \in Q^{n+m}$ and any permutation $(y_1, y_2, \ldots, y_{n+m})$ of $(x_1, x_2, \ldots, x_m)$. The $(n+m)$– ary relation $\rho \subseteq Q^{n+m}$ in this case is called *totally symmetric.*

The following theorem is proved in [7].

Theorem 1.3. *Every complete weighted projective plane* $S^{'}(2, n+1, n^2+n+1)$ *defines a totally symmetric* $(2, n-1)$*– quasigroup relation* $\rho \subseteq V^{n+1}$*, where*

$$(A_1, A_2, \ldots, A_{n+1}) \in \rho \Leftrightarrow \{A_1, A_2, \ldots, A_{n+1}\} \in B.$$

Conversely, any totally symmetric $(2, n-1)$– quasigroup relation $\rho \subseteq V^{n+1}$ satisfying $(A, A, \ldots, A) = (A^{n+1}) \in \rho$ for any $A \in V$, defines a complete weighted projective plane $S^{'}(2, n+1, n^2+n+1)$, where

$$\{A_1, A_2, \ldots, A_{n+1}\} \in B \Leftrightarrow (A_1, A_2, \ldots, A_{n+1}) \in \rho.$$

2 A construction weighted projective plane of order 11

Theorem 2.1. *A Frobenius group G of order 24 acts on a projective plane P of order 11 as a colineation group. Using this act the plane P may be constructed.*

Proof. Let

$$G = \langle \rho, \alpha / \rho^{12} = \alpha^2 = 1, \ \rho^{\alpha} = \rho^{-1} \rangle$$

be a Frobenius group of order 24 which acts on a projective plane P of order 11 as a colineation group. The plane P has $11^2 + 11 + 1 = 133$ points and same lines. From $133 = 12 \cdot 11 + 1$ and colineation $\langle \rho \rangle$ acts semiregular on a nonfixed points follows that $\langle \rho \rangle$ has 11 orbits of length 12 and one orbit of length 1. We may set that

$$\begin{aligned}\rho =&(\infty)(1_0, 1_1, 1_2, \ldots, 1_{11})(2_0, 2_1, 2_2, \ldots, 2_{11})(3_0, 3_1, 3_2, \ldots, 3_{11}) \ldots \\ &(11_0, 11_1, 11_2, \ldots, 11_{11})\end{aligned}$$

where $1_0, 1_1, 1_2, \ldots, 1_{11}, 2_0, 2_1, 2_2, \ldots, 2_{11}, 3_0, 3_1, 3_2, \ldots, 3_{11}, \ldots, 11_0, 11_1, 11_2, \ldots, 11_{11}$ are all points of plane P.

From theorem of orbite follows that $\langle \rho \rangle$ has same orbit structure of lines. We may set that

$$\begin{aligned}\rho =&(l_\infty)(l_1, l_1\rho, l_1\rho^2, \ldots, l_1\rho^{11})(l_2, l_2\rho, l_2\rho^2, \ldots, l_2\rho^{11}) \\ &(l_3, l_3\rho, l_3\rho^2, \ldots, l_3\rho^{11}) \ldots (l_{11}, l_{11}\rho, l_{11}\rho^2, \ldots, l_{11}\rho^{11})\end{aligned}$$

Where

$$\begin{aligned}&l_\infty, l_1, l_1\rho, l_1\rho^2, \ldots, l_1\rho^{11}, l_2, l_2\rho, l_2\rho^2, \ldots, l_2\rho^{11}, l_3, l_3\rho, l_3\rho^2, \ldots, l_3\rho^{11}, \ldots, \\ &l_{11}, l_{11}\rho, l_{11}\rho^2, \ldots, l_{11}\rho^{11}\end{aligned}$$

are all lines of plane P.

Let l_∞ be unique line fixed by $\langle \rho \rangle$. We may set that

$$l_\infty = \{1_0, 1_1, 1_2, \ldots, 1_{11}\}.$$

Let l_1 is a line through ∞. It is easy to see that l_1 occurs at one point from each orbits of points. Without a loss of generality, we may set that

$$l_1 = \{\infty, 1_0, 2_0, \dots, 11_0\}.$$

Other 11 lines of orbit of lines l_1 obtained by acting of $\rho, \rho^2, \rho^3, \dots, \rho^{11}$ on a line l_1. The lines l_1 and l_∞ through 1_0. Other 10 lines $l_2, l_3, \dots, l_{11}$ through 1_0 line in 10 remaining different $\langle\rho\rangle$– orbits of lines in P. If constracted these lines then the remaining lines of planes P are obtained by acting of $\rho, \rho^2, \rho^3, \dots, \rho^{11}$ on lines $l_2, l_3, \dots, l_{11}$. From statement

$$\left| l_i \cap l_1\rho^k \right| = 1, \ \ i = 2, 3, \dots, 11, \ \ k = 0, 1, 2, 3, \dots, 11$$

follows

$$l_i = \left\{1_0, 1'_1, 2'_2, 3'_3, \dots, 11'_{11}\right\}$$

where $1', 2', 3', 4', 5', 6', 7', 8', 9', 10', 11'$ are (unneccessary different) numbers from the set $\{2, 3, \dots, 11\}$.
We consider acting the involution α on a set points and set lines of plane P. The order of involution α is even follows involution α is elation. From $\rho^\alpha = \rho^{-1}$ follows that the point 1_0 is a center and the line l_1 is axis of involution α. Hence, involution α fixes 12 lines $l_\infty, l_1, l_2, l_3, \dots, l_{11}$ and 12 points $\infty, 1_0, 2_0, 3_0, \dots, 11_0$. From $133 = 2 \cdot 60 + 13$ follows α has 13 orbits of length 1 (13 fixed points) and 60 orbits of length 2. If we write α in a short way (writing only incides $0, 1, 2, 3, \dots, 11$) we may set that

$$\alpha = (0)(1)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$$

where (2,11) denoted that $2\alpha = 11$ from the same orbit of points, (3,10) denoted $3\alpha = 10$ from the same orbit of points, (4,9) denoted that $4\alpha = 9$ from the same orbit of points, (5,8) denoted that $5\alpha = 8$ from the same orbit of points, (6,7) denoted that $6\alpha = 7$ from the same orbit of points. From statement

$$l_i\alpha = l_i, \ \ i = 2, 3, \dots, 11$$

follows that $l_i, \ \ i = 2, 3, \dots, 11$ are of type

$$l_i = \{1_0, i_1, a_2, a_{11}, b_3, b_{10}, c_4, c_9, d_5, d_8, e_6, e_7\}$$

where a, b, c, d, e are pairwise different numbers from the set $\{2, 3, \dots, 11\}$. We may set that

$$l_2 = \{1_0, 2_1, 2_2, 2_{11}, 3_3, 3_{10}, 4_4, 4_9, 5_5, 5_8, 6_6, 6_7\}.$$

Now we constructed lines $l_i, \ i = 3, 4, \dots, 11$ which are of type

$$l_i = \{1_0, i_1, a_2, a_{11}, b_3, b_{10}, c_4, c_9, d_5, d_8, e_6, e_7\}$$

From statement

$$\left|l_i \cap l_2\rho^k\right| = 1,\ \ i = 3,4,\ldots,11,\ \ k = 0,1,2,3,\ldots,11$$

follows that only three from numbers i, a, b, c, d, e are from the set $\{2,3,4,5,6\}$ and other three are from the set $\{7,8,9,10,11\}$. From statement

$$\left|l_i\rho^s \cap l_j\rho^k\right| = 1,\ \ i \neq j,\ \ i,j = 3,4,\ldots,11,\ \ k,s = 0,1,2,\ldots,11$$

follows that the lines l_i and l_j, $i \neq j$, have exactly three common pairs of numbers i, a, b, c, d, e. Using these statements for the lines l_i, $i = 3,4,\ldots,11$ we obtain following unique solution for the lines:

$$\begin{aligned}
l_3 &= \{1_0, 3_1, 3_2, 3_{11}, 4_3, 4_{10}, 7_4, 7_9, 8_5, 8_8, 9_6, 9_7\}\\
l_4 &= \{1_0, 4_1, 4_2, 4_{11}, 5_3, 5_{10}, 8_4, 8_9, 9_5, 9_8, 10_6, 10_7\}\\
l_5 &= \{1_0, 5_1, 5_2, 5_{11}, 6_3, 6_{10}, 9_4, 9_9, 10_5, 10_8, 11_6, 11_7\}\\
l_6 &= \{1_0, 6_1, 3+6_2, 6_{11}, 7_3, 7_{10}, 10_4, 10_9, 11_5, 11_8, 2_6, 2_7\}\\
l_7 &= \{1_0, 7_1, 7_2, 7_{11}, 8_3, 8_{10}, 11_4, 11_9, 2_5, 2_8, 3_6, 3_7\}\\
l_8 &= \{1_0, 8_1, 8_2, 8_{11}, 9_3, 9_{10}, 2_4, 2_9, 3_5, 3_8, 4_6, 4_7\}\\
l_9 &= \{1_0, 9_1, 9_2, 9_{11}, 10_3, 10_{10}, 3_4, 3_9, 4_5, 4_8, 5_6, 5_7\}\\
l_{10} &= \{1_0, 10_1, 10_2, 10_{11}, 11_3, 11_{10}, 5_4, 5_9, 6_5, 6_8, 7_6, 7_7\}\\
l_{11} &= \{1_0, 11_1, 11_2, 11_{11}, 2_3, 2_{10}, 6_4, 6_9, 7_5, 7_8, 8_6, 8_7\}
\end{aligned}$$

The Theorem is proved. ⊣

Let $P = (V, B, \in)$ be projective plane of order 11 constructed in the theorem. The weighted projective plane $P' = (V', B', \in)$, where $V = V'$, $B' = B \cup B''$ where $B'' = \left\{\{A^{12}\} : A \in V\right\}$ is a complete weighted projective plane of order 11. The relation $\tau \subseteq V^{11+1}$ defined by

$$(A_1, A_2, A_3, \ldots, A_{11}, A_{11+1}) \in \tau \Leftrightarrow \{A_1, A_2, A_3, \ldots, A_{11}, A_{11+1}\} \in B \text{ or}$$
$$A_1 = A_2 = A_3 = \ldots = A_{11} = A_{11+1},$$

is a totally symmetric $(2, 11-1)$– quasigroup relation satisfuing the condition $(A, A, A, \ldots, A, A) = (A^{11+1}) \in \tau$ for all $A \in V$. The number of point is $|V| = 11^2 + 11 + 1 = 133$, the number of blocks is $\left|B'\right| = 11^2 + 11 + 1 + 133 = 266$ and $t_A = 12 + 12 = 24$.

3 Conclusion

This paper presents the results obtained by acting a colineation group on a set points and set lines of plane P which exists. Similar acting of a colineation group on a set points and set lines of plane P whose question of existence is open, can be studied.

References

[1] Beth,T., Jugnickel, D., Lenz, H.: *Design theory*, Manheim, Wien, Zurich, 1985.

[2] Čupona, $\dot{G}$., Stojaković, Z., Ušan, J.: *On finite multiquasigroups*, Publ. Inst. Math. (Beograd) 20(43) 1981, 53-59.

[3] Čupona, $\dot{G}$., Stojaković, Z., Ušan, J.: *Multiquasigroups and some related structures,* Prilozi MANU I/1, Skopje, 1980.

[4] Dimovski, D., Mandak, A.: *Incidence structures with n-metrics*, Zb. Rad. Fil. Fak. (Niš) 6(1992), 151-155.

[5] Dimovski, D., Mandak, A.: *Weighted block designs and Steiner systems,* Novi Sad J. Math. Vol.29, No.2 (1999), 163-169.

[6] Lenz, H.: *Vorlesungen uber projective geometrije*, Akademische Verlagsgesellschaft, Leipzing, 1965.

[7] Mandak, A.: On weighted block designs, Proc. Math. Conf. (Priština, 1994) 21-25.

[8] Mandak, A.: Multiquasigroups and weighted projective planes, Kragujevac J. Math.30 (2007) 211-219.

[9] Ušan, J.: <Nn,E>-seti s $(n+1)$- rastojaniem, Rew. Res. Fac. Sci. Univ. Novi Sad Ser. Math, 17(2) (1989), 65-87.

SEKCIJA ZA ANALIZU, VJEROVATNOĆU I STATISTIKU

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Konstrukcija rješenja graničnog zadatka sa dva konstantna kašnjenja i asimptotika sopstvenih vrijednosti

Milenko T. Pikula
Filozoski fakultet Pale
mpikula@paleol.net

Nataša M. Pavlović
Elektrotehnički fakultet u Istočnom Sarajevu
natasa.pavlovic@etf.unssa.rs.ba

Ljubica Ž. Diković
Visoka poslovno-tehnička škola, Užice
dikoviclj@gmail.com

Originalni nauči rad

Apstrakt

U radu se metodom koraka konstruiše rješenje graničnog zadatka

$$-y''(x) + q_1(x)y(x - \tau_1) + q_2(x)y(x - \tau_2) = \lambda y(x)$$

$$y(x - \tau_1) \equiv 0, x \in (0, \tau_1],$$

$$y(\pi) = 0,\ q_1(x),\ q_2(x) \in L_2[0, \pi],\ \tau_1, \tau_2 \in (0, \pi)$$

na segmentu $[0, \pi]$. Potom konstruišemo asimptotiku sopstvenih vrijednosti.

1 Uvod

Posmatrajmo diferencijalnu jednačinu

$$-y''(x) + q_1(x)y(x - \tau_1) + q_2(x)y(x - \tau_2) = \lambda y(x), \quad \tau_1, \tau_2 \in (0, \pi) \tag{1}$$

Rješavaćemo jednačinu za $x \in [0, \pi]$. Obzirom da u ovoj diferencijalnoj jednačini funkcija y uzima različite vrijednost argumenata, ona predstavlja jednu jednačinu sa pomjerenim argumentom. Jedan oblik jednačina sa pomjerenim argumentom je jednačina sa kašnjenjem i jednačina (1) je tog tipa.
Veličine τ_1, τ_2 su kašnjenja, a kako ne zavise od x ova jednačina ima konstantna kašnjenja. Funkcije $q_1(x)$ i $q_2(x)$ nazivamo potencijalima i ovdje smatramo poznatim. Posmatraćemo slučaj kada su $q_1(x)$, $q_2(x) \in L_2[0, \pi]$.
Jednačini (1) pridružićemo početni uslov

$$y(x - \tau_1) \equiv 0,\ x \in [0, \tau_1] \quad (2)$$

i granični uslov

$$y(\pi) = 0 \quad (3)$$

2 Konstrukcija rješenja

Ako u jednačini (1) stavimo $\lambda = z^2$, dobijamo

$$-y''(x) + q_1(x)y(x-\tau_1) + q_2(x)y(x-\tau_2) = z^2 y(x) \qquad (1')$$

Rješavanjem jednačine $(1')$ sa graničnim uslovom $y(0) = 0$ metodom varijacije konstanti dobijamo integralnu jednačinu

$$\begin{aligned} y(x,z) &= \sin zx + \frac{1}{z}\int_0^x q_1(t_1)\sin z(x-t_1)y(t_1-\tau_1,z)dt_1 + \\ &+ \frac{1}{z}\int_0^x q_2(t_1)\sin z(x-t_1)y(t_1-\tau_2,z)dt_1 \end{aligned} \qquad (5)$$

Metodom koraka odredićemo rješenje jednačine (5) koje zadovoljava početni uslov (2). Posmatrali smo slučaj $\tau_2 < \tau_1$, a kako su τ_1 , $\tau_2 \in (0,\pi)$ postoje $k_0, l_0 \in N$ takvi da je

$$0 < \tau_2 < 2\tau_2 < \ldots < k_0\tau_2 \le \tau_1 < (k_0+1)\tau_2 < \ldots < 2k_0\tau_2 \le 2\tau_1 < (2k_0+1)\tau_2 < \ldots$$

$$\ldots < l_0 k_0 \tau_2 \le l_0\tau_1 < \pi < (l_0 k_0 + 1)\tau_2.$$

Specijalno, ako stavimo da je $l_0 = k_0 = 2,\ 2\tau_2 = \tau_1$ dobijamo:

1. $x \in (0, \tau_2]$, a kako je $\tau_2 < \tau_1$ i $y(x-\tau_1) \equiv 0$ slijedi $y(x-\tau_2) \equiv 0$

   pa dobijamo $\quad y(x,z) = \sin zx$

2. $x \in (\tau_2, \tau_1]$, $2\tau_2 = \tau_1$ slijedi

$$y(x,z) = \sin zx + \frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1)y(t_1-\tau_2,z)dt_1$$

   kada primjenimo prethodni korak dobijemo

$$y(x,z) = \sin zx + \frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1)\sin z(t_1-\tau_2)dt_1$$

Uvedimo sledeće funkcije

$$b_1^{(i)}(x,z) = \int_{\tau_i}^x q_i(t_1) sinz(x-t_1)\sin z(t_1-\tau_i)dt_1$$

$$b_k^{(i)}(x,z)=\int\limits_{k\tau_i}^{x} q_i(t_1) sinz(x-t_1) b_{k-1}^{(i)}(t_1-\tau_i,z)dt_1;\ i=1,k=2;\ i=2,k=2,3,4.$$

$$b_2^{(1,2)}(x,z)=\int\limits_{\tau_1+\tau_2}^{x} q_1(t_1) sinz(x-t_1) \int\limits_{\tau_2}^{t_1-\tau_1} q_2(t_2) sinz(t_1-\tau_1-t_2)\sin z(t_2-\tau_2)dt_2dt_1$$

$$b_2^{(2,1)}(x,z)=\int\limits_{\tau_1+\tau_2}^{x} q_2(t_1) sinz(x-t_1) \int\limits_{\tau_1}^{t_1-\tau_2} q_1(t_2) sinz(t_1-\tau_2-t_2)\sin z(t_2-\tau_1)dt_2dt_1$$

$$b_3^{(1,2,2)}(x,z)=\int\limits_{2\tau_1}^{x} q_1(t_1) sinz(x-t_1) b_2^{(2)}(t_1-\tau_1,z)dt_1$$

$$b_3^{(2,2,1)}(x,z)=\int\limits_{4\tau_2}^{x} q_2(t_1) sinz(x-t_1) b_2^{(2,1)}(t_1-\tau_2,z)dt_1$$

$$b_3^{(2,1,2)}(x,z)=\int\limits_{4\tau_2}^{x} q_2(t_1) sinz(x-t_1) b_2^{(1,2)}(t_1-\tau_2,z)dt_1$$

Na razmaku $(\tau_1, \tau_1+\tau_2]$ rješenje $y(x,z)$ jednačine (5) sa uslovom (2) ima oblik

$$y(x,z)=\sin xz+\frac{1}{z}\sum_{i=1}^{2} b_1^{(i)}(x,z)+\frac{1}{z^2}b_2^{(2)}(x,z).$$

Dalje, za $x\in(\tau_1+\tau_2, 2\tau_1]=(3\tau_2, 4\tau_2]$ imamo

$$y(x,z)=\sin xz+\frac{1}{z}\sum_{i=1}^{2} b_1^{(i)}(x,z)+\sum_{p=2}^{3}\frac{1}{z^p}b_p^{(2)}(x,z)+\frac{1}{z^2}[b_2^{(1,2)}(x,z)+b_2^{(2,1)}(x,z)].$$

Konačno, na razmaku $(2\tau_1, \pi]=(4\tau_2, \pi]$ funkcija $y(x,z)$ prima formu:

$$y(x,z)=\sin xz+\frac{1}{z}\sum_{i=1}^{2} b_1^{(i)}(x,z)+\sum_{p=3}^{4}\frac{1}{z^p}b_p^{(2)}(x,z)+\sum_{i=1}^{2}\frac{1}{z^2}b_2^{(i)}(x,z)+$$
$$+\frac{1}{z^2}[b_3^{(2,1)}(x,z)+b_3^{(1,2)}(x,z)]+\frac{1}{z^3}\sum_{\pi\in S_3(2,1)} b_3^{(\pi)}(x,z). \qquad (6)$$

3 Karakteristična funkcija

Granični uslov $y(\pi, z) = 0$ definiše karakterističnu funkciju operatora L generisanog sa (1,2,3), koju označavamo sa F. Radi kraćeg zapisivanja koristićemo oznake

$$b_k^{(i)}(z) = b_k^{(i)}(\pi, z),\ b_3^{(i,j)}(z) = b_3^{(i,j)}(\pi, z).$$

Tada važi

$$F(z) = \sin \pi z + \sum_{p=1}^{2} \frac{1}{z^p} \sum_{i=1}^{2} b_p^{(i)}(z) + \sum_{p=3}^{4} \frac{1}{z^p} b_p^{(2)}(z) + \frac{1}{z^2}[b_3^{(2,1)}(z) + b_3^{(1,2)}(z)] +$$

$$+ \frac{1}{z^3} \sum_{\pi \in S_3(2,1)} b_3^{(\pi)}(z). \tag{7}$$

Funkcija F je cijela funkcija eksponencijalnog tipa. Poznato je (vidi npr.u [4]) da se nule z_n funkcije F asimptotski ponašaju kao cijeli brojevi.
Koristimo elementarni identitet

$$\sin z(\pi - t_1) \sin z(t_1 - \tau_i - t_2) \sin z(t_2 - \tau_j) = \frac{1}{4} \sin z(\pi - 2t_2 + \tau_j - \tau_i) +$$

$$+ \frac{1}{4} \left[\sin z(\pi - 2t_1 + 2t_2 + \tau_j - \tau_i) - \sin z(\pi - \tau_i - \tau_j) - \sin z(\pi - 2t_1 + \tau_j + \tau_i) \right].$$

Definišimo i funkcije

$$\beta_1^{(i,j)}(z) = \int_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int_{\tau_j}^{t_1-\tau_i} q_j(t_2) dt_2] \sin z(\pi - 2t_1 + \tau_j + \tau_i) dt_1$$

$$\beta_2^{(i,j)}(z) = \int_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin z(\pi - 2t_2 + \tau_j - \tau_i) dt_2] dt_1$$

$$\beta_3^{(i,j)}(z) = \int_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin z(\pi - 2t_1 + 2t_2 - \tau_j + \tau_i) dt_2] dt_1,\ i \neq j$$

$$\beta_k(z) = \beta_k^{(i,i)}(z),\ i = 1, 2,\ k = 1, 2, 3. \tag{8}$$

$$a^i(z) = \int_{\tau_i}^{\pi} q_i(t_1) \sin z(\pi - 2t_1 + \tau_i) dt_1$$

Uvedimo i sledeće veličine

$$J_1^{(i)} = \int\limits_{\tau_i}^{\pi} q_i(t_1)dt_1,\ J_2^{(i)} = \int\limits_{2\tau_i}^{\pi} q_i(t_1) \int\limits_{\tau_i}^{t_1-\tau_i} q_i(t_2)dt_2dt_1,\ i = 1,2$$

$$J_2^{(i,j)} = \int\limits_{\tau_1+\tau_2}^{\pi} q_i(t_1) \int\limits_{\tau_j}^{t_1-\tau_i} q_j(t_2)dt_2dt_1,\ i \neq j \tag{9}$$

Funkcija (7) pomoću (8) i (9) dobija oblik

$$F(z) = \sin \pi z + \frac{1}{2z}\sum_{i=1}^{2}[a^i(z) - J_1^{(i)} \cos z(\pi - \tau_i)]+$$

$$+\frac{1}{4z^2}\left\{\sum_{i=1}^{2}[\sum_{p=1}^{3} \beta_p(z) - J_2^{(i)} \sin z(\pi - 2\tau_i)] + \sum_{p=1}^{3}[\beta_p^{(1,2)}(z) + \beta_p^{(2,1)}(z)]\right\} -$$

$$-\frac{1}{4z^2}(J_2^{(1,2)}+J_2^{(2,1)}) \sin z(\pi - \tau_1 - \tau_2)+\frac{1}{z^3}\sum_{\pi\in S_3(2,1)} b_3^{(\pi)}(z)+\sum_{p=3}^{4}\frac{1}{z^p}a_p^{(2)}(z) \tag{10}$$

4 Asimptotika nula funkcije F

U ovom radu tražićemo asimptotiku nula z_n funkcije F u obliku

$$z_n = n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right), (n \to \infty) \tag{11}$$

Primjenićemo postupak analogan postupku koji se može vidjeti u npr. [4]. Ukoliko se ograničimo da važi $q_i(x) \in L_2[0, \pi]$ $i = 1, 2$, u asimptotskim procjenama koristićemo klasičnu činjenicu da se Furijeovi koeficijenti takvih funkcija ponašaju po zakonu $O\left(\frac{1}{n^s}\right)$, $s > \frac{1}{2}$ ili $o\left(\frac{1}{\sqrt{n}}\right)$.

Koristeći (11) dobijamo

$$\sin \pi z_n = (-1)^n \left[\frac{\pi C_1(n)}{n} + \frac{\pi C_2(n)}{n^2} + o\left(\frac{C_2(n)}{n^2}\right)\right] \tag{12}$$

Zbog $\frac{1}{z_n} = \frac{1}{n} + O\left(\frac{1}{n^3}\right)$, dovoljno je asimptotiku članova uz $\frac{1}{z_n}$ naći s tačnošću do koeficijenata uz $\frac{1}{n}$. Otuda dobijamo

$$\cos z_n(\pi-\tau_i)=(-1)^n\left[\cos n\tau_i+\frac{(\pi-\tau_i)C_1(n)}{n}\sin n\tau_i+O\left(\frac{C_2(n)}{n^2}\right)\right]$$

$$\sin z_n(\pi-2\tau_i)=(-1)^{n+1}\sin 2n\tau_i+O\left(\frac{C_1(n)}{n}\right) \tag{13}$$

$$\sin z_n(\pi-\tau_1-\tau_2)=(-1)^{n+1}\sin n(\tau_1+\tau_2)+O\left(\frac{C_1(n)}{n}\right)$$

Definišimo sledeće brojne nizove.

$$a_n^{(i)}=\int_{\tau_i}^{\pi} q_i(t_1)\cos n(2t_1-\tau_i)dt_1 \quad i=1,2,\ n=1,2,...$$

$$b_n^{(i)}=\int_{\tau_i}^{\pi} q_i(t_1)\sin n(2t_1-\tau_i)dt_1$$

$$\hat{b}_n^{(i)}=\int_{\tau_i}^{\pi} t_1 q_i(t_1)\sin n(2t_1-\tau_i)dt_1$$

Dalje, važi

$$a^{(i)}(z_n)=\int_{\tau_i}^{\pi} q_i(t_1)\cos\left(n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right)\right)(\pi-2t_1+\tau_i)dt_1=$$

$$=(-1)^n\left\{a_n^{(i)}+\left[\frac{C_1(n)(\pi+\tau_i)}{n}+\frac{C_2(n)(\pi+\tau_i)}{n^2}\right]b_n^{(i)}\right\}+$$

$$+(-1)^{n+1}\left[\frac{2C_1(n)}{n}+\frac{2C_2(n)}{n^2}\right]\hat{b}_n^{(i)}+o\left(\frac{b_n^{(i)}C_2(n)}{n^2}\right) \tag{14}$$

Uvedimo i nizove

$$b_{1,n}^{(i,j)}=\int_{\tau_1+\tau_2}^{\pi}[q_i(t_1)\int_{\tau_j}^{t_1-\tau_i} q_j(t_2)dt_2]\sin n(2t_1-\tau_1-\tau_2)dt_1$$

$$b_{2,n}^{(i,j)} = \int\limits_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int\limits_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin n(2t_2 - \tau_j + \tau_i) dt_2] dt_1$$

$$b_{3,n}^{(i,j)} = \int\limits_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int\limits_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin n(2t_1 - 2t_2 + \tau_j - \tau_i) dt_2] dt_1, \; n \in N, \qquad (15)$$

Tada važe relacije

$$\beta_p^{(i,j)} = (-1)^{n+1} b_{p,n}^{(i,j)} + O\left(\frac{C_1(n) b_{1,n}^{(i,j)}}{n}\right), \; p = 1, 2, 3 \qquad (16)$$

Koristeći jednakosti (11), (12), (13), (14) i (15), relacija (10) postaje

$$F(z_n) = \frac{(-1)^n}{n} \left[\pi C_1(n) + \frac{1}{2} \sum_{i=1}^{2} (a_n^{(i)} - J_1^{(i)} \cos n\tau_i)\right] +$$

$$+ \frac{(-1)^n}{n^2} \left\{\pi C_2(n) + \sum_{i=1}^{2} [\frac{C_1(n)(\pi + \tau_i)}{2} b_n^{(i)} - C_1(n) \hat{b}_n^{(i)} - (\pi - \tau_i) C_1(n) J_1^{(i)} \sin n\tau_i \right\} -$$

$$- \frac{(-1)^n}{4n^2} \left\{\sum_{i=1}^{2} \left(\sum_{p=1}^{3} b_{p,n}^{(i)} - J_2^{(i)} sin 2n\tau_i\right) + \sum_{p=1}^{3} [\beta_{p,n}^{(1,2)} + \beta_{p,n}^{(2,1)}]\right\} -$$

$$- \frac{(-1)^n}{4n^2} (J_2^{(1,2)} + J_2^{(2,1)}) \sin n(\tau_1 + \tau_2) + O\left(\frac{b_n^{(i)} C_2(n)}{n^3}\right) \qquad (17)$$

Odavde slijedi

$$C_1(n) = \sum_{i=1}^{2} \left(\frac{J_1^{(i)}}{2\pi} \cos n\tau_i - \frac{1}{2\pi} a_n^{(i)}\right) \qquad (18)$$

$$C_2(n) = \sum_{i=1}^{2} \left(\frac{C_1(n)}{\pi} \hat{b}_n^{(i)} + \frac{\pi - \tau_i}{\pi} C_1(n) J_1^{(i)} \sin n\tau_i - \frac{C_1(n)(\pi + \tau_i)}{2\pi} b_n^{(i)}\right) +$$

$$+ \frac{1}{4\pi} \sum_{i=1}^{2} \left[\sum_{p=1}^{3} b_{p,n}^{(i)} - J_2^{(i)} \sin 2n\tau_i\right] + \frac{1}{4\pi} \sum_{p=1}^{3} [b_{p,n}^{(1,2)} + b_{p,n}^{(2,1)}] -$$

$$-\frac{1}{4\pi}(J_2^{(1,2)}+J_2^{(2,1)})\sin n(\tau_1+\tau_2) \tag{19}$$

Koristeći (18) iz (19) imamo

$$C_2(n)=\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{2\pi^2}\left(J_1^{(j)}J_1^{(i)}\sin n\tau_j\sin n\tau_i+J_1^{(j)}\hat{b}_n^{(i)}-\frac{\pi+\tau_i}{2}J_1^{(j)}b_n^{(i)}\cos n\tau_j\right)+$$

$$+\frac{1}{4\pi}\sum_{i=1}^{2}\sum_{p=1}^{3}\left(b_{p,n}^{(i)}-J_2^{(i)}\sin 2n\tau_i\right)+\frac{1}{4\pi}\sum_{p=1}^{3}(b_{p,n}^{(1,2)}+b_{p,n}^{(2,1)})-$$

$$-\frac{J_2^{(1,2)}+J_2^{(1,2)}}{4\pi}\sin n(\tau_1+\tau_2)+O\left(\sum_{i=1}^{2}\sum_{j=1}^{2}a_n^{(j)}b_n\right) \tag{20}$$

Primedba 4.1. Pošto su potencijali q_1, q_2 iz prostora $L_2[0,\pi]$, asimptotika nula

$$z_n=n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right),(n\to\infty) \tag{21}$$

se odlikuje složenošću koeficijenata $C_1(n)$ i $C_2(n)$.

Naime, u svojoj strukturi ti koeficijenti imaju sabirke koji iščezavaju pri $n\to\infty$ i one sabirke koji ne iščezavaju nego osciluju. Tako npr. u $C_1(n)$ sabirak $\frac{J_1^{(i)}}{2\pi}\cos n\tau_i$ osciluje, a član $\frac{a_n^{(i)}}{2\pi}$ iščezava. Slično je i kod $C_2(n)$. Izraz

$$\zeta_2=\sum_{i=1}^{2}\left[\sum_{j=1}^{2}\frac{1}{2\pi^2}J_1^{(j)}J_1^{(i)}\sin n\tau_j\sin n\tau_i-\frac{1}{4\pi}J_2^{(i)}\sin 2n\tau_i\right]-$$

$$-(J_2^{(1,2)}+J_2^{(2,1)})\sin n(\tau_1+\tau_2)$$

osciluje dok izraz

$$\zeta_2^*=\sum_{i=1}^{2}\left[\sum_{j=1}^{2}\frac{1}{2\pi^2}\left(J_1^{(j)}\hat{b}_n^{(i)}-\frac{\pi+\tau_i}{2}J_1^{(j)}b_n^{(i)}\cos n\tau_j\right)+\frac{1}{4\pi}\sum_{p=1}^{3}b_{n,p}^{(i)}\right]+$$

$$+\frac{1}{4\pi}\sum_{p=1}^{3}(b_{p,n}^{(2,1)}+b_{p,n}^{(1,2)})+O\left(\sum_{i=1}^{2}\sum_{j=1}^{2}a_n^{(j)}b_n^{(i)}\right)$$

iščezava pri $n\to\infty$.

Primedba 4.2. Asimptotika tipa (21) sa koeficijentima (18) i (20) ima izuzetan značaj u teoriji inverznih zadataka.

Literatura

[1] Наймарк М. А., Линейные дифференциальные операторы, Москва, Наука, 1969.

[2] Гельфанд И. М., Левитан Б. М., Об определении дифференциального уравнения по его спектралной функции, Изв. АН СССР, Сер. мат(1951)Т. 15, С. 309-360.

[3] Садовничй В. А., Теория операторов, Издательство Москоского университета, 1979.

[4] Норкин С. Б., Дифференциаль ные уравнения второго порядка с запаздывающим аргументом, Москва, Наука 1969.

[5] Пикула М., О регуляризованных следах дифференциального оператора типа Штурма-Лиувилля с запаздывающим аргументом, Дифф. уравнения 1 (1990)103-109.

[6] Пикула М., О регуляризованных следах дифференциальны операторов высших порядков с запаздывающим аргументом, Дифф. уравнения 6 (1985)956-991.

[7] Pikula M., Asimptotika svojstvenih vrijednosti i regularizovani tragovi linearnih diferencijalnih operatora, doktorska disertacija, Sarajevo 1983.

[8] Lazovic R. and Pikula M., Regularized trace of the operator applied to solving inverse problems, Radovi Matematički 11 (2002)

[9] Pavlović N., Asimptotika svojstvenih vrijednosti linearne diferencijalne jednačine sa kašnjenjem, Zbornik radova sa naučnog skupa, Nauka i nastava na univerzitetu, Pale, 17-18. maj 2008., knjiga 3 tom 2, 591-604.

[10] Pikula M., Markovic O., Nedic D., Computations of first forced regularized trace Sturm-Liouville operator with homogeneous delay, Paper Proceedings Buisness-Technical Colleg, Uzice,(2012.), 185-187

[11] Freiling G. and Yurko V, Inverse problems for Sturm-Liouville diferential operators with a constant delay, Applied Mathematical Letters,(2012).

[12] Pikula M., Vladicic V. and Markovic O., A solution to the inverse problem for the Sturm-Liouville-type equation with a delay, Filomat 27:7(2013), 1237-1245

[13] Pikula M., Vladicic V. and Nedic D., Inverse problems for Sturm-Liouville differentials operators with a homogeneous delay,(accepted for publishing in Sib. Math. Journal)

THE THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013.

On Sets of Φ-type

Dojčin S. Petković
University of Priština, Faculty of Science and Mathematics
Mitrovica, Serbia.
dojcin.petkovic@gmail.com
Ivan D. Aranđelović
University of Belgrade - Faculty of Mechanical Engineering
iarandjelovic@mas.bg.ac.rs

Original scientific paper

Apstract

S. Kasahara [9], [10] in 1974, considered topological vector spaces over topological semifield, and introduced the notion of Φ-paranormed spaces. O. Hadžić [3], [4] introduced the notion of sets of Φ-type, and proved admissibility (in seance of V. Klee [11], [12]) of such sets. Sets of Zima's type were introduced by K. Zima [15] (for metric linear spaces) and O. Hadžić [5] (for arbitrary Hausdorff topological vector spaces).

In this talk (paper) we shall prove that each set of Φ-type is set of Zima's type, which implies that every the subset of Φ-type can be affinely embedded in a Hausdorff locally convex linear space.

1 Introduction

There are many approach to the theory of uniform spaces which are obtained by Tychonov, Kurepa, Weil, Efremovich, and many others. One of them is introduced by M. Ja. Antonovskii, V. G. Boltjanskii and T. A. Sarymsakov in 1960. They considered uniform spaces as "metric spaces" in which distance between points belongs to some topological semifield.

S. Kasahara [9], [10] in 1974, considered topological vector spaces over topological semifield, and introduced the notion of Φ-paranormed spaces. In [1] I. Aranđelović and M. Rajović proved that topological vector space are Φ-paranormable if and only if it is product of locally bounded spaces.

Kasahara's approach was applied in papers of O. Hadžić [3], [4], O. Hadžić and Lj. Gajić [8], Lj. Gajić [2] and S. Nešić [13], [14], in which were given some results in fixed point theory and related topics of nonlinear analysis.

O. Hadžić [3], [4] introduced the notion of sets of Φ-type, and proved admissibility (in seance of V. Klee [11], [12]) of such sets.

Sets of Zima's type were introduced by K. Zima [15] (for metric linear spaces) and O. Hadžić [5] (for arbitrary Hausdorff topological vector spaces). For its applications in nonlinear analysis see monograph [6]. H. Weber [14] proved that such set can be affinely embedded in a Hausdorff locally convex linear space.

In this talk (paper) we shall prove that each set of Φ-type is set of Zima's type, which implies that every the subset of Φ-type can be affinely embedded in a Hausdorff locally convex linear space.

2 Preliminary Notes

By $\mathbb{R}$ we shall denote the set of all real numbers. Further, let X be a Hausdorff topological vector space. Then there exists nonempty set Δ and family of metric linear spaces $\{X_i\}_{i\in\Delta}$ (see Klee [10]) such that

$$X = \prod_{i\in\Delta} X_i.$$

We shall denote by $\mathbf{R}_\Delta$ the set of all mappings from Δ into $\mathbf{R}$ with Tychonoff product topology and operations $+$ and scalar multiplication as usual. If $f, g \in \mathbb{R}_\Delta$ we shall say that $f \leq g$ if and only if $f(t) \leq g(t)$ for each $t \in \Delta$. By $\mathbb{P}_\Delta$ we shall denote the cone of nonnegative elements in $\mathbb{R}_\Delta$. For $q = (q_i)_{i\in\Delta} \in \mathbb{R}_\Delta$ we define the $i-$projection as:

$$p_i(q) = q_i.$$

The triplet $(X, \|.], \Phi)$ is a Φ-paranormed space if and only if X is the Hausdorff topological vector space, $\|.] : X \to \mathbb{P}_\Delta$, Φ is a linear, positive mapping from $\mathbb{R}_\Delta$ into $\mathbb{R}_\Delta$ such that the following conditions are satisfied:

1) $\|x] = 0$ if and only if $x = 0$;
2) $\|tx] = |t|\|x]$;
3) $\|x + y] \leq \Phi(\|x] + \|y])$.

Then for mapping $\|.]$ is said to be Φ paranorm.

Let $(X, \|.], \Phi)$ be a Φ-paranormed space. Then $K \subset X$ is set of Φ-type if and only if for every $n \in \mathbb{N}$, every $x_1, \ldots, x_n \in K - K$ and every $\lambda_1, \ldots, \lambda_n$ such that $\sum \lambda_i = 1$ we have:

$$\|\sum_{i=1}^{n} \lambda_i x_i] \leq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i]).$$

Let X, Y be topological spaces and $f : X \to Y$. f is *compact* if it is continuous an $\overline{f(X)}$ is a compact set. Let E be a topological vector space and $g : X \to Y$ be a single-valued function. A function $f : X \to E$ is *finite* if it is compact and $f(X)$ is a subset of some finite dimensional subspace of E. A subset $K \subseteq E$ is said to be *admissible* if for every compact subset $A \subseteq K$ and every U open neighborhoods of zero in E there exists a finite function $h : A \to K$ so that $h(x) - x \in U$ for each $x \in A$.

Let E be a topological vector space, $\mathcal{U}$ be a fundamental family of open neighborhoods of zero in E and $K \subseteq E$. We say that the set K is of Zima's type if and only if for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $co(U \bigcap (K - K)) \subseteq V$.

3 Results

Now we need the following Lemma.

Lemma 3.1. *Let $(X, \|.], \Phi)$ be a Φ-paranormed space. Then*

$$\|x] \leq \Phi(\|x]),$$

for any $x \in X$.

Proof: For any $x \in X$ we have

$$\|x] = \|\frac{x}{2} + \frac{x}{2}] \leq \Phi(\|\frac{x}{2}] + \|\frac{x}{2}]) = \Phi(2\|\frac{x}{2}]) = \Phi(\|x]).$$

$\diamondsuit$

Now we present our main result.

Theorem 3.1. *Let $(X, \|.], \Phi)$ be a Φ-paranormed space and $K \subset X$. If K is set of Φ-type then it is set of Zima's type.*

Proof: Let $\mathcal{U}$ be a fundamental family of open neighborhoods of zero in X and $V \in \mathcal{U}$. Kasahara [8],[9] proved that there exists real number $\epsilon > 0$ and an indecomposable idempotent $\rho \in \mathbb{P}_\Delta$ such that set

$$\{x \in X : \|x]\rho \ll \varepsilon\rho\}$$

is nonempty open subset of V. Let

$$U = \{x \in X : \Phi(\|x])\rho \ll \varepsilon\rho\}.$$

By Lemma 3.1 we get that

$$U \subseteq \{x \in X : \|x]\rho \ll \varepsilon\rho\}.$$

So we get that U is open subset of V because Φ is continuous and linear, which implies that U is open neighborhoods of zero.

Let $x_1, \ldots, x_n \in (K - K) \cap V$. For arbitrary real numbers $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$ we have:

$$\|\sum_{i=1}^{n} \lambda_i x_i] \leq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i]).$$

From $x_1, \ldots, x_n \in (K - K) \cap V$ it follows that

$$\Phi(\|x])\rho \ll \varepsilon\rho$$

for any $1 \leq i \leq n$ which implies

$$\| \sum_{i=1}^{n} \lambda_i x_i] \rho \leq \sum_{i=1}^{n} \lambda_i \Phi(\| x_i]) \rho \ll \varepsilon \rho.$$

So we get that

$$\sum_{i=1}^{n} \lambda_i x_i \in U,$$

which implies $co(U \bigcap (K - K)) \subseteq V$.

$\diamondsuit$

H. Weber [14] proved that set of Zima's type can be affinely embedded in a Hausdorff locally convex linear space. So we obtain:

Corollary 3.1. *Let $(X, \|.], \Phi)$ be a Φ-paranormed space and $K \subset X$. If K is set of Φ-type then it can be affinely embedded in a Hausdorff locally convex linear space.*

O. Hadžić, [5] proved that set of Zima's type is admissible. So we obtain:

Corollary 3.2. *(O. Hadžić [3], [4]) Let $(X, \|.], \Phi)$ be a Φ-paranormed space and $K \subset X$. If K is set of Φ-type then it is admissible.*

Acknowledgement. The second author was partially supported by Ministry of Education and Science Grant, No. 174002, Serbia.

References

[1] I. Aranđelović, M. Rajović, On locally bounded spaces and their products, Krag. J. Math. 27 (2005) 55–61.

[2] Lj. Gajić, On appromaxition of compact multivalued maps in topological vector spaces, Novi Sad J. Math. 26 (1996) 19-24.

[3] O. Hadžić, On the admissibility of topological vector spaces Acta Sci. Math. 42 (1980), 81–85.

[4] O. Hadžić, A fixed point theorem in topological vector spaces, Univ. u Novom Sadu Zb. rad. Prir.-Mat. Fak. Ser. Mat. 10 (1980), 23–29.

[5] O. Hadžić, Some fixed point and almost fixed point theorems for multivalued mappings in topological vector spaces, Nonlinear Anal. Theory, Methods, Appl. 5(9) (1981), 1009–1019.

[6] O. Hadžić, Fixed point theory in topological vector spaces, Institut za matematiku, Novi Sad 1984.

[7] O. Hadžić, Lj. Gajić, A fixed point theorem for multivalued mappings in topological vector spaces, Fund. Mat. 109 (1980) 163–167.

[8] S. Kasahara, On formulation of Topological Linear Spaces by Topological Semifields, Mathematics seminar notes 1 (1973) 11–29.

[9] S. Kasahara, On formulation of Topological Linear Spaces by Topological Semifields, Math. Japonica 19 (1974) 121–134.

[10] V. Klee, Shrinkable neighborhoods in Hausdorff linear spaces, Math. Ann. 141 (1960), 281–285.

[11] V. Klee, Leray-Schauder theory without local convexity, Math. Ann. 141 (1960), 286-296; V. Klee, Corrections to "Leray-Schauder theory without local convexity", Math. Ann. 145 (1961), 31–32.

[12] S. Č. Nešić, A fixed point theorem in Banach spaces over topological semifields, Mat. Vesnik 46,3-4 (1994) 77–79.

[13] S. Č. Nešić, Banach spaces over topological semifields and common fixed points, Mat. Vesnik 47,1-2 (1995) 19–22.

[14] H. Weber, Compact convex sets in non locally convex linear spaces, Note Mat. 271–289.

[15] K. Zima, On Schauder's fixed point theorem with respect to para-normed space, Comment. Math. Prace Mat. 19 (1977), 421-423.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Fourier Series of Functions with Infinite Discontinuities

Branko Sarić
Internacionalni Univerzitet Travnik, Bunar bb, 72 270 Travnik
saric.b@open.telekom.rs

Original scientific paper

Abstract

Using the the concept of the total *Kurzweil-Henstock* integral we shall try to show that functions, which can take not only finite but infinite values in the interval $(-\pi, \pi)$, can be expanded into a *Fourier* series over this interval.

1 Introduction

As is well-known to all of us, significant progress in *Fourier* analysis has gone hand in hand with progress in theories of integration, [3, 9]. Perhaps this can be best exemplified by using the so-called total value of the generalized *Riemann* integrals introduced by *Saric* in his works [4, 5, 6, 7]. This brand new theory of integration, which takes the notion of residues of real valued functions into account, gives us the opportunity to integrate real valued functions that was not integrable in any of the known integration methods until now. Accordingly, in the main part of this paper, we shall see that real-valued functions, with infinite discontinuities within the interval $(-\pi, \pi)$, can be expanded into a *Fourier* series over $[-\pi, \pi]$.

2 Preliminaries

The *Lebesgue* measure in the set of all real numbers $\mathbb{R}$ is denoted by μ, however, for $E \subset \mathbb{R}$ we write $|E|$ instead of $\mu(E)$. By $\mathbb{N}$ we denote the set of natural numbers. Given a compact interval $[-\pi, \pi]$ let the collection $\mathcal{I}([-\pi, \pi])$ be a family of all compact subintervals I of $[-\pi, \pi]$. Any real valued function defined on $\mathcal{I}([-\pi, \pi])$ is an interval function. For $f : [-\pi, \pi] \mapsto \mathbb{R}$ the associated interval function of f is an interval function $f : \mathcal{I}([-\pi, \pi]) \mapsto \mathbb{R}$, again denoted by f, [8]. A partition $P[-\pi, \pi]$ of $[-\pi, \pi]$ is a finite set of interval-point pairs $([a_i, b_i], x_i)$, $i = 1, ..., \nu$, such that the subintervals $[a_i, b_i]$ are non-overlapping ($(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, where (a_i, b_i) are the interiors of $[a_i, b_i]$), $\cup_{i \le \nu} [a_i, b_i] = [-\pi, \pi]$ and $x_i \in [a_i, b_i]$. The points $\{x_i\}_{i \le \nu}$ are the tags of $P[-\pi, \pi]$, [1, 2]. If E is a subset of $[-\pi, \pi]$, then the restriction of $P[-\pi, \pi]$ to E is a finite subset of $([a_i, b_i], x_i) \in P[-\pi, \pi]$ such that each pair of sets $[a_i, b_i]$ and E intersects in at least one point. In symbols, $P[-\pi,\pi]|_E = \{([a_i,b_i],x_i) \in P[-\pi,\pi] \mid [a_i,b_i] \cap E \neq \emptyset\}$. It is evident that a given partition of $[-\pi,\pi]$ can be tagged in infinitely many ways by choosing different points as tags. Given $\delta : [-\pi,\pi] \mapsto \mathbb{R}_+$, named a gauge, a partition $P[-\pi,\pi]$ is called *δ-fine* if $[a_i,b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$. Let $\mathcal{P}[-\pi,\pi]$ be the family of all partitions $P[-\pi,\pi]$ of $[-\pi,\pi]$. Then, by $\mathcal{P}_\delta[-\pi,\pi]$ we denote the family of all *δ-fine* partitions $P[-\pi,\pi]$ of $[-\pi,\pi]$ for some given $\delta : [-\pi,\pi] \mapsto \mathbb{R}_+$. In what follows we will use the following notations: $\Delta F(I) = F(v) - F(u)$, where u and v are the endpoints of $I \in \mathcal{I}([-\pi,\pi])$, $\sum_i \Delta F([a_i,b_i]) = \Delta F(P[-\pi,\pi])$ and $\sum_i f(x_i)|[a_i,b_i]| = (f\Delta x)(P[-\pi,\pi])$, whenever $([a_i,b_i],x_i) \in P[-\pi,\pi]$. In addition, if $t \in (0,\pi)$ and $\gamma(I,t)$ is an interval function associated to $\gamma(x,t)$, then $\sum_i [\gamma([a_i,b_i],t)\Delta F([a_i,b_i]) - \gamma(x_i,t)|[a_i,b_i]|] = (\gamma\Delta F - \gamma\Delta x)(P[-\pi,\pi],t)$.

The following two definitions come from [2].

Definition 2.1. A function $f : [-\pi,\pi] \mapsto \mathbb{R}$ is said to be *Kurzweil-Henstock* integrable to a real number $\mathcal{A}$ on $[-\pi,\pi]$ if for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that $|(f\Delta x)(P[-\pi,\pi]) - \mathcal{A}| < \varepsilon$, whenever $P[-\pi,\pi] \in \mathcal{P}_\delta[-\pi,\pi]$. In symbols, $\mathcal{A} = \mathcal{KH} - \int_{-\pi}^{\pi} f$.

Definition 2.2. Let $\gamma : \mathcal{I}[-\pi,\pi] \mapsto \mathbb{R}$ be an arbitrary interval function. Then, a function $g : [-\pi,\pi] \mapsto \mathbb{R}$ is the limit of γ on $E \subseteq [-\pi,\pi]$, if for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that $|\gamma([a_i,b_i]) - g(x_i)| < \varepsilon$, whenever $([a_i,b_i],x_i) \in P[-\pi,\pi]|_E$ and $P[-\pi,\pi] \in \mathcal{P}_\delta[-\pi,\pi]$.

For a primitive $F : [-\pi,\pi] \mapsto \mathbb{R}$, the derivative f could be defined as the limit of the interval function $\phi : \mathcal{I}[-\pi,\pi] \mapsto \mathbb{R}$ defined by

$$\phi(I) = \frac{\Delta F(I)}{\Delta x(I)} = \frac{\Delta F}{\Delta x}(I), \tag{1}$$

where $\Delta x(I) = |I|$. In this case, according to *Definition 2*, given $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that $|\Delta F([a_i,b_i]) - f(x_i)\Delta x([a_i,b_i])| \le \varepsilon \Delta x([a_i,b_i])$, whenever $([a_i,b_i],x_i) \in P[-\pi,\pi]|_E$ and $P[-\pi,\pi] \in \mathcal{P}_\delta[-\pi,\pi]$. Accordingly, if $E \subset [-\pi,\pi]$, more precisely if $F : [-\pi,\pi] \mapsto \mathbb{R}$ is a function that is not differentiable on $[-\pi,\pi]$, then for a given $\varepsilon > 0$ in the set

$$\Omega_\varepsilon^{\mathcal{KH}} = \{(x,I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } |\Delta F(I)| \ge \varepsilon |I|\}$$

we isolate two subsets:

$$\Omega_{<\varepsilon}^{\mathcal{KH}} = \{(x,I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } \varepsilon|I| \le |\Delta F(I)| < \varepsilon\} \text{ and}$$

$$\Omega_{\ge\varepsilon}^{\mathcal{KH}} = \{(x,I) : x \in [-\pi,\pi] \text{ is inside } I \text{ and } |\Delta F(I)| \ge \varepsilon\}.$$

Definition 2.3. Let $F : [-\pi,\pi] \mapsto \mathbb{R}$. The set $(vss)[-\pi,\pi] = \{x \in [-\pi,\pi] :$ for every $\varepsilon > 0$ there exists a δ-fine $(x,I) \in \Omega_{<\varepsilon}^{\mathcal{KH}}\}$ is said to be the set of seeming singular points of F on $[-\pi,\pi]$.

Definition 2.4. Let $F : [-\pi, \pi] \mapsto \mathbb{R}$. The set $(vs)[-\pi, \pi] = \{x \in [-\pi, \pi] :$ for every $\varepsilon > 0$ there exists a δ-fine $(x, I) \in \Omega_{\geq \varepsilon}^{\mathcal{KH}}\}$ is said to be the set of singular points of F on $[-\pi, \pi]$.

When working with functions, which have a finite number of discontinuities on $[-\pi, \pi]$, it does not really matter, from the point of view of totalization of the *Kurzweil-Henstock* integral, how these functions will be defined on the set E of discontinuities. Hence, we adopt the convention that such functions are equal to 0 at all points at which they can take values $\pm\infty$ or not be defined at all. Accordingly, we may define point functions $F_{ex} : [-\pi, \pi] \mapsto \mathbb{R}$ and $f_{ex} : [-\pi, \pi] \mapsto \mathbb{R}$ by extending F and its derivative f from $[-\pi, \pi] \backslash E$ to E by $F_{ex}(x) = 0$ and $f_{ex}(x) = 0$ for $x \in E$, so that

$$F_{ex}(x) = \begin{cases} F(x), \text{ if } x \in [-\pi, \pi] \backslash E \\ 0, \text{ if } x \in E \end{cases} \quad \text{and} \tag{2}$$

$$f_{ex}(x) = \begin{cases} f(x), \text{ if } x \in [-\pi, \pi] \backslash E \\ 0, \text{ if } x \in E \end{cases} .$$

The following two definitions come from [5].

Definition 2.5. Let $\gamma : \mathcal{I}[-\pi, \pi] \mapsto \mathbb{R}$ be an arbitrary interval function and for $F : [-\pi, \pi] \mapsto \mathbb{R}$ let $\phi : \mathcal{I}[-\pi, \pi] \mapsto \mathbb{R}$ be an interval function defined by (1), that converge, according to Definition 2, to $g(x)$ and $f(x)$, respectively, almost everywhere on $[-\pi, \pi]$. A point function $g(x)$ is totally *Kurzweil-Henstock* integrable, with respect to the differential form $dF(x) = f(x)\,dx$, to a real point $\mathcal{F}$ on $[-\pi, \pi]$, if for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi, \pi]$ such that

$$|(\gamma\Delta F)(P[-\pi, \pi]) - \mathcal{F}| < \varepsilon, \tag{3}$$

whenever $P[-\pi, \pi] \in \mathcal{P}_\delta[-\pi, \pi]$. In symbols, $\mathcal{F} := \mathcal{KH} - vt\int_{-\pi}^{\pi} g dF$.

Remark 2.1. In case, any of the point functions g and f above is the limit of the corresponding interval function on $[-\pi, \pi]$, then in the previous definition (3) can be replaced by

$$|(g\Delta F)(P[-\pi, \pi]) - \mathcal{F}| < \varepsilon \text{ or } |(\gamma f \Delta x)(P[-\pi, \pi]) - \mathcal{F}| < \varepsilon,$$

respectively.

Definition 2.6. Let $F : [-\pi, \pi] \mapsto \mathbb{R}$ and $E \subset (-\pi, \pi)$ be a set of *Lebesgue* measure zero such that $E = (vs)[-\pi, \pi]$. The linear differential form $dF(x) = f(x)\,dx$, as the limit of $\Delta F(I)$ on $[-\pi, \pi]$, where $I \in \mathcal{I}([-\pi, \pi])$, is said to be basically summable (BS_δ) to a real number $\Re$ on E if for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi, \pi]$ such that

$$|(\Delta F - f_{ex}\Delta x)(P[-\pi, \pi]|_E) - \Re| < \varepsilon,$$

whenever $P[-\pi, \pi] \in \mathcal{P}_\delta[-\pi, \pi]$. If in addition E can be written as a countable union of sets on each of which the linear differential form $f(x)\,dx$ is BS_δ, then $f(x)\,dx$ is said to be BSG_δ on E. In symbols, $\Re := \sum_{x \in E} f(x)\,dx$.

3 Main results

It is an old result (see [9]) that if $\gamma : [-\pi, \pi] \mapsto \mathbb{R}$ is a point function defined by $\gamma(x,t) = \sum_{k=1}^{+\infty} \Gamma_k(x,t) + (x-t)/2$, where $\Gamma_k(x,t) = \sin[k(x-t)]/k$, for every fixed $t \in (0,\pi)$, then the dispersion of function values on $[-\pi,\pi]$ is as follows

$$\gamma(x,t) = \begin{cases} -\frac{\pi}{2}, \text{ if } x \in [-\pi, t) \\ 0, \text{ if } x = t \\ \frac{\pi}{2}, \text{ if } x \in (t, \pi] \end{cases} . \tag{4}$$

Let $\gamma : \mathcal{I}([-\pi,\pi]) \mapsto \mathbb{R}$ be the associated interval function of γ. If $I \in \mathcal{I}([-\pi,\pi])$ and $\gamma(I,t) = \gamma(v,t) - \gamma(u,t)$, where u and v are the endpoints of I, then

$$\gamma(I,t) = \sum_{k=1}^{+\infty} \Gamma_k(I,t) + \frac{I}{2} = \begin{cases} \pi, \text{ if } t \in int.I \\ \frac{\pi}{2}, \text{ if } t \text{ is the endpoint of } I \\ 0, \text{ if } t \notin I \end{cases} , \tag{5}$$

where $\Gamma_k(I,t) = \Gamma_k(v,t) - \Gamma_k(u,t)$. In addition, let $E \subset (-\pi,\pi)$ be a set of *Lebesgue* measure zero at whose points an arbitrary point function F, defined and differentiable to f on $[-\pi,\pi] \backslash E$, can take values $\pm\infty$ or not be defined at all and $t \notin E$. If we introduce into the analysis the interval function $\gamma(I,t)\, \Delta F_{ex}(I)$, as the product of the two interval functions $\gamma(I,t)$ defind by (5) and $\Delta F_{ex}(I)$, whenever $I \in \mathcal{I}([-\pi,\pi])$, then, according to *Definition 5*,

$$\mathcal{KH} - vt \int_{-\pi}^{\pi} [\sum_{k=1}^{+\infty} G_k(x,t) + \frac{1}{2}] f(x)\, dx = \pi f(t), \tag{6}$$

where $G_k(x,t) = \cos[k(x-t)]$ is the limit of $\Gamma_k(I,t)/\Delta x(I)$, since $f(t)$ is the limit of the interval function $\phi(I) = (\Delta F_{ex}/\Delta x)(I)$ at the point t and therefore for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that

$$|(\boldsymbol{\gamma\phi}_{ex})(P[-\pi,\pi], t) - \pi f(t)| = \pi\, |\phi_{ex}([a_i,b_i]_t) - f(t)| < \pi\varepsilon, \tag{7}$$

whenever $P[-\pi,\pi] \in \mathcal{P}_\delta[-\pi,\pi]$, where $[a_i,b_i]_t$ are the subintervals $[a_i,b_i]$ to which the point t belongs.

On the other hand, considering the fact that $\sum_{k=1}^{+\infty} \Gamma_k(I,t)$ converges on $[-\pi,\pi]$ it follows that $\sum_{k=1}^{+\infty} (\mathbf{\Gamma}_k/\Delta x)(P[-\pi,\pi], t) = (\sum_{k=1}^{+\infty} \mathbf{\Gamma}_k/\Delta x)(P[-\pi,\pi], t)$, for every $P[-\pi,\pi] \in \mathcal{P}[-\pi,\pi]$. Hence, (6) becomes the *Fourier* series of f at the point t, as follows

$$\frac{1}{\pi} \sum_{k=1}^{+\infty} \mathcal{KH} - vt \int_{-\pi}^{\pi} G_k(x,t) f(x)\, dx + \frac{1}{2\pi} \mathcal{KH} - vt \int_{-\pi}^{\pi} f(x)\, dx = f(t). \tag{8}$$

The following theorem give us the opportunity to compute the *Fourier* coefficients for a function that can take not only finite but infinite values within $[-\pi,\pi]$, using the *Kurzweil-Henstock* integral.

Theorem 3.1. *For $[-\pi,\pi]\in\mathbb{R}$ let $E\subset(-\pi,\pi)$ be a set of Lebesgue measure zero at whose points a primitive F that is defined and differentiable on $[-\pi,\pi]\setminus E$ and its derivative f can take values $\pm\infty$ or not be defined at all. If $G_k(x,0)\,dF(x)=G_k(x,0)\,f(x)\,dx$, as the limit of the point-interval function $G_k(x,0)\,\Delta F_{ex}(I)$ on $[-\pi,\pi]$, where $I\in\mathcal{I}([-\pi,\pi])$, is basically summable (BS_δ) on E to the sum $\Re_k$ and $G_k(x,0)\,f_{ex}(x)$ is Kurzweil-Henstock integrable to a real number $\mathcal{A}_k$ on $[-\pi,\pi]$, for each $k\in\mathbb{N}$, then*

$$\mathcal{KH}-vt\int_{-\pi}^{\pi}G_k(x,0)\,f(x)\,dx=\mathcal{KH}-\int_{-\pi}^{\pi}G_k(x,0)\,f_{ex}(x)\,dx+\Re_k,\qquad(9)$$

for each $k\in\mathbb{N}$.

Proof. Let F_{ex} and f_{ex} be defined by (2). Since the point function $G_k(x,0)\,f_{ex}(x)$ is *Kurzweil-Henstock* integrable to a real number $\mathcal{A}_k$ on $[-\pi,\pi]$ and $G_k(x,0)\,dF(x)$ is (BS$_\delta$) on E to $\Re_k$, for each $k\in\mathbb{N}$, it follows from *Definitions 2 and 6* that for every $\varepsilon>0$ there exist a gauge δ_1 on $[-\pi,\pi]$ such that

$$\left|(G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,,0)-\mathcal{A}_k\right|<\varepsilon,$$

whenever $P\,[-\pi,\pi]\in\mathcal{P}_{\delta_1}[-\pi,\pi]$, and a gauge δ_2 on $[-\pi,\pi]$ such that

$$\left|(G_k\Delta F_{ex}-G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,|_E\,,0)-\Re_k\right|<\varepsilon,$$

whenever $P\,[-\pi,\pi]\in\mathcal{P}_{\delta_2}[-\pi,\pi]$. In addition, $f_{ex}(x)\equiv0$ on E and $G_k(x,0)\,dF(x)$ is the limit of $G_k(x,0)\,\Delta F(I)$ on $[-\pi,\pi]\setminus E$, that means that for every $\varepsilon>0$ there exists a gauge δ_3 on $[-\pi,\pi]$ such that

$$\left|(G_k\Delta F-G_kf\Delta x)\,(P\,[-\pi,\pi]\setminus P\,[-\pi,\pi]\,|_E\,,0)\right|\le2\pi\varepsilon,$$

whenever $P\,[-\pi,\pi]\in\mathcal{P}_{\delta_3}[-\pi,\pi]$. A gauge δ on $[-\pi,\pi]$ may be chosen so that $\delta(x)=\min(\delta_1(x)\,,\delta_2(x)\,,\delta_3(x))$. Hence, for every $\varepsilon>0$ there exists a gauge δ on $[-\pi,\pi]$ such that

$$\begin{aligned}\left|(G_k\Delta F_{ex})\,(P\,[-\pi,\pi]\,,0)-\mathcal{A}_k-\Re_k\right|&\le\left|(G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,,0)-\mathcal{A}_k\right|+\\&+\left|(G_k\Delta F_{ex}-G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,,0)-\Re_k\right|,\end{aligned}$$

that is,

$$\begin{aligned}&\left|(G_k\Delta F_{ex}-G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,,0)-\Re_k\right|\le\\&\le\left|(G_k\Delta F-G_kf\Delta x)\,(P\,[-\pi,\pi]\setminus P\,[-\pi,\pi]\,|_E\,,0)\right|+\\&+\left|(G_k\Delta F_{ex}-G_kf_{ex}\Delta x)\,(P\,[-\pi,\pi]\,|_E\,,0)-\Re_k\right|\le(2\pi+1)\,\varepsilon,\end{aligned}$$

whenever $P\,[-\pi,\pi]\in\mathcal{P}_\delta[-\pi,\pi]$, so that given $\varepsilon>0$ there exists a gauge δ on $[-\pi,\pi]$ such that $\left|(G_k\Delta F_{ex})\,(P\,[-\pi,\pi])-\mathcal{A}_k-\Re_k\right|\le2\,(1+\pi)\,\varepsilon$, whenever $P\,[-\pi,\pi]\in\mathcal{P}_\delta[-\pi,\pi]$. Therefore,

$$\mathcal{KH}-vt\int_{-\pi}^{\pi}G_k(x,0)\,f(x)\,dx=\mathcal{KH}-\int_{-\pi}^{\pi}G_k(x,0)\,f_{ex}(x)\,dx+\Re_k.$$

$\dashv$

Let $H(x)$ be the so-called *Heaviside* (unit) step function. It is easy to see that $dH(x) = \delta(x)\,dx$, where $\delta(x)$ is the *Dirac* delta function that is zero everywhere except at zero, is the limit of the interval function $\Delta H(I)$, associated to $H(x)$, on $[-\pi,\pi]\setminus E_0$, where $E_0 = \{0\}$. Since for every $\varepsilon > 0$ there exists a gauge δ on $[-\pi,\pi]$ such that $|\Delta H(P[-\pi,\pi]) - 1| < \varepsilon$ and $|(G_k\Delta H)(P[-\pi,\pi],0) - 1| < \varepsilon$, whenever $P[-\pi,\pi] \in \mathcal{P}_\delta[-\pi,\pi]$ and $k \in \mathbb{N}$, it follows from *Definition 5* that

$$\mathcal{KH} - vt \int_{-\pi}^{\pi} G_k(x,0)\,dH(x) = 1, \tag{10}$$

for each $k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By *Theorem 1* and *Definition 6*, we see that $\mathcal{KH} - \int_{-\pi}^{\pi} G_k(x,0)\,\delta_{ex}(x)\,dx \equiv 0$, for each $k \in \mathbb{N}_0$, since $G_k(x,0)\,dH(x)$ are basically summable (BS_δ) to 1 on E_0, for each $k \in \mathbb{N}_0$. Finally, it follows from (8) that

$$\sum_{k=1}^{+\infty} G_k(0,t) + \frac{1}{2} = \pi\delta(t), \tag{11}$$

at every point t belonging to the set $[-\pi,\pi]\setminus E_0$. This confirms that $\sum_{k=1}^{+\infty} G_k(x,t)$ is the limit of $\sum_{k=1}^{+\infty} \Gamma_k(I,t)/\Delta x(I)$ on $[-\pi,\pi]\setminus E_t$, where $E_t = \{t\}$ and $t \in (-\pi,\pi)$. In addition, for any real-valued periodic function $f(x)$ of period 2π, which is defined at a point $t \in (-\pi,\pi)$, it follows that

$$\pi f(t) = \mathcal{KH} - vt \int_{-\pi}^{\pi} \pi\delta(x-t)\,f(x)\,dx = \tag{12}$$

$$= \mathcal{KH} - vt \int_{-\pi}^{\pi} [\sum_{k=1}^{+\infty} G_k(x,t) + \frac{1}{2}] f(x)\,dx = \sum_{k=1}^{+\infty} \mathcal{KH} - vt \int_{-\pi}^{\pi} G_k(x,t)\,f(x)\,dx +$$

$$+ \frac{1}{2}\mathcal{KH} - vt \int_{-\pi}^{\pi} f(x)\,dx.$$

References

[1] R. G. Bartle, *A Modern Theory of Integration*, Graduate Studies in Math., Vol. 32, AMS, Providence, 2001.

[2] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Math., Vol. 4, AMS, Providence, 1994.

[3] J. Marcinkiewicz and A. Zygmund, Two theorems on trigonometrical series, *Rec. Math.* [*Mat. Sbornik*] *N.S.*, Vol. **2 (44)**, No **4**, (1937), 733–737.

[4] B. Sarić, The *Fourier* series of one class of functions with discontinuities, Doctoral dissertation defended on 20th of October 2009. at the University of Novi Sad, Faculty of Science, Department of Mathematics and Informathics.

[5] B. Sarić, *Cauchy's* residue theorem for a class of real valued functions, *Czech. Math. J.* Vol. **60**, No.**4**, (2010), 1043–1048.

[6] B. Sarić, On totalization of the *Henstock-Kurzweil* integral in the multidimensional space, *Czech. Math. J.*, Vol. **61**, No. **4**, (2011), pp. 1017-1022.

[7] B. Sarić, On totalization of the H_1-integral, *Taiw. J. Math.*, Vol. **15**, No. **4**, (2011), 1691-1700.

[8] V. Sinha and I. K. Rana, On the continuity of associated interval functions, *Real Analysis Exchange* Vol. **29**(**2**) (2003/2004), 979-981.

[9] A. Zygmund, *Trigonometric series*, University Press, Cambridge, 2003.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Dispersion under Iteration of Transformations in Some Classes of Discrete Time Dynamical Systems

Berina Fatkić
University of Sarajevo
bfatkic@yahoo.com

Jasmina Fatkić
University of Sarajevo
noviimidz@yahoo.com

Alem Čolaković
University of Sarajevo
alem.colakovic@gmail.com

Huse Fatkić
University of Sarajevo
hfatkic@etf.unsa.ba

Original scientific paper

Abstract

In the present paper we study some metric properties and dispersive effects of weakly mixing (WM) measure-preserving transformations on general metric spaces endowed with a probability measure; in particular, we investigate connections of WM discrete time dynamical systems with the theory of probabilistic metric spaces. Further, continuing the work begun by B. Schweizer and A. Sklar [*Z. Wahrsch. Verw. Geb.* 26 (1973), 235 – 239], in [Huse Fatkić, Slobodan Sekulović, and Hana Fatkić, Probabilistic metric spaces determined by weakly mixing transformations, *Proceedings of the 2nd Mathematical Conference of Republic of Srpska - Section of Applied Mathematics* (2012), 195-208] is proven that if (S, d) is a separable metric space endowed with a probability measure P and if T is a transformation on S that is weakly mixing with respect to P, then for any $x > 0$ and almost all pairs of points (p, q) in S^2, there is a distribution function F such that the average number of times in first $(n-1)$ iterations of T that the distance between points $T^n(p)$ and $T^n(q)$ is less than x converges to $F(x)$ as n go to infinite. The collection of these distribution functions is almost an equilateral probabilistic pseudometric space and the transformation T is (probabilistic-) distance-preserving on this space. We apply this result to establish several facts, e.g. the fact that under iteration of WM transformation T, k-tuples of distinct distances $d(p, q)$ behave asymptotically as independent, identically distributed random variables.

1 Introduction and preliminaries

In this work we shall study actions of the group $\mathbf{Z}$ of integers on a probability space X , i.e., we study a transformation $T : X \to X$ and its iterates $T^n, n \in \mathbf{Z}$.

Specifically, we shall investigate some metric properties and dispersive effects of weakly mixing (WM) transformations on general metric spaces endowed with a probability measure; in particular, we shall study connections of WM discrete time dynamical systems with the theory of probabilistic metric (PM) spaces. We shall generally refer to Billingsley [1], Cornfeld, Fomin and Sinai [3], Fatkić [6], Hille [14], Schweizer and Sklar [21] and Walters [24].

Suppose $(X, \mathcal{A}, P)$ is a probability space. As usual, a transformation $T : X \to X$ is called:

(i) *measurable (P - measurable)* if, for any A in $\mathcal{A}$, the inverse image $T^{-1}(A)$ is in $\mathcal{A}$;

(ii) *measure-preserving* if T is measurable and $P(T^{-1}(A)) = P(A)$ for any A in $\mathcal{A}$ (or, equivalently, measure P is said to be *invariant* under T);

(iii) *ergodic* if the only members A of $\mathcal{A}$ with $T^{-1}(A) = A$ satisfy $P(A) = 0$ or $P(X \setminus A) = 0$;

(iv) *weakly mixing* (or *weak-mixing*) (with respect to P) if T is P - measurable and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |P(T^{-i}(A) \cap B) - P(A)P(B)| = 0$$

for any two P- measurable subsets A, B of X.

(v) *strongly mixing* (or *mixing, strong-mixing*) (with respect to P) if T is P - measurable and

$$\lim_{n\to\infty} P(T^{-n}(A) \cap B) = P(A)P(B) \tag{1}$$

for any two P- measurable subsets A, B of X.

We say that the transformation $T : X \to X$ is *invertible* if T is one-to-one (monic) and such that $T(A)$ is P - measurable whenever A is P - measurable subset of X.

A transformation T on a probability space $(X, \mathcal{A}, \mu)$ is said to be *measurability - preserving* if $T(\mathcal{A}) \subseteq \mathcal{A}$ (i.e., if $T(A)$ is P - measurable whenever A is P - measurable. In this case we also say that the transformation T *preserves P - measurability.*

If $(X, \mathcal{A}, P)$ is a probability space, and $T : X \to X$ is a measure-preserving transformation (with respect to P), then we say that $\Phi := (X, \mathcal{A}, P, T)$ is an *abstract dynamical system*. An abstract dynamical system is often called a *dynamical system with discrete time* or a *measure-theoretic dynamical system* or an *endomorphism*. We shall say that the abstract dynamical system Φ is: (i) *invertible* if T is invertible; (ii) *ergodic* if T is ergodic; (iii) *weakly* (resp. *strongly*) *mixing* if T is weakly (resp. strongly) mixing (see [2, pp. 6 - 26]).

2010 *Mathematics Subject Classification*: 60B05, 28A10, 54E35.

If T is a strongly mixing transformation of a probability space $(X, \mathcal{A}, P)$, then, as is well-known, T is both measure-preserving and ergodic. Furthermore, if $T : X \to X$, in addition (to being strongly mixing on X with respect to P), is invertible, then (1.1) is equivalent to (the well- known result):

$$\lim_{n\to\infty} P(T^n(A) \cap B) = P(A)P(B) \tag{2}$$

for any μ - measurable subsets A, B of X.

In this work we consider a metric space/(extended metric spaces, i.e., if we allow d to take values in $[0, \infty]$ (the nonnegative lower reals) instead of just in $[0, \infty)$, then we get extended metric spaces) (X, d) on which a probability measure P is defined. The domain of P, a σ - algebra $\mathcal{A}$ of subsets of X , is assumed to include all Borel sets in (X, d); in particular, therefore, all open balls in (X, d) are P - measurable (see [10]).

The (ordinary) *diameter* of a subset A of X , i.e., the supremum of the set $\{d(x, y)|x, y \in A\}$, will be denoted by diam(A). We can define the diameter of the empty set (the case $A = \varnothing$) as 0 or $-\infty$, as we like. But, like many other authors, we prefer to treat the empty set as a special case, assigning it a diameter equal to 0, i.e., diam$(\varnothing) = 0$, which corresponds to taking the codomain of d to be the set of all nonnegative real numbers (if the distance function d is viewed here as having codomain $\mathbf{R}$, this implies that diam $(\varnothing) = -\infty$) (see [10]). We call P *nonsingular* if there exist two open balls K_1, K_2, and a positive number x_0 such that $P(K_1) > 0, P(K_2) > 0$, and $d(w_1, w_2) > 0$ for all w_1 in K_1, w_2 in K_2. We call P *pervasive* if $P(K) > 0$ for all open balls K in X. Note that any pervasive measure is nonsingular if X has more than one point.

If A is P-measurable, then the (ordinary) *essential diameter* of A, denoted by ess diam (A), is the infimum of the set of diameters of all P-measurable sets B such that $B \subseteq A$ and $P(B) = P(A)$.

Let (X, d) be a metric space/(extended metric space), and let A be a subset of X. For any positive integer $k \geq 2$, we define the the *geometric diameter of order* k of A (kth diameter of A), denoted by $\delta_k(A)$, to be the quantity

$$\delta_k(A) := sup \left\{ \sqrt[\binom{n}{2}]{\prod_{1\leq i<j\leq k} d(x_i, x_j)} \mid x_1, \ldots, x_k \in A \right\}. \tag{3}$$

Note that $\delta_2(A)$ is the (ordinary) diameter of the set A. The sequence $(\delta_k(A))$ can be shown to be decreasing (see, e.g., [14] and [19]), and therefore has a limit as k tends to infinity. By definition, the *(geometric) transfinite diameter* of A is

$$\tau(A) := \lim_{k\to\infty} \delta_k(A). \tag{4}$$

Note that $0 \leq \tau(A) \leq \delta_k(A) \leq \text{diam}(A)$, and that $B \subseteq A$ implies $\tau(B) \leq \tau(A)$.

Example 1.1. [19, p. 167] Let A be the closed unit disk (or the unit circle). Then

$$\delta_k(A) := \sqrt[k-1]{k}, \tau(A) = 1.$$

Example 1.2. [19, p. 169] The closed set $\{0\} \cup \{1/k : k = 1, 2, \ldots\}$ has transfinite diameter zero.

If A is μ-measurable subset of X, then, for any positive integer $k \geq 2$, the *essential geometric diameter of order* k of A, denoted by ess $\delta_k(A)$, is the infimum of the set of geometric diameters of all μ-measurable sets B such that $B \subseteq A$ and $\mu(B) = \mu(A)$, i.e.,

$$\text{ess } \delta_k(A) := \inf\{\delta_k(B) | B \text{ is } \mu \text{ - measurable}, B \subseteq A, \mu(B) = \mu(A)\}. \qquad (5)$$

The general theory of geometric diameters and transfinite diameters plays an important role in complex analysis. It is related to the logarithmic potential theory with applications to approximation theory and the Čebyšev constant (see, e.g., [19]).

Investigations in [3 - 12], [18], [21] and [22] have shown, however, that many important consequences of (1.2) persist in the absence of invertibility and/or the strongly mixing property.

Many problems in mathematical analysis can be dealt with within the framework of metric spaces (see [10]). This very fact compelled mathematicians to introduce the spaces as general as *topological spaces* as well as the spaces such as, *probabilistic metric spaces* and eve *probabilistic information spaces* (*probabilistic topological spaces, intuitionistic fuzzy* (*quasi-*) *metric spaces, quasi-fuzzy topological spaces etc.*). In that respect, in 1942 K. Menger [16] proposed *probabilistic/statistical generalization* of the *theory of metric spaces*. Specifically, he proposed replacing the number $d(p, q)$ (the distance between p and q, elements/points of a nonempty set X) by a real-value function F_{pq} whose value $F_{pq}(x)$, for any real number x, is interpreted as the probability that the distance between p is less than x. From it follows that F_{pq} (for all p, q in X) is *probability distribution function.*

For the sake of convenience, we recall some of the basic concepts related to *the theory of probabilistic metric spaces* (for further details, see [6, 10, 12, 21]).

Definition 1.1. A real function F defined on the extended real line $\overline{\mathbf{R}} := [-\infty, +\infty]$ is called a distribution function briefly, a d.f.) if it is nondecreasing and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$.

The set of all d.f.'s that are left-continuous on the unextended real line $\mathbf{R} := (-\infty, +\infty)$ is denoted by Δ and the subset of all F's in Δ satisfying $F(0) = 0$ is denoted by Δ^+ (the set of *distance functions*). Let D be the subclass of Δ formed by all functions $F \in \Delta$ such that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. The subset of all F's in D satisfying $F(0) = 0$ is denoted by D^+.

The sets Δ, Δ^+, D and D^+ are partially ordered by the usual pointwise partial ordering of functions.

Definition 1.2. For any a in $\overline{\mathbf{R}}$, ε_a, the unit step at a, is the function in Δ given by

$$\varepsilon_a(x) = \begin{cases} 0, & -\infty \le x \le a, \\ & \text{for } -\infty \le a < +\infty \\ 1, & a < x \le +\infty; \end{cases} \tag{6}$$

$$\varepsilon_\infty(x) = \begin{cases} 0, & -\infty \le x < +\infty, \\ 1, & x = +\infty. \end{cases} \tag{7}$$

Note that $\varepsilon_a \le \varepsilon_b$ if and only if $b \le a$; that ε_∞ is the minimal element of both Δ and Δ^+; and that $\varepsilon_{-\infty}$ is the maximal element of Δ, and ε_0 the maximal element of Δ^+.

Definition 1.3. A *triangle function* is a binary operation τ on Δ^+ that is commutative, associative, nondecreasing in each place, and has ε_0 as an identity element.

Continuity of a triangle function means uniform continuity with respect to the natural product topology on $\Delta^+ \times \Delta^+$. Typical (continuous) triangle functions are convolution and the operations τ_T, which are given by $\tau_T(F,G)(x) = \sup\{T(F(u), G(v)) | u + v = x\}$, for all F, G in Δ^+ and all $x \in \mathbf{R}$. Here, T is a *continuous t - norm*, i.e., a continuous binary operation on [0, 1] that is commutative, associative, nondecreasing in each place, and has 1 as identity.

Example 1.3 (21, pp. 70 - 71)**.** The most important t - norms are the functions W, Prod and M which are defined, respectively, by

$$W(a,b) = \max\{a+b-1, 0\},\ \mathrm{Prod}(a,b) = ab,\ M(a,b) = \min\{a,b\}.$$

Their corresponding t - conorms are given, respectively, by

$$W^*(a,b) = \min\{a+b, 1\},\ \mathrm{Prod}^*(a,b) = a+b-ab,\ M^*(a,b) = \max\{a,b\}.$$

In the following we shall define some functions, say F, on $\mathbf{R}$ and consider them automatically extended to $\overline{\mathbf{R}}$ by $F(-\infty) = 0$ and $F(\infty) = 1$.

$\{F_i : i \in I\}$ is a family of functions in Δ^+, then the function $F : \overline{\mathbf{R}} \to [0,1]$ defined by

$$F(x) = \sup\{F_i(x) : i \in I\}, x \in \mathbf{R},$$

is the supremum of the family $\{F_i : i \in I\}$ in the order set $(\Delta^+, \le) : F = \sup_{i \in I} F_i$.

To define the infimum of the family $\{F_i : i \in I\}$ put

$$\Gamma(x) = \inf\{F_i(x) : i \in I\}, x \in \mathbf{R}.$$

Since the function Γ is nondecreasing, but not necessarily left continuous on $\mathbf{R}$, we have to regularize it by taking the left limit

$$G(x) = l^-\Gamma(x) := \lim_{x' \to x} \Gamma(x') = \sup_{x' < x} \Gamma(x'), x \in \mathbf{R}$$

Then $G(x) \le \Gamma(x), \forall x \in \mathbf{R}$, the function G belongs to Δ^+ and $G = \inf_{i \in I} F_i$ - the infimum of the family $\{F_i : i \in I\}$ in the order set $(\Delta^+, \le)$.

Definition 1.4. A *probabilistic metric* (briefly, PM) *space* is a triple $(S,\mathfrak{F},\ \tau)$, where S is a nonempty set, τ is a triangle function, and $\mathfrak{F}$ is a mapping from $S\times S$ into Δ^+ such that, if F_{pq} denotes the value of $\mathfrak{F}$ at the pair (p,q), the following conditions hold for all p,q,r in S:

(PM1a) $F_{pq}=\varepsilon_0$;

(PM1b) $F_{pq}\neq\varepsilon_0$ *if* $p\neq q$;

(PM2) $F_{pq}=F_{qp}$;

(PM3) $F_{pr}\geq\tau(F_{pq},F_{qr})$.

If (PM1a), (PM2) and (PM3) are satisfied, then (S,F,τ) is a probabilistic pseudometric space.

The mapping F is called the *probabilistic metric* on S and (PM3) is the probabilistic analogue of the triangle inequality.

Every metric space can be regarded as a special kind of PM space. For if (S,d) is a metric space, if $F:S\times S\to\Delta^+$ is defined via $F_{pq}=\varepsilon_{d(p,q)}$, and τ is a triangle function such that $\tau(\varepsilon_a,\varepsilon_b)\leq\varepsilon_{a+b}$ for all $a,b\geq 0$ – e. g., if τ is given by $\tau_T(F,G)(x)=\sup_{u+v=x}T(F(u),G(v))$, for all F,G in Δ^+ and all x in $\mathbf{R}$, where T is a continuous t - norm – then (S,F,τ) is a PM space from which the original metric space can be immediately recovered.

It is well known that the simplest metric spaces are discrete metric spaces, and by analogy are defined the simplest probabilistic metric spaces: PSM space $(S,\mathfrak{F})$ is said to be equilateral if there exists (distance) distribution function G, which is different from step functions ε_0 and ε_∞, such that

$$(\forall p,q\in S,p\neq q)\ \mathfrak{F}(p,q)=G.$$

Although equilateral spaces look trivial, they naturally appear as the spaces generated by strongly mixing transformations and weakly mixing transformations on metric spaces.

Let A be a class of PSM spaces and let τ be a triangular function. We say that τ is universal for A if each PSM space in A is PM space relative to τ.

In the second part of this introductory paragraph we define notions and bring out the basic facts on a very important, special class of generalized metric spaces, so called transformation-generated spaces. Such spaces were introduced and studied from the ergodic theory point of view in [20, 21, 12]. They represent the limits of PPM spaces, each of is up to isometry the so called E space (see [21, §9.1 and §11.1; Theorem 11.1.1]) constructed in the following way:

Let (S,d) be a metric space, and let ϕ be transformation on S, i.e. a function from S into S. The iterates of ϕ are defined recursively by:

$$\phi^0(p)=p \text{ and } \phi^{n+1}(p)=\phi(\phi^n(p)) \text{ for each } p\in S \text{ and for each } n\in\mathbf{N}_0.$$

For brevity, we denote $\phi^n(p)$ by $\phi^n p$.

For an arbitrary $p\in S$, sequence $\{\phi^n(p)\}_{n=0}^{\infty}$ is trajectory of p under the transformation ϕ.

Further on, for an arbitrary pair of points p, q in S, an arbitrary natural number n and an arbitrary real number x, let

$$\chi(p, q, x, n) = |\{0 \leq m < n : d(\phi^m p, \phi^m q) < x\}|,$$

where $|A|$ denotes the number of elements of a (finite) set A.

Hence, $\chi(p, q, x, n)$ denotes how many times, in the first $n-1$ iterations, is the distance between nth iterations of ϕ in p and q less than x.

Let, for $n \in \mathbf{N}$ and $(p, q) \in S^2, F_{pq}^{(n)}$ denote a function defined by

$$F_{pq}^{(n)}(-\infty) = 0,\ F_{pq}^{(n)}(\infty) = 1,\ F_{pq}^{(n)}(x) = \frac{1}{n}\chi(p, q, x, n)$$

for each $x \in \mathbf{R}$. Hence, $F_{pq}^{(n)}(n \in \mathbf{N})$ are the function for which

$$F_{pq}^{(1)} = \varepsilon_{d(p,q)}, F_{pq}^{(2)} = \frac{1}{2}(\varepsilon_{d(p,q)} + \varepsilon_{d(\phi p, \phi q)}), ..., F_{pq}^{(n)} = \frac{1}{n}\sum_{m=0}^{n-1}\varepsilon_{d(\phi^m p, \phi^m q)},$$

where, for each fixed $t \in \overline{\mathbf{R}}$, ε_t is a unit step function in t. Hence, for an arbitrary $x > 0$, $F_{pq}^{(n)}(x)$ is the average number of times in the first $(n-1)$ iterations of ϕ that the distance $d(\phi^n p, \phi^n q)$ is less then x.

Clearly, for every fixed pair $(p, q) \in S^2$ and for every fixed $n \in \mathbf{N}, F_{pq}^{(n)}$ is non-decreasing function, it has minimal value 0 (which takes for every non-positive value of argument x), has maximum value 1 (which takes for every value of argument x which is larger of the largest among the numbers $\varepsilon_{d(\phi^n p, \phi^n q)}$; $n = 0, 1, ..., n-1$), and it is continuous from the left on $\mathbf{R}$. Hence, $F_{pq}^{(n)}$ is probability distribution function, continuous from the left, and for an arbitrary real number x, the value $F_{pq}^{(n)}(x)$ can be interpreted as the probability that distance between initial segments (size n with respect length), which belong to trajectories of the points p and q, be less than x. Consequently, $F_{pq}^{(n)}$ belongs to Δ^+, and if metric d never takes value ∞ then $F_{pq}^{(n)}$ belongs to the set Δ^+.

For any $n \in \mathbf{N}$ function $\mathfrak{F}^{(n)}$, defined on $S \times S$ by

$$\mathfrak{F}_{(p,q)}^{(n)} = F_{pq}^{(n)},$$

satisfies the conditions

$$\mathfrak{F}_{(p,q)}^{(n)} = \varepsilon_0,\ \mathfrak{F}_{(p,q)}^{(n)} = \mathfrak{F}_{(q,p)}^{(n)},$$

i.e. each of the spaces $(S, \mathfrak{F}^{(n)})$ is probabilistic metric space. It has been proved in [21, §11.1] that each of the spaces $(S, \mathfrak{F}^{(n)})$ is isometric with E - space, and therefore $(S, \mathfrak{F}^{(n)})$ is probabilistic pseudometric space with triangular function τ_W.

However, our primary interest is not the sequence $\{(S, \mathfrak{F}^{(n)})\}$ of probabilistic pseudometric spaces itself, but its limits, which exhibits information about the sequence of distances $\{d(\phi^n p, \phi^n q)\}$ behavior over the asymptotic average. It has been proved (see [21, p. 176]) that, in weak sense, this limit always exists. Hence, we are interested in asymptotic behavior of the sequences $\{F^{(n)}_{(p,q)}(x)\}$, and in that sense, for each $x \in \mathbf{R}$, we take

$$F_{pq}(x) = \lim_{n\to\infty} \inf F^{(n)}_{pq}(x) (= \lim_{n\to\infty}(\inf\{F^{(m)}_{pq} : m \geq n\})) \tag{8}$$

and

$$F^*_{pq}(x) = \lim_{n\to\infty} \sup F^{(n)}_{pq}(x). \tag{9}$$

For arbitrary p, q in S, functions F_{pq} and F^*_{pq} are probabilistic distribution functions such that $F_{pq}(x) \leq F^*_{pq}(x)$ for each $x \in \mathbf{R}$. Without loss of generality, we can assume that these distribution functions are normalized in such a way that they are continuous from the left on $\mathbf{R}$ so from $F_{pq} < F^*_{pq}$ it follows that $F_{pq}(x) < F^*_{pq}(x)$ not only for some x but for all x from some positive interval with positive length. We call function F_{pq} the lower distribution of p and q, whereas F^*_{pq} is the upper distribution of p and q.

It causes no great difficulty to prove (see [21, §11.1; Theorem 11.1.2]) that the lower distribution satisfies (one version of Menger's triangular inequality) inequality

$$F_{pq}(u + v) \geq W(F_{pq}(u), F_{qp}(v))$$

for all real numbers u, v (where $W(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$). It follows that, if $\mathfrak{F}$ is mapping from $S \times S$ into the space of distribution functions, defined by

$$\mathfrak{F}(p, q) = F_{pq},$$

for all p, q in S, then the pair $(S, \mathfrak{F})$ is probabilistic pseudometric space (with triangular function τ_W). We call this space *probabilistic metric space determined by transformation* (or *transformation-generated space*, defined by the metric space (S, d) and transformation ϕ), and we denote it by $[S, d, \phi]$.

Example 1.4. Strongly mixing transformations exhibit dispersive effects. Thus, for example, the dispersive character of the functions C_n defined by

$$C_n = 2\cos(n \text{ arc } \cos\frac{x}{2}), \tag{10}$$

i.e., $C_n = 2\check{C}_n(x/2)$ (where $\check{C}_n$ is the standard nth-degree Čebyšev polynomial), is brought out by the fact that if I is a subinterval of $[-2, 2]$ with $P_c(I) > 0$ (where P_c is the Lebesgue-Stieltjes F_c measure on $[-2, 2]$ determined by $F_c(x) = 1/2 + (1/\pi) \text{ arc } \sin(x/2))$) if $n \geq 2$ and m is any integer such that $m \geq (\log(2/P_c(I))/\log n + 2)$, then $C^m_n(I) = [-2, 2]$; i.e., the mth image of I is the entire space (see [21, p.184] and [6]).

A strongly mixing transformation T that is one to one cannot exhibit such extreme behavior, for in this case T^{-1} is also measure preserving. Nevertheless, as was shown in [3] and [18], under iteration, all strongly mixing transformations tend to spread sets out. Indeed any weakly mixing transformation cannot exhibit such extreme behavior; for as H. Fatkié [5] has shown, we have the following result:

Theorem 1.1. [5, Theorem 3]. *Let (X, d) be a metric space, let $\mathcal{A}$ be a σ-algebra of subsets of X and μ a normalized (probability) measure on $\mathcal{A}$. Suppose further that every open ball in (X, d) is μ-measurable and has positive measure. Let T be a transformation on X that is weakly mixing with respect to μ and suppose that A is μ-measurable subset of X with positive measure. Then*

$$\lim_{n\to\infty} \sup \delta_k(T^n(A)) = \delta_k(X), \tag{11}$$

where δ_k is the geometric diameter of order k given by (1.3).

Note that the hypotheses of Theorem 2.4 are such that ess diam (X) = diam (X).

Theorem 1.4. shows that any set A of positive measure necessarily spreads out, not only in (ordinary) diameter, but also in geometric diameter of any finite order. Thus, even though A may not spread out in "volume" (measure), there is a very definite sense in which A does not remain small.

2 Main results

In this section of the paper we continue work from a previous paper [12], where is proven that if (S, d) is a separable metric space endowed with a probability measure P and if T is a transformation on S that is weakly mixing with respect to P, then for any $x > 0$ and almost all pairs of points (p, q) in S^2, there is a distribution function F such that the average number of times in first $(n-1)$ iterations of T that the distance between points $T^n(p)$ and $T^n(q)$ is less than x converges to $F(x)$ as n go to infinite. The collection of these distribution functions is almost an equilateral probabilistic pseudometric space and the transformation T is (probabilistic-) distance-preserving on this space. In that direction, we have the following result, which is Theorem 2.1 in [12] and is a substantial improvement of Theorem 11.3.4 in [21]:

Theorem 2.1. [12, Theorem 2.1]. *Let $[X, d, T]$ be a transformation-generated space. Suppose that the following conditions hold:*

(i) *The metric space (X, d) is separable.*

(ii) *There is a probability measure P, defined on σ-algebra F of subsets of X.*

(iii) *Every open ball in X belongs to $\mathcal{A}$.*

(iv) *$T : X \to X$ is weakly mixing with respect to P.*

Then there is a unique distribution function G_T with the following properties:

(a) *For all $x \in [-\infty, \infty)$,*

$$G_T(x) = P^{(2)}(D(x)), \tag{12}$$

where $P^{(2)}$ is the product measure on the set $X \times X$, while, for any $x \in \overline{\mathbf{R}}$, $D(x)$ denote the set of all pairs (p, q) in $X \times X$ such that $d(p, q) < x$.

(b) *There is a subset* A_0 *of* $X \times X$ *with* $P^{(2)}(A_0) = 1$ *such that for all* $(p, q) \in A_0$ *the sequence of distribution functions* $(F_{pq}^{(n)})_{n=1}^{\infty}$ *defined by*

$$F_{pq}^{(n)}(-\infty) = 0, F_{pq}^{(n)}(\infty) = 1 \text{ and } F_{pq}^{(n)}(x) = \frac{1}{n}\chi(p, q, x, n) \; (x \in \mathbf{R}), \quad (13)$$

where

$$\chi(p, q, x, n) = \operatorname{card}\{0 \le m < n : d(\phi^m p, \phi^m q) < x\},$$

converges to G_T, *i.e. for each point* $x \in \overline{\mathbf{R}}$ *holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \varepsilon_{d(T^m p, T^m q)}(x) = G_T(x), \quad (14)$$

where, for each fixed point $t \in \overline{\mathbf{R}}$, ε_t *is a unit step function at* t (see (1.6) and (1.7)).

Technique developed in the proof of this theorem surpasses the needs of this proof and, it can be successfully employed in further investigations.

We apply this result to establish several facts about the connections of WM discrete time dynamical systems with the theory of PM spaces, e.g. the fact that under iteration of weakly mixing transformation T, k-tuples of distinct distances $d(p, q)$ behave asymptotically as independent, identically distributed random variables.

The space $[X, d, T]$ in Theorem 2.1 is called *the weakly mixing transformation-generated space* determined by the separable metric space (X, d) and the weakly mixing transformation T and is denoted by $[X, d, T]_{WM}$. It follows that under the hypotheses of Theorem 2.1, the associated weakly mixing transformation-generated space $[X, d, T]$ is almost an equilateral probabilistic pseudo-metric space. If the probability measure P is nonsingular, then there exist two open balls Q_1, Q_2, and a positive number x_0 such that $P(Q_1) > 0, P(Q_2) > 0$ and $d(q_1, q_2) > x_0$ for all $q_1 \in Q_1, q_2 \in Q_2$. This yields

$$G_T(x_0) \le 1 - P^{(2)}(Q_1 \times Q_2) = 1 - P(Q_1) \cdot P(Q_2) < 1, \quad (15)$$

whence $G_T \ne \varepsilon_0$ and $[X, d, T]_{WM}$ is an equilateral probabilistic metric space.

Remark 2.1. If P in Theorem 2.1 is pervasive, then we note that x_0 in (2.4) can be taken to by any number less than ess diam (X). Also, it follows that under the hypotheses of Theorem 2.1, the associated weakly mixing transformation-generated space $[X, d, T]$ is almost an equilateral probabilistic metric space if, instead of the conditions that the measure P is nonsingular, assume that there is an $x > 0$ such that $P^{(2)}(D(x)) < 1$, because then $G_T \ne \varepsilon_0$. Note that in this case A_0 (in Theorem 2.1) cannot contain any pair of the form (p, p), whence

$$P^{(2)}(\{(p, p) : p \in X\}) = 0. \quad (16)$$

Theorem 2.2. *Under hypotheses of Theorem 2.1, for any $\alpha > 0$ there is a subset A_α of $X \times X$ with $P^{(2)}(A_\alpha) = 1$ such that the sequence of α-moments $(m^{(\alpha)} F_{pq}^{(n)})$,*

$$m^{(\alpha)} F_{pq}^{(n)} := \frac{1}{n} \sum_{m=0}^{n-1} [d(T^m(p), T^m(q))]^\alpha, \tag{17}$$

converges for all pairs (p, q) in A_α. If, in addition, there is an $\alpha_0 \in (0, \infty)$ such that $|x|^{\alpha_0}$ is uniformly integrable in $(F_{pq}^{(n)})$, then for all $\alpha \in (0, \alpha_0]$ and for all pairs (p, q) in $A_\alpha \cap A_0$ (where A_0 as in Theorem 2.1)

$$\lim_{n\to\infty} m^{(\alpha)} F_{pq}^{(n)} = m^{(\alpha)} G_T < \infty, \tag{18}$$

where $m^{(\beta)} G_T$ is α-th moment of G_T as in Theorem 2.1.

Proof. For each fixed $x \in [-\infty, +\infty]$ let $D(x)$ be the subset of $X \times X$ defined by

$$D(x) = \{(p, q) \mid d(p, q) < x\}.$$

If $x \in [-\infty, 0]$, then $D(x)$ is empty, therefore automatically $P^{(2)}$ (where $P^{(2)}$ is the product measure on the set $X \times X$) measurable with $P^{(2)}(D(x)) = G_T(x) = 0$. If $x \in (0, +\infty)$, then it readily follows from the separability of (X, d) that $D(x)$ may be expressed as a countable union of open Cartesian rectangles of the form $A \times B$, where A is an open ball of small diameter, compared to x, and B is the set of all points t such that $d(t, t') < x$ for all points $t' \in A$. Hence $D(x)$ is $P^{(2)}$-measurable. Now, the $P^{(2)}$-measurability of the sets $D(x)$ and the continuity of the metric d together imply that for any α in $(0, \infty)$, the function d^α is $P^{(2)}$-measurable. Since T is weakly mixing with respect to P, the product transformation $T \times T$ (defined on $X \times X$ by $(T \times T)(p, q) = (T(p), T(q)))$ is weakly mixing with respect to $P^{(2)}$. Hence $T \times T (= T^{(2)})$ is ergodic on $X \times X$ and he Birkhoff ergodic theorem can be applied to the sums in (2.6) to yield that for all $\alpha \in (0, \alpha_0)$ and for all pairs (p, q) in $A_\alpha \cap A_0$ the limit in (2.7) exists and is equal to $m^{(\alpha)} G_T$. ⊣

For any positive integer n, we denote the n-fold product $T \times T \times \cdots \times T$ of a transformations T with itself by $T^{(n)}$, and the n-fold product of a probability measure P with itself by $P^{(n)}$. Then the next lemma is readily established:

Lemma 2.1. *Let $(X, \mathcal{A}, P)$ be a probability space. For any positive integer n, let $(X^n, \mathcal{A}^{(n)}, P^{(n)})$ be the probability space in which $\mathcal{A}^{(n)}$ is the product σ-field $\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$ and $P^{(n)}$ the corresponding product measure. For any transformation T on X, let $T^{(n)}$ be the n-fold product of T with itself. If T is weakly mixing with respect to P, then $T^{(n)}$ is weakly mixing with respect to $P^{(n)}$.*

Next, using Lemma 2.1 and the techniques developed in the proof of Theorem 2.1, by working with $T^{(2k)}$ rather than $T^{(2)}$ we can extend Theorem 2.1 to k-tuples of pairs of points as follows:

Theorem 2.3. *Under the hypotheses of Theorem* 2.1*, for any* $k \geq 0$ *there is a subset* A_k *of* X^{2k} *with* $P^{(2k)}(A_k) = 1$ *such that, for any* $2k$*-tuple* $(p_1, q_1, \ldots, p_k, q_k) \in A_k$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \chi_{D(x_1)\times\cdots\times D(x_k)}[(T^{(2)})^m(p_1, q_1), \ldots, (T^{(2)})^m(p_k, q_k)] \\ = G_T(x_1) \cdot \ldots \cdot G_T(x_k). \tag{19}$$

Note that Theorem 2.3 shows, loosely speaking, that under iteration of weakly mixing transformation T, k-tuples of distinct distances $d(p, q)$ behave asymptotically as independent, identically distributed random variables.

Let (X, d) be a metric space with a probability measure P defined on a σ-algebra $\mathcal{A}$ of subsets of X including the Borel sets, and $T : X \to X$ be a transformation that is measure preserving with respect to P. The *α-harmonic diameter* $(0 < \alpha < \gamma)$ *of order* $k\,(k \geq 2)$ of a set $A(\subseteq X)$ is defined by

$$D_k^{(\alpha)}(A) := \sup \left\{ \binom{k}{2} \left(\sum_{1\leq i<j\leq k} [d(p_i, p_j)]^{\alpha-\gamma-1} |\, p_1, \ldots, p_k \in A \right) \right\}. \tag{20}$$

Note that $D_2^{(\alpha)}(A) = \text{diam}\,(A)$.

Comparison of (2.9) with (1.3) and (2.8) and the proof of Theorem 3 in [5] quickly leads to the following conjecture (open problem):
Problem 2.1. Does Theorem 1.1. remain valid when "geometric diameter of order k" is replaced by "α-harmonic diameter of order k"?

References

[1] P. Billingsley, *Ergodic theory and information*, John Wiley & Sons, Inc., New York - London - Sydney, 1965.

[2] I. P. Cornfeld, S. V. Fomin,YA. G. Sinai, *Ergodic Theory*, Springer Verlag, New York - Heidelberg - Berlin, 1982.

[3] T. Erber, B. Schweizer, A. Sklar, *Mixing transformations on metric spaces*, Comm. Math. Phys. **29** (1973), 311 - 317.

[4] H. Fatkić, *On mean values and mixing transformations*, in: Proc. Twenty - fourth Internat. Symposium on Functional Equations, Aequationes Math. **32** (1987), 104 - 105.

[5] H. Fatkić, *Note on weakly mixing transformations*, Aequationes Math. **43** (1992), 38 - 44.

[6] H. Fatkić, *On probabilistic metric spaces and ergodic transformations*, Ph. D. dissertation, Univ. Sarajevo, 2000, 170 pp; Rad. Mat. **10(2)** (2001), 261 - 263.

[7] H. Fatkić, *Characterizations of measurability-preserving ergodic transformations*, Sarajevo J. Math. **1(13)** (2005), 49 - 58.

[8] H. Fatkić, *On measurability-preserving weakly mixing transformations*, The First Bosnian-Herzegovinian Mathematical Conference, The Mathematics Department of the Faculty of Natural Sciences and Mathematics of the University of Sarajevo , July 1, 2006; in: Resume of the Bosnian-Herzegovinian mathematical conference, Sarajevo J. Math., **2(15)** (2006), 259 - 260.

[9] H. Fatkić, *Measurability-preserving weakly mixing transformations*, Sarajevo J. Math. **2(15)** (2006), 159 - 172.

[10] H. Fatkić, M. Brkić, *Strongly mixing transformations and geometric diameters*, Sarajevo J. Math. **8(21)** (2012), no. 2, 245 - 257.

[11] H. Fatkić, S. Sekulović, Hana Fatkić, *Further results on the ergodic transformations that are both strongly mixing and measurability preserving*, The sixth Bosnian-Herzegovinian Mathematical Conference, International University of Sarajevo, June 17, 2011, Sarajevo, B&H; in: Resume of the Bosnian-Herzegovinian mathematical conference, Sarajevo J. Math. **7(20)** (2011), 310 - 311.

[12] H. Fatkić, S. Sekulović, Hana Fatkić, *Probabilistic metric spaces determined by weakly mixing transformations*, in: Proceedings of the 2nd Mathematical Conference of Republic of Srpska - Section of Applied Mathematics, June 8&9, 2012, Trebinje, B&H, pp. 195-208.

[13] M. Grandcolas, *Regular polygons and transfinite diameter*, Bull. Austral. Math. Soc. **62** (2000), 67- 74.

[14] E. Hille, *Topics in classical analysis*, in: Lectures on Modern Mathematics (T. L. Saaty, ed.), Vol. 3, Wiley, New York, 1965, pp. 1-57.

[15] A. Kolesárová, R. Mesiar, C. Sempi, *Measure-preserving transformations, copulæ and compatibility*, Mediterr. J. Math. **5(3)** (2008), 325 - 339.

[16] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. USA. **28** (1942), 535 - 537.

[17] R. Pikuła, *On some notions of chaos in dimension zero*, Colloq. Math. **107** (2007), 167 -177.

[18] R. E. Rice, *On mixing transformations*, Aequationes Math. **17** (1978), 104 - 108.

[19] E. B. Saff, *Logarithmic potential theory with applications to approximation theory*, Surv. Approx. Theory **5** (2010), 165 - 200.

[20] B. Schweizer, A. Sklar, *Probabilistic metric spaces determined by measure-preserving transformations*, Z. Wahrsch. Verw. Geb. **26** (1973), 235 - 239.

[21] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland Ser. Probab. Appl. Math., North-Holland, New York, 1983; second edition, Dover, Mineola, NY, 2005.

[22] C. Sempi, *On weakly mixing transformations on metric spaces*, Rad. Mat. **1** (l985), 3 -7.

[23] S. V. Tikhonov, *Homogeneous spectrum and mixing transformations*, (Russian) Dokl. Akad. Nauk **436 (4)** (2011), 448 - 451; translation in Dokl. Math. **83** (2011), 80 - 83.

[24] P. Walters, *An Introduction to Ergodic Theory*, (GTM, Vol. 79), Springer - Verlag, New York - Heidelberg - Berlin, 1982.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013.

The Hilbert Transform on the Spaces $\mathcal{S}(\mathbb{R})$ and $L^p(\mathbb{R}), 1 < p < \infty$

Amina Šahović
University "Džemal Bijedić" Mostar
amina.sahovic@unmo.ba

Fikret Vajzović
University of Sarajevo

Sead Peco
University "Džemal Bijedić" Mostar
sead.peco@unmo.ba

Original scientific paper

Abstract

We give a new and simple formula for the Hilbert transform on the Schwartz space $S(\mathbb{R})$. Using this result we obtain a simple formula for the Hilbert transform on the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. At the end of this paper, we give a result "stronger" than the previously obtained one for the Hilbert transform on a Hilbert space.

1 Introduction

The Hilbert's result for the Hilbert transform on the space $L^2(\mathbb{R})$ is well-known. In [8] we have studied the Hilbert transform on some Banach spaces. Particularly, in [8] we have obtained the following result (Theorem 1.1) for the Hilbert transform on a Hilbert space.

If $\mathbb{H}$ is a Hilbert space, $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group of isometries in $B(H)$, then the infinitesimal generator A of the group $\{U(t)\}_{t\in\mathbb{R}}$ can be written as $A = i(B_+ - B_-)$, where B_+ and B_- are positive hermitian operators such that $B_+B_- = B_-B_+ = 0$ on $D(A)$.

Theorem 1.1. *(proved in [8]) Let $\mathbb{H}$ be a Hilbert space and let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous group of isometries in $B(\mathbb{H})$ with the infinitesimal generator A. If 0 is not an eigenvalue of the operator A^2, then the Hilbert transform H is defined on the whole $\mathbb{H}$ and*

$$Hx = i(E_+x - E_-x), \quad \text{for all } x \in \mathbb{H},$$

where the orthogonal projections from the space H onto the closed linear hull of H spanned by $R(B_\pm)$ is denoted by $E_\pm$.

Theorem 1.1 is the generalization of the Hilbert's result for the Hilbert transform on the space $L^2(\mathbb{R})$:

- Theorem 1.1 is valid on any Hilbert space (not only on $L^2(\mathbb{R})$);
- in Theorem 1.1, $\{U(t)\}_{t\in\mathbb{R}}$ is any group of isometries (not only the group of translations).

Furthermore, the representation formula for the Hilbert transform in Theorem 1.1. is simple.
In this paper we give a new and simple representation formula for the Hilbert transform on the space $\mathcal{S}(\mathbb{R})$ (Theorem 2.1). Using this result we obtain the equivalence of the formula in Theorem 1.1 for the Hilbert transform on the spaces $L^p(\mathbb{R}), 1 < p < \infty$ (Theorem 2.2). At the end of this paper we formulate and prove a theorem "stronger" than Theorem 1.1(Remark 2.2, Theorem 2.3).

First, we recall some notations and basic notions. Let X be complex Banach space, and let $B(X)$ denote the complex Banach algebra of all bounded linear operators on X.

Definition 1.1. If the map $t \to U(t)$ from the real axis $\mathbb{R}$ into the Banach algebra $B(X)$ satisfies the conditions

i) $U(t_1 + t_2) = U(t_1)U(t_2), \ \ (t_1, t_2 \in \mathbb{R})$,

ii) $U(0) = I, \ \ (I = \text{identity operator})$,

then the family $\{U(t)\}_{t\in\mathbb{R}}$ is called a *one-parameter group of operators in* $B(X)$. The group is said to be *strongly continuous* if $\lim\limits_{t\to 0_+} U(t)f = f$ for every $f \in X$ in the norm of X (for short, in X). If there is a constant $M \geq 1$ such that $\|U(t)\| \leq M$ for all $t \in \mathbb{R}$, then the group $\{U(t)\}_{t\in\mathbb{R}}$ is said to be *bounded.*

The infinitesimal generator A of the group $\{U(t)\}_{t\in\mathbb{R}}$ is defined by

$$Af = \lim_{t\to 0} \frac{U(t)f - f}{f}$$

whenever the limit exists in X. A is a closed linear operator with domain $D(A)$ dense in X. The range of A is denoted by $R(A)$.

Definition 1.2. (see [3]) Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous group of operators in $B(X)$. A continuous linear operator $H_{\epsilon,N}$ $(0 < \epsilon < N < \infty)$ on X is defined as follows

$$H_{\epsilon,N}f := \frac{1}{\pi} \int\limits_{\epsilon\leq|t|\leq N} \frac{U(t)f}{f} dt \ \ (f \in X).$$

If $\lim\limits_{\substack{\epsilon\to 0\\ N\to\infty}} H_{\epsilon,N}f$ exists in X, we denote it by Hf, and call it a *Hilbert transform of* f, i. e.

$$Hf := \lim_{\substack{\epsilon\to 0\\ N\to\infty}} H_{\epsilon,N}f.$$

Let $\{U(t)\}_{t\in\mathbb{R}}$ be a bounded strongly continuous group in $B(X)$ such that $\|U(t)\| \le 1$ for all $t \in \mathbb{R}$ with the infinitesimal generator A.

Note that in this case $\{U(t)\}_{t\in\mathbb{R}}$ is a group of isometries in $B(X)$, i.e. $U(t), t \in \mathbb{R}$ is a bounded linear operator on X that has a bounded inverse linear operator (defined on the whole X), and such that $\|U(t)x\| = \|x\|$ for all $x \in X$.
Namely, from $\|U(t)x\| \le \|U(t)\|\|x\| \le \|x\|$ and
$\|x\| = \|U(-t)U(t)x\| \le \|U(-t)\|\|U(t)x\| \le \|U(t)x\|$ for all $x \in X$ and $t \in \mathbb{R}$ it follows $\|U(t)x\| = \|x\|$ for all $x \in X$ and $t \in \mathbb{R}$.
Let us set

$$C(t) := \frac{1}{2}\left[U(t) + U(-t)\right], \quad t \in \mathbb{R}. \tag{1}$$

One readily verifies that $C(\cdot)$ is a bounded strongly continuous cosine operator function on X, such that $\|C(t)\| \le 1$ for all $t \in \mathbb{R}$, and that its infinitesimal generator is A^2 , i. e. $C(\cdot)$ is a function from $\mathbb{R}$ into $B(X)$ satisfying

a) $C(0) = I$, $(I =$ identity operator),

b) $C(t+s) + C(t-s) = 2C(t)C(s)$, $t, s \in \mathbb{R}$,

c) the function $C(\cdot)f$ is continuous on $\mathbb{R}$ for every $f \in X$,

d) $\|C(t)\| \le 1$ for every $t \in \mathbb{R}$.

The infinitesimal generator of $C(\cdot)$ is defined by the limit in norm as $t \to 0$ of $2\frac{C(t)f-f}{t^2}$, with natural domain (see [1], [2], [6]).
Let $F_a, a \ge 0$ a family introduced in [10] as

$$F_a x := \lim_{\alpha \searrow 0} F_{a,\alpha} x, \quad x \in X, a \ge 0 \tag{2}$$

where

$$F_{a,\alpha} x := \frac{1}{\pi i}\int_0^a du \int_{\alpha+i0}^{\alpha+iu} \left[\lambda R(\lambda^2, A^2) + \overline{\lambda} R(\overline{\lambda}^2, A^2\right] d\lambda, \quad \lambda = \alpha + iy, \quad i = \sqrt{-1}.$$

Here A^2 is the infinitesimal generator of $C(\cdot)$ defined by (1), the resolvent of A^2 is denoted by $R(\lambda^2, A^2)$, i.e. $R(\lambda^2, A^2) = (\lambda^2 - A^2)^{-1} \in B(X)$.

We remark that the family of bounded linear operators $F_a, a \ge 0$ exists for every bounded strongly continuous cosine operator function on X. In [4], it is proved that the limit in (2) exists for all $x \in X$ and $a \ge 0$, and that

$$F_a x = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin at}{t}\right)^2 C(2t)x dt = \frac{2a}{\pi}\int_0^\infty \left(\frac{\sin at}{t}\right)^2 C\left(\frac{2t}{a}\right) x dt. \tag{3}$$

Since $\|C(t)\| \le 1$ for all $t \in \mathbb{R}$, (3) implies that $\|F_a\| \le a$ for all $a \ge 0$ and that the function $a \longmapsto F_a$ is strongly continuous on $[0, +\infty)$.
Let us note some further properties of operators $F_a, a \ge 0$ (proved in [4]):

i)

$$\lim_{a\to+\infty} \frac{F_a x}{a} = x, \quad x \in X \tag{4}$$

ii)

$$F_a F_b x = F_b F_a x = 2\int_0^a F_u x du + (b-a) F_a x, \quad x \in X, 0 \le a \le b. \tag{5}$$

iii) $A^2 F_a x = F_a A^2 x$ for all $x \in D(A^2), a \ge 0$

iv) $F_a x \in D(A^{2k}),\ k = 1, 2, 3, \ldots,\ a \ge 0, x \in X$

v)

$$\text{the set } \bigcup_{a\ge 0} \overline{F_a(X)} \text{ is dense in } X. \tag{6}$$

Let A_0 be an operator defined by $A_0 F_a x := a F_a x - F_a^2 x,\ x \in X, a \ge 0$, where $F_a, a \ge 0$ is a family defined by (2). The operator A_0 has a closed extension denoted by A_+ (proved in [8]). In [8] the operator A_+ is called *the positive square root of* $-A^2$. A detailed investigation of some of its properties is given in [8] and [9].

Theorem 1.2. *(proved in [8]) If X is a complex Banach space, $\{U(t)\}_{t\in\mathbb{R}}$ is a bounded strongly continuous group in $B(X)$ such that $\|U(t)\| \le 1$ for all $t \in \mathbb{R}$ with the infinitesimal generator A, $C(\cdot)$ is defined by (1), A_+ is the positive square root of $-A^2$, Hilbert transform H is defined by Definition 1.2, then*

$$HAx = -A_+ x \text{ for all } x \in D(A^2). \tag{7}$$

2 The Hilbert transform on the spaces $\mathcal{S}(\mathbb{R})$ and $L^p(\mathbb{R}), 1 < p < \infty$

We will need the following lemma.

Lemma 2.1. *If X is a complex Banach space, $\{U(t)\}_{t\in\mathbb{R}}$ is a bounded strongly continuous group in $B(X)$ such that $\|U(t)\| \le 1$ for all $t \in \mathbb{R}$ with the infinitesimal generator A,$C(\cdot)$ is defined by (1), $F_a, a \ge 0$ is defined by (2), Hilbert transform H is defined by Definition 1.2, then*

$$\begin{aligned} HF_a x &= \frac{1}{\pi}\int_0^\infty \frac{at - \sin at}{t^2}\left[U(t) - U(-t)\right] x dt = \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{at - \sin at}{t^2} U(t) x dt \end{aligned} \tag{8}$$

for all $x \in R(A), a \ge 0$.

Proof. By definition of the operator A_+, and using (5) and (3), we get

$$A_+F_ax := aF_ax - F_a^2x = aF_ax - 2\int\limits_0^a F_uxdu \text{ for } a \geq 0,$$

and

$$2\int\limits_0^a F_udu = 2\cdot\frac{2}{\pi}\int\limits_0^a\left[\int\limits_0^\infty \frac{\sin^2 ut}{t^2}C(2t)dt\right]du =$$

$$= \frac{4}{\pi}\int\limits_0^\infty \frac{C(2t)}{t^2}\left[\int\limits_0^a \sin^2 utdu\right]dt =$$

$$= \frac{4}{\pi}\int\limits_0^\infty \left(\frac{a}{2t^2} - \frac{\sin 2at}{4t^3}\right)C(2t)dt$$

Using (1) and the following relation

$$\frac{d}{dt}U(t)x = AU(t)x = U(t)Ax, \quad x \in D(A),$$

we then obtain

$$-A_+F_ax = -a\cdot\frac{2}{\pi}\int\limits_0^\infty \frac{\sin^2 at}{t^2}C(2t)xdt + \frac{4}{\pi}\int\limits_0^\infty\left(\frac{a}{2t^2} - \frac{\sin 2at}{4t^3}\right)C(2t)xdt =$$

$$= -\frac{4}{\pi}\int\limits_0^\infty \frac{a\sin^2\frac{at}{2}}{t^2}C(t)xdt + \frac{4}{\pi}\int\limits_0^\infty\left(\frac{a}{t^2} - \frac{\sin at}{t^3}\right)C(t)xdt =$$

$$= -\frac{2}{\pi}\int\limits_0^\infty \frac{2\sin at - at(1+\cos at)}{t^3}C(t)xdt$$

for $x \in D(A), a \geq 0$.
The last integral is finite and

$$-\frac{2}{\pi}\int\limits_0^\infty \frac{2\sin at - at(1+\cos at)}{t^3}C(t)xdt = -\frac{2}{\pi}\int\limits_0^\infty\left(\frac{at - \sin at}{t^2}\right)'C(t)xdt =$$

$$= -\frac{1}{\pi}\int\limits_0^\infty\left(\frac{at - \sin at}{t^2}\right)'[U(t) + U(-t)]\,xdt.$$

This, and

$$-\frac{1}{\pi}\int\limits_0^\infty\left(\frac{at - \sin at}{t^2}\right)'[U(t) + U(-t)]\,xdt =$$

$$= -\frac{1}{\pi} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int_{\epsilon}^{N} \left(\frac{at - \sin at}{t^2} \right)' [U(t) + U(-t)] \, x dt =$$

$$= -\frac{1}{\pi} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \frac{at - \sin at}{t^2} [U(t) + U(-t)] \, x \Big|_{\epsilon}^{N} +$$

$$+ \frac{1}{\pi} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int_{\epsilon}^{N} \left(\frac{at - \sin at}{t^2} \right) [U(t) - U(-t)] \, Axdt$$

and $\frac{aN - \sin aN}{N^2} \to 0$ as $N \to +\infty$ and $\frac{a\epsilon - \sin a\epsilon}{\epsilon^2} \to 0$ as $\epsilon \to 0$ imply that $\frac{1}{\pi} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int_{\epsilon}^{N} \left(\frac{at - \sin at}{t^2} \right) [U(t) - U(-t)] \, Axdt$ exists, i.e. that the integral

$$\frac{1}{\pi} \int_{0}^{\infty} \left(\frac{at - \sin at}{t^2} \right) [U(t) - U(-t)] \, Axdt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{at - \sin at}{t^2} U(t) Axdt$$

is finite.
So,
$-A_+ F_a x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{at - \sin at}{t^2} U(t) Axdt$ for $x \in D(A), a \geq 0$.
Since $AF_a x = F_a Ax$ for all $x \in D(A), a \geq 0$ and (6), from the last equality and from (7) (Theorem 1.2.), we get (8) for all $x \in R(A)$ and $a \geq 0$. The lemma is proved. ⊣

From now, $X = L_p(\mathbb{R}), 1 < p < \infty$, $\{U(t)\}_{t \in \mathbb{R}}$ is the group of right translations in $B(X)$ defined by

$$[U(t) f](x) := f(x - t), \quad f \in X, t \in \mathbb{R}, x \in \mathbb{R}. \tag{9}$$

It is well-known that $\{U(t)\}_{t \in \mathbb{R}}$ is a strongly continuous group of isometries in $B(X)$ ($\|U(t)\| = 1$ for all $t \in \mathbb{R}$) . We denote its infinitesimal generator by A. $\mathcal{S}(\mathbb{R})$ will denote the space of functions ϕ on $\mathbb{R}$ having derivates of all order and satisfying $\sup_{x \in \mathbb{R}} \left| x^k \phi^{(n)}(x) \right| < \infty$ for all indicies $k, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, where $\phi^{(n)}$ denotes the derivate of n order of the function ϕ. It is well-known that $\mathcal{S}(\mathbb{R})$ is a Fréchet space with the system of semi-norms $\left\{ \sup_{x \in \mathbb{R}} \left| x^k \phi^{(n)}(x) \right| ; k, n \in \mathbb{N}_0 \right\}$.

Definition 2.1. The Fourier transformation $\hat{f}$ of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{f}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) dx,$$

and the inverse of the Fourier transformation $\check{f}$ of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\check{f}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx.$$

Remark 2.1. a) It is known (see [5]) that the Fourier transformation "$\hat{}$" and the inverse of the Fourier transformation "$\check{}$" from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ are linear continuous in the topology of $\mathcal{S}(\mathbb{R})$, and that $\check{\hat{f}} = f,\ f \in \mathcal{S}(\mathbb{R})$.

b) $f \in \mathcal{S}(\mathbb{R})$ can be written in the following form (see [7])

$$f(x) = \int_0^\infty [a(z)\cos zx + b(z)\sin zx]\, dz, \tag{10}$$

where

$$a(z) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos zudu \ \text{ and } \ b(z) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin zudu. \tag{11}$$

Theorem 2.1. *The Hilbert transform H on the space $\mathcal{S}(\mathbb{R})$ has the representation*

$$(Hf)(x) = \int_0^\infty [a(z)\sin zx - b(z)\cos zx]\, dz, \quad f \in \mathcal{S}(\mathbb{R}), \tag{12}$$

where $a(z)$ and $b(z)$ are defined by (11).

Proof. Let us recall that 0 is not an eigenvalue of the infinitesimal generator A of the group $\{U(t)\}_{t\in\mathbb{R}}$ defined by (9) in $B(X)$, $X = L_p(\mathbb{R}), 1 < p < \infty$. Therefore, by lemma 2.1. and (9), we obtain

$$(HF_a f)(x) = \frac{1}{\pi}\int_0^\infty \frac{at - \sin at}{t^2}[f(x-t) - f(x+t)]\, dt \tag{13}$$

for $f \in \mathcal{S}(\mathbb{R})$ and $a \geq 0$.
But, by remark 2.1 b),

$$f(x+t) - f(x-t) =$$
$$= \int_0^\infty \{a(z)[\cos z(x+t) - \cos z(x-t)] + b(z)[\sin z(x+t) - \sin z(x-t)]\}\, dz =$$
$$= \int_0^\infty [-2a(z)\sin zx \sin zt + 2b(z)\cos zx \sin zt]\, dz$$

for $f \in \mathcal{S}(\mathbb{R})$ and $a \geq 0$.
Thus, from (13), we get

$$(HF_a f)(x) = -\frac{2}{\pi}\int_0^\infty \frac{at - \sin at}{t^2}\int_0^\infty [b(z)\cos zx \sin zt - a(z)\sin zx \sin zt]\, dzdt =$$

$$= -\frac{2}{\pi}\int\limits_0^\infty [b(z)\cos zx - a(z)\sin zx]\, dz \int\limits_0^\infty \frac{at-\sin at}{t^2}\sin ztdt$$

for $f \in \mathcal{S}(\mathbb{R})$ and $a \geq 0$.
Since

$$\int\limits_0^\infty \frac{at-\sin at}{t^2}\sin ztdt = \begin{cases} 0 & \text{for } \ a \leq z \\ (a-z)\frac{\pi}{2} & \text{for } \ a > z \end{cases},$$

now we have

$$(HF_a f)(x) = -\int\limits_0^a [b(z)\cos zx - a(z)\sin zx]\,(a-z)dz$$

for $f \in \mathcal{S}(\mathbb{R})$ and $a \geq 0$.
Since the operator H is continuous on X (Riesz's result) and (4), from the last equality it follows, by dividing the last equality by a and by taking the limit as $a \to \infty$ on both of its sides, that (12) holds. The theorem is proved. ⊣

Now, using this result (Theorem 2.1) we can obtain the equivalence of the formula in Theorem 1.1. for the Hilbert transform on the spaces $L_p(\mathbb{R})\,, 1 < p < \infty$.

Theorem 2.2. *If $X = L_p(\mathbb{R})\,, 1 < p < \infty$ then there are subspaces L_1 and L_2 of the space X such that $X = L_1 \dot{+} L_2$ (i.e. X is a direct sum of subspaces L_1 and L_2 of the space X), and the Hilbert transform H has the representation*

$$Hf = i(P_- f - P_+ f)\,, \quad f \in X,$$

where P_- and P_+ are continuous projections from X onto L_1 and L_2 respectively.

Proof. From (12) (Theorem 2.1) and (10) it follows

$$f(x) + i(Hf)(x) = \int\limits_0^\infty [a(z)\cos zx + b(z)\sin zx]\, dz +$$

$$+\, i\int\limits_0^\infty [a(z)\sin zx - b(z)\cos zx]\, dz =$$

$$= \int\limits_0^\infty \left[a(z)e^{izx} - ib(z)e^{izx}\right] dz = \int\limits_0^\infty e^{izx}\,[a(z) - ib(z)]\, dz =$$

$$= \int\limits_0^\infty e^{izx}\left[\frac{1}{\pi}\int\limits_{-\infty}^\infty f(u)\cos zudu - i\frac{1}{\pi}\int\limits_{-\infty}^\infty f(u)\sin zudu\right] dz =$$

$$= \int_0^\infty e^{izx} \left[\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-izu} f(u) du \right] dz =$$

$$= \int_0^\infty e^{izx} \sqrt{\frac{2}{\pi}} \hat{f}(z) dz = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{izx} \hat{f}(z) dz$$

for $f \in \mathcal{S}(\mathbb{R})$.
So,

$$f(x) + i(Hf)(x) = 2\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{izx} \hat{f}(z) dz, \quad f \in \mathcal{S}(\mathbb{R}). \tag{14}$$

Let $X = L_p(\mathbb{R}), 1 < p < \infty$.
Set $L^{'} := \left\{ f \in \mathcal{S}(\mathbb{R} \middle| \text{supp} \hat{f} \subseteq (-\infty, 0] \right\}$, and $L^{''} := \left\{ f \in \mathcal{S}(\mathbb{R} \middle| \text{supp} \hat{f} \subseteq [0, +\infty) \right\}$.
Let us denote the closure (in X) of $L^{'}$ and $L^{''}$ by L_1 and L_2 respectively. It is well-known from the theory of Hardy spaces that the set $L^{'} + L^{''}$ is dense in X. Let f be any element belonging to X. Then there are $f_n^{'} \in L^{'}$ and $f_n^{''} \in L^{''}$, $n = 1, 2, \ldots$, such that

$$f_n^{'} + f_n^{''} \xrightarrow{L^p} f \ \ (n \to \infty). \tag{15}$$

Since H is continuous on X, it follows

$$H(f_n^{'} + f_n^{''}) \xrightarrow{L^p} Hf \ \ (n \to \infty). \tag{16}$$

From (14) we get

$$f^{'}(x) + i(Hf^{'})(x) = 0 \ \text{ for all } \ f^{'} \in L^{'},$$

and

$$f^{''}(x) + i(Hf^{''})(x) = 2f^{''}(x) \ \text{ for all } \ f^{''} \in L^{''},$$

i.e.

$$(Hf^{'})(x) = if^{'}(x) \ \text{ for all } \ f^{'} \in L^{'}, \text{ and } (Hf^{''})(x) = -if^{''}(x) \ \text{ for all } \ f^{''} \in L^{''}.$$

Thus,

$$H\left(f_n^{'} + f_n^{''}\right) = i\left(f_n^{'} - f_n^{''}\right), \ \text{ for every } \ n = 1, 2, \ldots. \tag{17}$$

Hence,

$$2if_n^{'} = i\left(f_n^{'} + f_n^{''}\right) + H\left(f_n^{'} + f_n^{''}\right), \ \text{ for every } \ n = 1, 2, \ldots.$$

From this, (15), and (16) it follows that the sequence $\left\{f_n^{'}\right\}_{n=1}^{\infty}$ converges to a limit point $f_1 \in L_1$ as $n \to \infty$, and then from (15) it follows that the sequence $\left\{f_n^{''}\right\}_{n=1}^{\infty}$ converges to a limit point $f_2 \in L_2$ as $n \to \infty$. Therefore, from (15),

(16) and (17) we obtain
$f = f_1 + f_2$, and $Hf = H(f_1 + f_2) = i(f_1 - f_2) = i(P_- f - P_+ f)$,
where P_- and P_+ are continuous projections from X onto L_1 and L_2 respectively. Note, the uniqueness of f_1 and then of f_2 immediately follows from $2if_1 = Hf + if$. Also from this, since the operator H is continuous, it easily follows that P_- and P_+ are continuous. The theorem is proved.

$\dashv$

Remark 2.2. It is easy to see that the following "stronger" than Theorem 1.1 Theorem 2.3 is valid.

Theorem 2.3. *If $\mathbb{H}$ is a Hilbert space and $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group of isometries in $B(\mathbb{H})$ with the infinitesimal generator A, then there are subspaces $\mathbb{H}_0, \mathbb{H}_+, \mathbb{H}_-$ of the space $\mathbb{H}$ satisfying:*

a) $\mathbb{H}$ is a direct and orthogonal sum of subspaces $\mathbb{H}_0, \mathbb{H}_+, \mathbb{H}_-$ of the space H, i.e.

$$\mathbb{H} = \mathbb{H}_0 \oplus \mathbb{H}_+ \oplus \mathbb{H}_-;$$

b) the Hilbert transform H is defined (and continuous) on the whole $\mathbb{H}$ and

$$Hx = i(E_+x - E_-x) \text{ for all } x \in \mathbb{H},$$

where E_0, E_+ and E_- are orthogonal projections from $\mathbb{H}$ onto $\mathbb{H}_0, \mathbb{H}_+$ and $\mathbb{H}_-$ respectively.

The proof of Theorem 2.3 immediately follows from Theorem 1.1 when applied on $\mathbb{H} \ominus \mathbb{H}_0$, where $\mathbb{H} = \mathbb{H}_0 \oplus (\mathbb{H} \ominus \mathbb{H}_0)$, and $\mathbb{H}_0 = \{x \in H;\ Ax = 0\}$, and from the fact that $Hx = 0, x \in \mathbb{H}_0$ (it easily follows from $U(t)x = x, x \in \mathbb{H}_0$).

References

[1] Fattorini H. O., Ordinary Differential Equations in Linear Topological Spaces I, J. Diff. equ. 5 (1968), 72-105.

[2] Goldstein J., Semigroup of Linear Operators and Applications, Oxford 1985.

[3] Ishikawa S., Hilbert Transforms on One-parameter Groups of Operators, Tokyo, J. Math. 9 (1986), 383-393.

[4] Radić M., Vajzović F., On the Bounded Cosine Operator Function in Banach Space, Mat. Vesnik, 39 (1987), 187-204.

[5] Reed M., Simon B., Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness, Academis Press New York San Francisco London, 1978.

[6] Sova, M., Cosine Operator Functions, Rozprawy Mat. J. 49 (1966), 1-47.

[7] Stein E. M., Сингуларне интеграли и дифференциалне своиства функции, "Мир", 1978.

[8] Šahović A., Vajzović F., Cosine Operator Functions and Hilbert Transforms, NSJOM, Vol. 35., No. 2. (2005), 41-55.

[9] Vajzović F., Šahović A., Cosine Operator Functions and Hilbert Transforms II, Proc. III congress of mathematicians of Macedonia, Struga, Macedonia (2005), 347-356.

[10] Vajzović F., On the Cosine Operator Function, Glasnik Mat. 22 (42) (1987), 381-40

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Konstrukcija rješenja graničnog zadatka sa linearnim kašnjenjem

Milenko Pikula
Univerzitet u Istočnom Sarajevu, Filozofski fakultet Pale
pikulam1947@gmail.com

Ismet Kalčo
Politehnički fakultet univerziteta u Zenici
hanasim@windowslive.com

E. F. Jakupović
Panevropski univerzitet Aperion Banja Luka
esad.j@apeiron-uni.eu

Originalni naučni rad

Apstrakt

U ovom radu se konstruiše rješenje i karakteristična funkcija graničnog zadatka generisanog diferencijalnom jednačinom sa linearnim kašnjenjem, potom se uspostavlja važna relacija između potencijala q na segmentu $[\xi_1, \pi]$ i takozvane prelazne funkcije $\widetilde{q}$ na segmentu $[-\pi, \pi]$. Taj dobijeni rezultat otvara mogućnost rješenja inverznog zadatka.

1 Uvod

Ovaj rad je posvećen uspostavljanju relacije između potencijala q i takozvane prelazne funkcije $\widetilde{q}$ spektralnog zadatka sljedećeg tipa

$$-y''(x) + q(x)\, y(x - \tau(x)) = \lambda y(x) \tag{1}$$

$$y(x - \tau(x)) \equiv 0 \quad x < \tau(x) \tag{2}$$

$$y'(0) - hy(0) = 0 \tag{3}$$

$$y'(\pi) + Hy(\pi) = 0 \tag{4}$$

Pretpostavimo da je $q \in L_2[0, \pi]$. Osim toga, neka je $\tau(x) = \alpha x + \beta, \quad \alpha, \beta \in R^+$ i $\alpha < 1$. Funkcija $\gamma(x) = x - \tau(x) = (1-\alpha)x - \beta$ je strogo rastuća, jer je $\gamma'(x) = 1 - \alpha > 0$. Njena inverzna funkcija je

$$\gamma^{-1}(y) = \frac{1}{1-\alpha} y + \frac{\beta}{1-\alpha} \quad \text{i vrijedi } \gamma^{-1}(0) = \frac{\beta}{1-\alpha} = \xi_1.$$

Uslov (2) ispunjen je na razmaku $[0, \xi_1)$.

Ograničimo se na slučaj $h = H = \infty$. Tada iz (1) i (3) dobijamo integralnu jednačinu

$$y(x,z) = \sin zx + \frac{1}{z}\int_0^x q(t_1)\sin z(x-t_1)y(t_1-\tau(t_1),z)\,dt_1 \quad z^2=\lambda \tag{5}$$

Rješenju zadatka (1-3) posvećena je monografija [4].

2 Rješenje zadatka (1,2,3)

Za $x\in[0,\xi_1)$ iz (5) dobijamo

$$y(x,z) = \sin zx$$

Na razmaku $(\xi_1,\pi]$ jednačina (5) postaje

$$y(x,z) = \sin zx + \frac{1}{z}\int_{\xi_1}^x q(t_1)\sin z(x-t_1)y(t_1-\tau(t_1),z)\,dt_1, \quad z^2=\lambda \tag{6}$$

Lema 2.1. Postoji $k_0\in N_0$ takav da važi

$$\xi_{k_0} < \pi \le \xi_{k_0+1}; \quad \xi_l = \gamma^{-l}(0).$$

Dokaz 2.1. Ako je $\frac{\beta}{1-\alpha}\ge\pi$ tada je $k_0 = 0$ pa se nema šta dokazivati. Pretpostavimo da je $\xi_1 = \gamma^{-1}(0) = \frac{\beta}{1-\alpha} < \pi$. Konstruišimo niz

$$\xi_l = \gamma^{-l}(0) = \gamma^{-1}\left(\gamma^{-l+1}(0)\right), l = 2,3,\ldots$$

Za $l = 2$ dobijamo

$$\xi_2 = \gamma^{-l}(\xi_1) = \gamma^{-1}\left(\frac{\beta}{1-\alpha}\right) = \frac{\beta}{(1-\alpha)^2} + \frac{\beta}{1-\alpha}.$$

Nastavljajući proceduru konstrukcije članova niza ξ_l dobijamo

$$\xi_l = \sum_{k=1}^{l}\frac{\beta}{(1-\alpha)^k} = \frac{\beta}{1-\alpha}\left[1 + \frac{1}{1-\alpha} + \cdots + \frac{\beta}{(1-\alpha)^{l-1}}\right].$$

Zbog $0 < \alpha < 1$ važi $\frac{1}{1-\alpha} > 1$ pa geometrijski red

$$\sum_{k=0}^{\infty}\frac{1}{(1-\alpha)^k} \text{ divergira za } +\infty.$$

Stoga postoji najmanji $k_0\in N_0$ takav da važi $\xi_{k_0+1}\ge\pi$ dok je $\xi_{k_0} < \pi$.

Posledica 2.0.1.

$$(\xi_1, \pi] = \bigcup_{l=2}^{k_0} (\xi_{l-1}, \xi_l] \cup (\xi_{k_0}, \pi] \qquad (*)$$

Na osnovu (*) rješenje integralne jednačine (6) se konstruiše metodom promjenjivih koraka na razmaku $(\xi_{l-1}, \xi_l]$.

Uvedimo sljedeće funkcije

$$a_{s^2}(x,z) = \int_{\xi_1}^{x} q(t_1) \sin z(x-t_1) \sin z\gamma(t_1)\, dt_1$$

$$a_{s^{l+1}}(x,z) = \int_{\xi_l}^{x} q(t_1) \sin z(x-t_1) a_{s^l}(\gamma(t_1), z)\, dt_1, \quad l = \overline{1, k_0}$$

Lema 2.2. Rješenje $y_l(x,z)$ na razmaku $(\xi_{l-1}, \xi_l]$, $l = \overline{1, k_{0-1}}$ dato je sa

$$y_l(x,z) = \sin zx + \sum_{k=1}^{l} \frac{1}{z^k} a_{s^{k+1}}(x,z) \qquad (7)$$

Dokaz se izvodi metodom matematičke indukcije. Radi ilustracije raspišimo rješenje za $l = 2$.

$$y_2(x,z) = \sin zx + \frac{1}{z} \int_{\xi_1}^{x} q(t_1) \sin z(x-t_1) \sin z\gamma(t_1)\, dt_1$$
$$+ \frac{1}{z^2} \int_{\xi_2}^{x} q(t_1) \sin z(x-t_1) \int_{\xi_1}^{\gamma(t_1)} q(t_2) \sin z(\gamma(t_1 - t_2)\, sinz\gamma(t_2)\, dt_2 dt_1$$

Na završnom razmaku $(\xi_{k_0}, \pi] \subset (\xi_{k_0}, \xi_{k_0+1}]$ rješenje ima oblik

$$y_{k_0}(x,z) = \sin zx + \frac{1}{z} a_{s^2}(x,z) + \sum_{k=2}^{k_0} \frac{1}{z^k} a_{s^{k+1}}(x,z) \qquad (8)$$

Primedba 2.1. U [5] je posmatran slučaj bilinearnog kašnjenja sa ciljem konstrukcije asimptotike svojstvenih vrijednosti.

3 Karakteristična funkcija

Stavljajući $x = \pi$ i koristeći granični uslov $y(\pi, z) = 0$ dobijamo karakterističnu funkciju $F(z) = y(\pi, z)$, $z \in C$ operatora L.

Dakle,

$$F(z) = \sin \pi z + \frac{1}{z} a_{s^2}(\pi, z) + \sum_{k=2}^{k_0} \frac{1}{z^k} a_{s^{k+1}}(\pi, z) \tag{9}$$

Funkcija F je cijela funkcija eksponencijalnog tipa sa prividnim singularitetom u nuli. Zapravo $\lim_{z\to 0} F(z) = 0$.

Koristeći elementarni identitet

$$\sin z(\pi - t_1) \sin z(t_1 - \tau(t_1)) = \frac{1}{2}\left[\cos z(\pi - 2t_1 + \tau(t_1) - \cos z(\pi - \tau(\tau_1)\right].$$

imamo,

$$\begin{aligned} a_{s^2}(\pi, z) &= \frac{1}{z} \int_{\xi_1}^{\pi} q(t_1) \cos z(\pi - 2t_1 + \tau(t_1))\, dt_1 - \frac{1}{z} \int_{\xi_1}^{\pi} q(t_1) \cos z(\pi - \tau(t_1))\, dt_1 \\ &= \frac{1}{2} \int_{\xi_1}^{\pi} q(t_1) \cos z(\pi + (\alpha - 2)t_1 + \beta)\, dt_1 - \frac{1}{2} \int_{\xi_1}^{\pi} q(t_1) \cos z(\pi - \alpha t_1 - \beta)\, dt_1 \end{aligned}$$

Neka je $\theta = \pi + 2\beta + (\alpha - 2)t_1,\ dt_1 = \frac{d\theta}{\alpha - 2} \frac{t_1|\xi_1|\pi}{\theta|\pi - \xi_1|\gamma(\pi)}$
odnosno $\theta = \pi - \alpha t_1 - \beta,\ dt_1 = \frac{-d\theta}{\alpha} \frac{t|\xi_1|\pi}{\theta|\pi - \xi_1|\gamma(\pi)}$.
Tada važi

$$a_{s^2}(\pi, z) = \frac{1}{2} \int_{-\gamma(\pi)}^{\pi - \xi_1} \frac{1}{2 - \alpha} q\left(\frac{\pi + \beta - \theta}{2 - \alpha}\right) \cos\theta d\theta - \frac{1}{2} \int_{\gamma(\pi)}^{\pi - \xi_1} \frac{1}{\alpha} q\left(\frac{\pi - \beta - \theta}{\alpha}\right) \cos\theta d\theta$$

Definišimo takozvanu prelaznu funkciju $\widetilde{q}$ na sljedeći način

$$\widetilde{q} = \begin{cases} 0, & \theta \in (-\pi, -\gamma(\pi)) \cup (\pi - \xi_1, \pi) \\ \frac{1}{2-\alpha} q\left(\frac{\pi+\beta-\theta}{2-\alpha}\right), & \theta \in (-\gamma(\pi), \gamma(\pi)) \\ \frac{1}{2-\alpha} q\left(\frac{\pi+\beta-\theta}{2-\alpha}\right) - \frac{1}{\alpha} q\left(\frac{\pi-\beta-\theta}{\alpha}\right), & \theta \in (\gamma(\pi), \pi - \xi_1) \end{cases} \tag{10}$$

Iz (10) vidimo da poznati potencijal q na segment $[\xi_1, \pi]$ u pri poznatom kašnjenju $\tau(x) = \alpha x + \beta, \xi_1 = \frac{\beta}{1-\alpha}$ jednoznačno definiše prelaznu funkciju $\widetilde{q}$ na segmentu $[-\pi, \pi]$. Zapravo, dato kašnjenje znači poznavanje funkcije $\gamma(x)$ pa su poznati i brojevi $-\gamma(\pi)$ i $\gamma(\pi)$.

Postavimo obrnuto pitanje: da li data prelazna funkcija $\widetilde{q}$ dobro definiše potencijal q?

Parametre $\xi_1, \pm\gamma(\pi), \alpha$ i β smatramo da su poznati. Radi davanja odgovora na postavljeno pitanje, izvršimo particiju segmenata $[\xi_1, \pi]$ i $[-\gamma(\pi), \pi - \xi_1]$ u skladu sa prirodom preslikavanja koji ostvaruju funkcije $\theta_1 = \pi + \beta - (2 - \alpha)t_1$ odnosno $t_1 = \frac{\pi + \beta - \theta_1}{2 - \alpha}$.

Funkcija $\theta_1 : [\xi_1,\pi] \to [-\gamma(\pi), \pi-\xi_1]$ je strogo opadajuća. Istovremeno funkcija $\theta_1^* = \pi - \beta - \alpha t_1 : [\xi_1,\pi] \to [\gamma(\pi), \pi-\xi_1]$ je takođe strogo opadajuća. Stavimo $\delta_1 = \frac{2\beta+\alpha\pi}{2-\alpha}$. Tada imamo

$$\begin{aligned}\theta_1([\delta_1,\pi]) &= [-\gamma(\pi), \gamma(\pi)], \quad \theta_2([\delta_1,\pi]) = [\gamma(\pi), \eta_1], \\ \eta_1 &= \theta_2(\delta_1) = \pi - \beta - \frac{\alpha^2\pi + 2\alpha\beta}{2-\alpha}\end{aligned}$$

Neka je $\delta_0 = \pi, \eta_0 = \gamma(\pi)$. Dalje, $\theta_1(\delta_2) = \eta_1, \theta_1^*(\delta_2) = \eta_2$

U opštem slučaju

$$\theta_1(\delta_k) = \eta_{k-1}, \quad \theta_1^*(\delta_k) = \eta_k \quad K \in N_0$$

Navedimo vrijednosti nekoliko prvih članova niza δ_k :

$$\delta_2 = \frac{4\beta+\alpha^2\pi}{(2-\alpha)^2}, \ \delta_3 = \frac{8\beta - 4\alpha\beta + \alpha^3\pi + 2\alpha^2\pi}{(2-\alpha)^3}, \ \delta_4 = \frac{16\beta - 16\alpha\beta + \alpha^4\pi + 2\alpha^2\pi}{(2-\alpha)^4}$$

$$\delta_5 = \frac{32\beta - 48\alpha\beta - 8\alpha^3\pi + 32\alpha^2\pi + 2\alpha^4\pi + \alpha^5\pi}{(2-\alpha)^5}$$

Indukcijom dokazujemo relacije

$$\begin{aligned}\delta_k - \xi_1 &= \tfrac{\alpha^k}{(2-\alpha)^k(1-\alpha)}\gamma(\pi) \\ \pi - \xi_1 - \eta_k &= \tfrac{\alpha^{k+1}}{(2-\alpha)^k(1-\alpha)}\gamma(\pi), \quad K \in N\end{aligned} \tag{11}$$

Na osnovu (11) možemo pisati

$$[\xi_1,\pi] = \bigcup_{k=1}^{\infty}[\delta_k,\delta_{k-1}], \quad [\gamma(\pi), \pi-\xi_1] = \bigcup_{k=1}^{\infty}[\eta_{k-1},\eta_k] \tag{12}$$

Iz (10) slijedi

$$q(t_1) = (2-\alpha)\widetilde{q}(\theta_1) = (2-\alpha)\widetilde{q}(\pi+\beta-(2-\alpha)t_1), \quad t_1 \in [\delta_1,\delta_0] \tag{12$'$}$$

Dakle, na segmentu $[\delta_1,\pi] = [\delta_1,\delta_0]$ potencijal q je jednoznačno određen pomoću prelazne funkcije $\widetilde{q}$ na segmentu $[\gamma(\pi), \eta_0]$. Za $t_1 \in [\delta_2, \delta_1)$ imamo $\theta_1 \in (\eta_0, \eta_1]$ i vrijedi jednakost

$$q(t_1) = (2-\alpha)\widetilde{q}(\theta_1) + \frac{2-\alpha}{\alpha}q(t_1^*), \quad t_1^* = \frac{-2\beta + (2-\alpha)t_1}{\alpha} \in [\delta_1,\delta_0] \tag{12$''$}$$

Vrijednosti funkcije q na razmaku $[\delta_1, \delta_0)$ su poznate iz (12′), a $\widetilde{q}(\theta_1)$ je unaprijed data pa je funkcija q pomoću (12″) dobro definisana i na $[\delta_2, \delta_1)$.

Potpuno analogno nastavljamo proces konstrukcije potencijala q na svakom razmaku $[\delta_{k+1}, \delta_k)$, $k \in N$ na osnovu poznatog potencijala na razmaku $[\delta_k, \delta_{K-1})$, $k \in N$.

Dobijeni rezultat znači da se rješavanje obrnutog zadatka za posmatranu jednačinu svodi na određivanje prelazne funkcije operatora.

Primedba 3.1. U [11] i [12] koristi se prelazna funkcija za rješavanje inverznih zadataka pri homogenom kašnjenju to jest pri otklonjenom argumentu αx, $0 < \alpha < 1$.

Literatura

[1] Наймарк М. А., Линейные дифференциальные операторы, Москва, Наука, 1969.

[2] Гельфанд И. М., Левитан Б. М., Об определении дифференциального уравнения по его спектралной функции, Изв. АН СССР, Сер. мат(1951)Т. 15, С. 309-360.

[3] Садовничй В. А., Теория операторов, Издательство Москоского университета, 1979.

[4] Норкин С. Б., Дифференциаль ные уравнения второго порядка с запаздывающим аргументом, Москва, Наука 1969.

[5] Пикула М., О регуляризованных следах дифференциального оператора типа Штурма-Лиувилля с запаздывающим аргументом, Дифф. уравнения 1 (1990)103-109.

[6] Пикула М., О регуляризованных следах дифференциальны операторов высших порядков с запаздывающим аргументом, Дифф. уравнения 6 (1985)956-991.

[7] Pikula M., Asimptotika svojstvenih vrijednosti i regularizovani tragovi linearnih diferencijalnih operatora, doktorska disertacija, Sarajevo 1983.

[8] Lazovic R. and Pikula M., Regularized trace of the operator applied to solving inverse problems, Radovi Matematički 11 (2002)

[9] Pavlović N., Asimptotika svojstvenih vrijednosti linearne diferencijalne jednačine sa kašnjenjem, Zbornik radova sa naučnog skupa, Nauka i nastava na univerzitetu, Pale, 17-18. maj 2008., knjiga 3 tom 2, 591-604.

[10] Pikula M., Markovic O., Nedic D., Computations of first forced regularized trace Sturm-Liouville operator with homogeneous delay, Paper Proceedings Buisness-Technical Colleg, Uzice,(2012.), 185-187

[11] Freiling G. and Yurko V, Inverse problems for Sturm-Liouville diferential operators with a constant delay, Applied Mathematical Letters,(2012).

[12] Pikula M., Vladicic V. and Markovic O., A solution to the inverse problem for the Sturm-Liouville-type equation with a delay, Filomat 27:7(2013), 1237-1245

[13] Pikula M., Vladicic V. and Nedic D., Inverse problems for Sturm-Liouville differentials operators with a homogeneous delay,(accepted for publishing in Sib. Math. Journal)

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Zero Solution of Complex Differential Equation of First Order

Jelena Vujaković
University in Kosovska Mitrovica, Faculty of Sciences and Mathematics,
enav@ptt.rs

Original scientific paper

Abstract

The study of complex differential equations in recent years has open numerous questions regarding the determination of frequencies of zero solutions, dispositions of zero solutions, oscillatory of solutions, asymptotic behavior, growth rank and so on. This paper presents the result of research seeking for model of determination of location of zero solutions of complex homogenous differential equation of first order with analytic coefficients. Using the sequence-iteration method, which seemed to us better in applications, the new approach to problem solving has been developed and opened the perspectives in further research.

1 Introduction

Complex homogeneous linear differential equation (CHLDE) of the first order is equation

$$\Phi\left(z, w\left(z\right), w\prime\left(z\right)\right) = 0 \tag{1.1}$$

where Φ is given function, $w\left(z\right) = F\left(z\right) = u + iv$ is complex function of complex variable $z = x + iy$, where $u = u\left(x, y\right)$ and $v = v\left(x, y\right)$ are continuously differentiable functions in area G (for the further details, see [1]).
As derivative of a function $w\left(z\right)$ depends on four partial derivatives $u\prime_x, u\prime_y, v\prime_x,\ v\prime_y$, (1.1) transforms into equation

$$\Phi\left(x, y; u\left(x, y\right), v\left(x, y\right); u\prime_x, u\prime_y, v\prime_x, v\prime_y\right) = 0 \tag{1.2}$$

Here, according to the definition of equality of complex numbers we obtain the real system

$$\begin{aligned} \Phi_1\left(x, y; u\left(x, y\right), v\left(x, y\right); u\prime_x, u\prime_y, v\prime_x, v\prime_y\right) &= 01.3 \\ \Phi_2\left(x, y; u\left(x, y\right), v\left(x, y\right); u\prime_x, u\prime_y, v\prime_x, v\prime_y\right) &= 0 \end{aligned} \tag{1}$$

of partial differential equations (PDEs) of the first order with two unknown real functions $u(x,y)$ and $v(x,y)$.
From the normal form of CHLDE of the first order (1.1)

$$w'(z) = F(z, w(z)) \tag{1.4}$$

we obtain complex integral $w(z) = \int_L F(z, w(z))\, dz + c$, where L is path that connects points z_0 and z and where $c = a + ib$ is orbitrary complex constant. Independence from the path is valid only if at (1.1) Φ is analytical function from its arguments. Then for solutions $w(z) = u(x,y) + iv(x,y)$ Cauchy–Riemann conditions must be applied in advance.
The idea is to observe at the system of PDE (1.3) and assume that $w(z)$ is an analytical function. The problem is to find a zero solution $w(z) = 0$, ie zero real functions $u(x,y) = 0$ and $v(x,y) = 0$ of a region of G (see [7])
Let G is circular sector $|z| = R,\ 0 \leq \arg z \leq \frac{\pi}{2}$. There may or may not be a certain number of zero solutions $w(z) = 0$. Those zeros can be "included", if we draw a direction $y = kx,\ k = \tan\varphi$, where k continuously changes. Direction $y = kx$ shall encompass all zeros from G. At a certain direction there shall be one zero, and at another one more zeros, but isolated and finitely many zeros, at certain direction not a single one zero. Therefore, the number of zeros shall depend on k, and locations of zeros shall depend on zeros of a function

$$u(x,y) = u(x,kx) = U(x)\,,\ v(x,y) = v(x,kx) = V(x)\,. \tag{1.5}$$

From Cauchy–Riemann conditions and from real functions (1.5) by substitution of $y = kx$ we obtain one complete real homogenous linear differential equation of the second order $U''(x) + A(x)U'(x) + B(x)U(x) = 0$, for which Sturm theorems apply.
In the following text we consider first order complex homogenous linear differential equations in form

$$\frac{dw}{dz} + a(z)\, w(z) = 0 \tag{1.6}$$

where $a(z) = \alpha(x,y) + i\beta(x,y)$ is given analytical function.
It is easy to prove that non-trivial solution of CHLDE (1.6), has no zeros. Only trivial solution has zeros. If $a(z) = 0$ then solution of CHLDE $\frac{dw}{dz} = 0$ has only one zero at point $z = 0$.These discussion can be easily generalized on the case of CHLDE of n-th order $\frac{d^n w}{dz^n} = 0$, because after successive integrations we obtain polynomial $P_{n-1}(z)$, which has exactly $n-1$ zeros, real or complex, simple or multiple. Also, we can link it with small Picard theorem. Namely, equation $f(z) = A$, where $f(z)$ is analytical function, A complex constant, has solution if $A = 0$, that is, if $A = 0 + i0 = (0,0)$. On the other hand, solution of CHLDE

of the first order $\frac{dw}{dz} = 0$ is arbitrary constant $w(z) = c = a_1 + ib_1$. Therefore, infinite number of solutions is possible, for each real pair (a_1, b_1).

2 Main results

2.1 Behavior along direction

Theorem 2.1. *Real and imaginary part of the solution $w(z) = u(x,y) + iv(x,y)$ of complex homogenous linear differential equation of the first order (1.6), with analytical coefficient $a(z) = \alpha(x,y) + i\beta(x,y)$, each has countless number of zeros. Those zeros, except at trivial solution $w(z) \equiv 0$, are never common. Zeros for $u(x,y)$ and for $v(x,y)$, along arbitrary direction $y = kx$, $0 \leq k < \infty$, are respectively in the solutions of the equations*

$$\int_0^x (\beta(x,kx) + \alpha(x,kx)k)\,dx = -\arctan\frac{a_1}{b_1} \tag{2.1.1}$$

$$\int_0^x (\beta(x,kx) + \alpha(x,kx)k)\,dx = \arctan\frac{b_1}{a_1} \tag{2.1.2}$$

where a_1 and b_1 are arbitrary constants.

Dokaz 2.1. From general solution $w(z) = C\exp\left(-\int_L a(z)\,dz\right) = u(x,y) + iv(x,y)$, where L is path that connects points z_0 and z, and $C = a_1 + ib_1$ is arbitrary constants, according to the definition of equality of complex numbers we obtain

$$u(x,y) = \exp\left(-\int \alpha(x,y)\,dx - \beta(x,y)\,dy\right)\cdot$$

$$\cdot\left[a_1\cos\int(\beta(x,y)\,dx + \alpha(x,y)\,dy) + b_1\sin\int(\beta(x,y)\,dx + \alpha(x,y)\,dy)\right],$$

$$v(x,y) = \exp\left(-\int \alpha(x,y)\,dx - \beta(x,y)\,dy\right)\cdot$$

$$\cdot\left[b_1\cos\int(\beta(x,y)\,dx + \alpha(x,y)\,dy) - a_1\sin\int(\beta(x,y)\,dx + \alpha(x,y)\,dy)\right].$$

Area G (circular sector $|z| = R$, $0 \leq \arg z \leq \frac{\pi}{2}$) is covered by directions $y = kx$, $0 \leq k < \infty$. Then real function $u(x, y = kx)$ and $v(x, y = kx)$, transform into functions from x and parameter k.

These functions do not have a common zero, but each can have its own zero. Zero is determined from the equations

$$a_1 \cos \int_0^x (\beta(x,kx) + k\alpha(x,kx))\, dx + b_1 \sin \int_0^x (\beta(x,kx) + k\alpha(x,kx))\, dx = 0$$

and

$$b_1 \cos \int_0^x (\beta(x,kx) + k\alpha(x,kx))\, dx - a_1 \sin \int_0^x (\beta(x,kx) + k\alpha(x,kx))\, dx = 0,$$

or if we express through the tangent we obtain formula (2.1.1) and (2.1.2). As tanges unambiguous function to be within a period of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the value of the tangent (2.1.1) or (2.1.2) occur only once, and they can coincide in a common point. Therefore, $u(x,y)$ and $v(x,y)$ may have each of their zero, but have in common. So, $u(x,y) = v(x,y)$ only in $(0,0)$.$\Diamond$

Example 2.1. Non-trivial solution $w(z) = u(x,y) + iv(x,y)$ of CHLDE of the first order $w'(z) - w(z) = 0$ does not have real zeros for each z. Zeros of the functions for $u(x,y)$ and $v(x,y)$, along arbitrary direction $y = kx,\ 0 \le k < \infty$ are respectively in points $x_u = \frac{1}{k} \arctan \frac{a_1}{b_1},\ x_v = -\frac{1}{k} \arctan \frac{b_1}{a_1}$.
Along $x-$axis is $k = 0$. Here, zeros of function $u(x,0)$ are at $x_u = \infty$, and zeros $v(x,0)$ are at $x_v = -\infty$, which means that there are no finite zeros. Along direction $y = x$, therefore along line where $k = 1$, zeros for $u(x,x)$ are at $x_u = \arctan \frac{a_1}{b_1}$, and for $v(x,x)$ are at $x_v = -\arctan \frac{b_1}{a_1}$. Along $y-$axis is $k = \infty$, and zeros for u are at $x_u = 0$, and for v are at $x_v = 0$. Thus, there is one zero $O(0,0)$.
Now observe the sector $|z| = R,\ 0 \le \arg z < \frac{\pi}{2}$, and let the constants are $a_1, b_1 > 0$. Then $x_u = \frac{1}{\tan\varphi} \arctan \frac{a_1}{b_1},\ y_u = kx_u = \arctan \frac{a_1}{b_1}$ and $x_v = -\frac{1}{\tan\varphi} \arctan \frac{b_1}{a_1}$, $y_u = kx_u = -\arctan \frac{b_1}{a_1}$, so the zeros $u(x,y)$ are at points $M_u(x_u, y_u)$, and zeros $v(x,y)$ are at points $M_v(x_v, y_v)$.
If $0 < x_u \le R$, then $0 < \frac{1}{\tan\varphi} \arctan \frac{a_1}{b_1} \le R$. If ordinate $y_u = \arctan \frac{a_1}{b_1} \le R$, then there is only one intersection at point M_u. Hence, for one $k = \tan\varphi$, that is for one φ and for $\frac{a_1}{b_1} \le \tan(Rk)$, there exists only one zero M_u. However, if $\frac{a_1}{b_1}$ increases, for example from R to $2R$, then ordinate $y_u = \arctan \frac{a_1}{b_1}$ is greater than R, and lower than $2R$, and for same φ there is one more zero. We conclude, for given $k = \tan\varphi$, constant and positive, for given R, which is variable parameter and for a_1,b_1 positive and variable constants following apply:

- for $0 < \arctan \frac{a_1}{b_1} \le R$, there is one zero x_u of function $u(x,y)$,
- for $R < \arctan \frac{a_1}{b_1} \le 2R$, there are two zeros x_u and x'_u of function $u(x,y)$,
- for $2R < \arctan \frac{a_1}{b_1} \le 3R$, there are three zeros x_u, x'_u,x''_u of function $u(x,y)$,

$\vdots$

- for $(n-1)R < \arctan \frac{a_1}{b_1} \le nR$, there are n zeros x_u, x'_u,$x''_u, \ldots, x_u^{(n-1)}$ of function $u(x,y)$ and no zeros of function $v(x,y)$.$\Diamond$

Example 2.2. For CHLDE of the first order $w'(z) + (1-2z)\,w(z) = 0$ from analytical function $a(z) = 1-2z$ follows $\alpha(x,y) = 1-2x$ and $\beta(x,y) = -2y$. General solution of the equation is
$w(z) = (\alpha_1 + i\beta_1)\exp\left(-\int (1-2x)\,dx + 2ydy\right)\cdot$
$\cdot\left[\cos\left(\int (2x-1)\,dy + 2ydx\right) + i\sin\left(\int (2x-1)\,dy + 2ydx\right)\right].$
Zeros for $u(x,y)$ are solutions of equation $-kx^2 + k\left(x - x^2\right) = -\arctan\frac{a_1}{b_1}$, while from (2.1.2) follows, zeros for $v(x,y)$ are in solutions of equation $-kx^2 + k\left(x - x^2\right) = \arctan\frac{b_1}{a_1}$. Here we obtain quadratic equtions for abscissas of zeros, for $u(x,y)$ we have $x_{1,2} = \frac{1}{4k}\left(k \pm \sqrt{k^2 + 8k\arctan\frac{a_1}{b_1}}\right)$, and for $v(x,y)$ we obtain $x_{3,4} = \frac{1}{4k}\left(k \pm \sqrt{k^2 - 8k\arctan\frac{b_1}{a_1}}\right)$. Ordinates of zeros are respectively $y_{1,2} = kx_{1,2}$ and $y_{3,4} = kx_{3,4}$.
Due to simplicity of the coefficient $a(z)$, this CDE can be directly solved.$\diamondsuit$

2.2 On rationality of increasing of order at resolving of the system of PDEs

CHLDE of the first order (1.6), with analytical coefficient $a(z) = \alpha(x,y) + i\beta(x,y)$ can be written in the form

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + (\alpha(x,y)\,u(x,y) - \beta(x,y)\,v(x,y)) +$$

$$+i\,(\beta(x,y)\,u(x,y) + \alpha(x,y)\,v(x,y)) = 0.$$

From here we obtain the real system of PDEs of the first order

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\alpha(x,y)\,u(x,y) + \beta(x,y)\,v(x,y),\\ \frac{\partial v}{\partial x} &= -\beta(x,y)\,u(x,y) - \alpha(x,y)\,v(x,y)\,.2.2.1\end{aligned} \tag{2}$$

The system (2.2.1) can be transformed to the system of one PDE of the second order and one algebraic equation. For example,

$$\frac{\partial^2 u}{\partial x^2} + \left(2\alpha(x,y) - \frac{1}{\beta(x,y)}\frac{\partial\beta}{\partial x}\right)\frac{\partial u}{\partial x} +$$

$$+\left(\beta(x,y)\frac{\partial}{\partial x}\left(\frac{\alpha(x,y)}{\beta(x,y)}\right) + \alpha^2(x,y) + \beta^2(x,y)\right)u(x,y) = 0, \tag{2.2.2}$$

$$\beta(x,y)\,v(x,y)=\frac{\partial u}{\partial x}+\alpha(x,y)\,u(x,y)\,.$$

During the elimination of the function $u(x,y)$ or $v(x,y)$, it was notice that is better not to increase order of DE, but is better to stay at lower order and seak for separate zeros of mentioned real functions, especially along the direction $y=kx,\ 0\leq k<\infty$.

Theorem 2.2. *General solution of CHLDE of the first order (1.6) with analytical coefficient* $a(z)=\alpha(x,y)+i\beta(x,y)$ *is*

$$\Phi\left(\varphi_1(x,y,u)\,,\varphi_2(x,y,u)\right)=0, \tag{2.2.3}$$

where Φ *is arbitrary differential function and* φ_1,φ_2 *linearly independet first integrals of characteristic system of equations.*

Dokaz 2.2. Multiply the first equation of system (2.2.1) with $\alpha(x,y)$ and the second with $\beta(x,y)$ and sum them. Since $w(z)=u(x,y)+iv(x,y)$ is the analytic function, it follows that Cauchy–Riemann conditions apply and in the end we obtain equation

$$\alpha(x,y)\,\frac{\partial u}{\partial x}-\beta(x,y)\,\frac{\partial v}{\partial x}+\left(\alpha^2(x,y)+\beta^2(x,y)\right)u(x,y)=0. \tag{2.2.4}$$

This is the quasi-linear first order PDE which is equivalent in terms of finding solutions to the system of ordinary differential equations in symmetric form

$$\frac{dx}{\alpha(x,y)}=\frac{dy}{-\beta(x,y)}=\frac{du}{-\left(\alpha^2(x,y)+\beta^2(x,y)\right)}. \tag{2.2.5}$$

Since $\alpha(x,y)$ and $\beta(x,y)$ are given, follows $\varphi_1(x,y,u)=c_1$ and $\varphi_2(x,y,u)=c_2$ are linearly independent two first integrals, which depends on $\alpha(x,y)$ and $\beta(x,y)$. The general solution of PDE (2.2.4) is precisely the expression (2.2.3).$\Diamond$

Remark 2.1. It is known that if the differential equation has an analytical solution $w(z)=u(x,y)+iv(x,y)$, then for the $u(x,y)$ and $v(x,y)$ are valid Cauchy-Riemann conditions. Therefore, the imaginary part $v(x,y)$ of the solution $w(z)$, of CHLDE of the first order (1.6), has been characterized by a system of ordinary differential equations

$$\frac{dx}{\alpha(x,y)}=\frac{dy}{-\beta(x,y)}=\frac{dv}{-\left(\alpha^2(x,y)+\beta^2(x,y)\right)}. \tag{2.2.5'}$$

Theorem 2.3. *General solution $w(z) = u(x,y) + iv(x,y)$ of CHLDE of the first order (1.6), with analytical coefficient $a(z) = \alpha(x,y) + i\beta(x,y)$, along arbitrary direction $y = kx,\ 0 \le k < \infty$ is given with*

$$w(z) = -(1+i)\int_0^x \frac{\alpha^2(x,kx) + \beta^2(x,kx)}{\alpha(x,kx)}dx + c_1 + ic_2. \qquad (2.2.6)$$

Dokaz 2.3. From the system (2.2.5), for the function $u(x,y)$, we obtain the DE $\frac{dx}{dy} = \frac{\alpha(x,y)}{-\beta(x,y)}$. It follows that for an arbitrary direction $y = kx,\ 0 \le k < \infty$ we can determine $k = \frac{dy}{dx}$, that is $k = \frac{\beta(x,y)}{-\alpha(x,y)}$. From the second equation of system (2.2.5), that is from $\frac{dx}{\alpha(x,y)} = \frac{du}{-(\alpha^2(x,y)+\beta^2(x,y))}$, in the same direction we can find a solution

$$u(x,k) = -\int_0^x \frac{\alpha^2(x,kx) + \beta^2(x,kx)}{\alpha(x,kx)}dx + c_1. \qquad (2.2.7)$$

This formula gives the behavior of the real part of the solution $w(z) = u(x,y) + iv(x,y)$.
Similarly, from the second equation system (2.2.5'), in the direction $y = kx$ we have

$$v(x,k) = -\int_0^x \frac{\alpha^2(x,kx) + \beta^2(x,kx)}{\alpha(x,kx)}dx + c_2. \qquad (2.2.8)$$

Based on (2.2.7) and (2.2.8) follows (2.2.6).$\diamondsuit$

Corollary 2.1. *Only trivial solution $w(z) \equiv 0$ of CHLDE of the first order (1.6), with analytical coefficient $a(z) = \alpha(x,y) + i\beta(x,y)$, along arbitrary direction $y = kx,\ 0 \le k < \infty$ has zero.*

Dokaz 2.4. From (2.3.6), for $z = x + iy = 0$, that is, for $x = 0$ and $y = kx = 0$, it follows that the integral is equal to zero. Then $c_1 + ic_2 = 0$, that is $c_1 = u(0,0) = 0$ and $c_2 = v(0,0) = 0$.
Hence, the functions $u(x,y)$ and $v(x,y)$ have their zeros, but common zeros are possible only for $c_1 = c_2 = 0$. Accordingly, only the trivial solution $w(z) \equiv 0$ has zero $z = 0$.$\diamondsuit$

The following is a formula for the exact number of zeros along direction if we already know the location.

Theorem 2.4. *Number of zeros of sine solution of CHLDE of the first order (1.6), with analytical coefficient* $a(z) = \alpha(x,y) + i\beta(x,y)$*, along arbitrary direction* $y = kx,\ 0 \le k < \infty$*, from* $O(0,0)$ *to the point* (R, kR) *is*

$$n = E\left[\frac{1}{\pi}\int_0^R (k\alpha(x,kx) + \beta(x,kx))\,dx\right], \tag{2.2.9}$$

$E[.]$ *denotes the entire part of argument. Number of zeros of cosine solution is a for one less.*

Dokaz 2.5. From the general solution CHLDE of of the first order (1.6), with analytical coefficient $a(z)$ after elementary transformations we obtain that the zeros of the real and imaginary part of the solution is possible only for

$$\cos\int_0^x (\beta(x,y)\,dx + \alpha(x,y)\,dy) = 0, \sin\int_0^x (\beta(x,y)\,dx + \alpha(x,y)\,dy) = 0.$$

It follows that the zeros of cosine and sine solution, along arbitrary direction $y = kx,\ 0 \le k < \infty$ are given by

$$\begin{aligned}\int_0^R (\beta(x,kx) + k\alpha(x,kx))\,dx &= (2n-1)\frac{\pi}{2},\ n = 1,2,\ldots\\ \int_0^R (\beta(x,kx) + k\alpha(x,kx))\,dx &= n\pi,\ n = 0,1,\ldots.\end{aligned}$$

Considering the sine function characteristic which within one period 2π has three zeros $0, \pi, 2\pi$, and of the cosine function characteristic to have a two zeros $\frac{\pi}{2}, \frac{3\pi}{2}$ at the same period, it is concluded that the cosine solution has one less zero than the sine solution. ◊

Remark 2.2. If we now move along Ox-axis, then $z = x$, so $w(z) = w(x) = u(x) + iv(x)$. Here, by substitution in system (2.2.1) we obtain ordinary system of DE, with two unknown functions $u(x)$ and $v(x)$. Eliminating one function, for example $v(x)$, from first equation of the system and substituting in second equation of the system, we obtain complete HLDE of the second order

$$u''(x) + \left(2\alpha(x) - \frac{\beta'(x)}{\beta(x)}\right)u'(x) +$$

$$+\left(\alpha'(x) - \frac{\alpha(x)\beta'(x)}{\beta(x)} + \alpha^2(x) + \beta^2(x)\right)u(x) = 0,$$

which by substitution

$$u(x) = \exp\left(-\frac{1}{2}\int\left(2\alpha(x) - \frac{\beta'(x)}{\beta(x)}\right)dx\right)S(x)$$

where $S(x)$ is new unknown function transforms into canonical DE of the second order $S''(x) + \Psi(x)S(x) = 0$, for which applies Sturm theorem, (for further details, see [2],[3],[4],[5],[6],[7]).
It is well known that solution of CHLDE (1.6) does not have mutual zeros, except $w(z) = w(x) = 0$, that is, only for $u(y) = v(y) = 0$.

Remark 2.3. An important direction, Oy-axis till now has been excluded from observed directions, because for $k = \infty$ it couldn't be encompassed by expression $y = kx,\ 0 \le k < \infty$. This direction is called singular direction. Along it, CHLDE of the first order (1.6), with analytical coefficient $a(z)$ can have solution, even zeros of solution.
Indeed, along Oy-axis is $x = 0$, so CHLDE (1.6), written in developed form, must contain derivatives from y. Since CHLDE has analytical solution, that is, Cauchy–Riemann conditions apply, then system (2.2.1) becomes

$$\frac{dv}{dy} + \alpha(y)u(y) - \beta(y)v(y) = 0,$$

$$\beta(y)u(y) + \alpha(y)v(y) = 0. \tag{2.2.10}$$

This is mixed algebraic-differential system, which is also called singular. Eliminating $u(y)$ from the second equation of the system, and substituting into first one, we obtain DE of the first order with separated variables. Its general solution is $v(y) = c_1\exp\left(\int\frac{\alpha^2(y)+\beta^2(y)}{\beta(y)}dy\right)$. Solutions for $u(y)$ are obtained without quadratures. Since system (2.2.10) is mixed, this case is treated separately. Note that system (2.2.10) also has trivial solution for $u(y) = v(y) = 0$.

Example 2.3. General solution of CHLDE of the first order $w'(z) + zw(z) = 0$ along direction $y = kx,\ 0 \le k < \infty$, is

$$w(x, kx) = c\exp\left(\frac{x^2\left(k^2-1\right)}{2}\right)\left(\cos\left(kx^2\right) - i\sin\left(kx^2\right)\right).$$

Zeros of sine solution are $x = \pm\sqrt{\frac{n\pi}{k}},\ n = 0, 1, \ldots,$ and their number along direction, to point $x = R$ is $n = E\left[\frac{k}{\pi}R^2\right]$. Zeros of cosine solution are $x = \pm\sqrt{(2n-1)\frac{\pi}{2k}},\ n = 1, 2, \ldots,$ and number of zeros to abscissa $x = R$ is $n =$

$E\left[\frac{1}{2}+\frac{k}{\pi}R^2\right]$. We note that separate sine and cosine zeros there are more if k is growing.

3 Conclusion

Question on location of zeros even the simplest complex differential equations, is very difficult and certainly not elementary. Moreover, it has been solved only for certain classes of differential equations. In this short presentation a goal was determination of number of zeros solutions of complex linear homogenous differential equations of first order. The idea on location of isolated zeros according to arbitrary direction in first quadrant, have proved useful,where with simple application of elementary theorems on substitution of independent variable a problem transferred to ordinary differential equations.

References

[1] E. Hille, Ordinary differential Equations in the Complex Domains, Wiley-Interscience, 1976.

[2] D. Dimitrovski, M. Rajovic, R. Stojiljkovic, On types, form and supremum of the solutions of the linear differential equation of the second order with entire coefficients, App. Analy. And Disscrete Mathematics, 1 (2007), 360-370.

[3] M. Lekić, Šturmove teoreme kroz iteracije, Doktorska disertacija, Kosovska Mitrovica, 2007.

[4] J. Vujaković, D. Dimitrovski , M. Rajović, On location of zeros solution of second order complex differential equation, Computational and Informational Technologies for Science, Engineering and Education, Almaty, Novosibirsk, 2010, Vol. 15, No. 4, 27-30.

[5] J. Vujaković, M. Rajovć, An idea for determination of zeros of complex differential equations, Computational and Informational Technologies for Science, Engineering and Education, Almaty, Novosibirsk, 2010, Vol. 15, No. 3, 8-12.

[6] J. Vujaković, M. Rajović, D. Dimitrovski, Some new results on linear equation of the second order, Computers and Mathematics with Applications, 61 (2011), 1837-1843, doi: 10.1016/j.camwa.2011.02.012.

[7] J. Vujaković, Nule rešenja kompleksnih diferencijalnih jednačina, Doktorska disertacija, Kosovska Mitrovica, 2012.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Generalized Contractive Mappings on Compact Metric Spaces

Mirko S. Jovanović
Faculty of Electrical Engineering, University of Belgrade
msj@sbb.rs

Original scientific paper

Abstract

In this paper we generalize the theorems of Немыцкий and Edelstein.

1 Introduction

Let (X,d) be a metric space and $T : X \to X$. A mapping T is *contractive* if it fulfills the condition ([1]):

$$d(Tx,Ty) < d(x,y),\ x \neq y,\ x,y \in X. \tag{1}$$

A mapping T is a generalized contractive mapping if it fulfills the condition (where $x \neq y,\ x,y \in X$)

$$d(Tx,Ty) < \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \tag{2}$$

Obviously, if T is a contractive mapping, then it is also a generalized contractive mapping (i.e. (1) $\Rightarrow$ (2)), but not vice versa.

Example 1.1. Let $X = [0,4]$ be a set of real numbers with the standard metric $d(x,y) = |x-y|$ and $T : X \to X$ a mapping defined by

$$Tx = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{x}{3}, & x \in (0,3] \\ \frac{x}{4}, & x \in (3,4]. \end{cases} \tag{3}$$

Then the mapping T fulfills the condition (2) but not (1). Also [1] confirms that the condition (2) is fulfilled. One can conclude that the condition (1) does not hold based on the fact that the mapping T defined by (3) is not continuous on X (it has discontinuities at the points $x = 0$ and $x = 3$) while contractive mappings (i.e. mappings fulfilling condition (1)), are necessarily continuous.

2 Main results

In this section we prove the generalized Немыцкий theorem ([3], [1]).

Theorem 2.1. *Let $T : X \to X$ be a continuous mapping of a compact space (X, d). If T fulfills the condition (2), then T has a unique fixed point.*

Proof. Define a function $f : X \to \mathbb{R}^+$ by $f(x) = d(x, Tx)$, $x \in X$. Then f is a continuous function, because T is a continuous mapping. Since X is a compact metric space, there exists $u \in X$ such that

$$f(u) = d(u, Tu) = \min\{f(x) : x \in X\}. \tag{4}$$

Assuming that for each $x \in X$ it holds that $x \neq Tx$, which is equivalent with $f(u) > 0$ i.e. $d(u, Tu) > 0$, we can apply the condition (2):

$$\begin{aligned} d(TuT^2u) &= d(Tu, T(Tu)) \\ &< \max\left\{d(u, Tu), d(u, Tu), d(Tu, T(Tu)), \frac{d(u, T(Tu)) + d(Tu, Tu)}{2}\right\} \\ &= \max\left\{d(u, Tu), d(Tu, T^2u), \frac{d(u, T^2u)}{2}\right\}. \end{aligned}$$

Since

$$d(u, T^2u) \leq d(u, Tu) + d(Tu, T^2u) \leq 2\max\{d(u, Tu), d(Tu, T^2u)\}$$

i.e.

$$\frac{d(u, T^2u)}{2} \leq \max\{d(u, Tu), d(Tu, T^2u)\},$$

it holds that

$$d(Tu, T^2u) < \max\{d(u, Tu), d(Tu, T^2u)\}. \tag{5}$$

Based on (4) and (5) we conclude that

$$d(Tu, T^2u) < d(Tu, T^2u)$$

which is a contradiction. Therefore, $f(u) = 0$, i.e. $Tu = u$. This proves the existence of a fixed point. Let us now prove that the fixed point is unique. Suppose that T has another fixed point v, $(Tv = v)$ and $v \neq u$. Then, based on (2), we have:

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv) + d(v, Tu)}{2}\right\} \\ &= \max\left\{d(u, v), d(u, u), d(v, v), \frac{d(u, v) + d(v, u)}{2}\right\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction. Therefore, the mapping T has a unique fixed point. □

The condition that the mapping T must be continuous in Theorem 2.1 can't be ignored, as shown by Example 1.1. Here a discontinuous mapping T of a compact metric space $X = [0, 4]$ has no fixed points. The following theorem reduces the continuity condition using that T should be orbitally continuous, and the compactness is replaced by the existence of an iterated sequence that has a convergent subsequence which must fulfill the condition (2), and not (1), as stated by the Edelstein theorem ([2]).

Theorem 2.2. *Let (X, d) be a metric space and $T : X \to X$ orbitally continuous mapping that fulfills the condition (2). If there exists $x_0 \in X$ such that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ contains a convergent subsequence $(T^{n_i} x_0)_{i\in\mathbb{N}}$, then $u = \lim_{i\to\infty} T^{n_i} x_0$ is a unique fixed point of the mapping T.*

Proof. We consider two mutually exclusive cases:

(i) There exists $n_0 \in \mathbb{N}$ such that $T^{n_0+1}x_0 = T^{n_0}x_0$;

(ii) For each $n \in \mathbb{N}$ it holds that $T^{n+1}x_0 \neq T^n x_0$, i.e. $d(T^{n+1}x_0, T^n x_0) > 0$.

In the case (i) $T^{n_0}x_0 = u$ is a fixed point, because of $Tu = u$. We now consider the case (ii)

$$\begin{aligned}
&d(T^{n+1}x_0, T^{n+2}x_0) = d(T(T^n x_0), T(T^{n+1}x_0))\\
&< \max\Big\{d(T^n x_0, T^{n+1}x_0), d(T^n x_0, T(T^n x_0)), d(T^{n+1}x_0, T(T^{n+1}x_0)),\\
&\qquad \frac{d(T^n x_0, T(T^{n+1}x_0)) + d(T^{n+1}x_0, T(T^n x_0))}{2}\Big\}\\
&= \max\Big\{d(T^n x_0, T^{n+1}x_0), d(T^n x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^{n+2}x_0),\\
&\qquad \frac{d(T^n x_0, T^{n+2}x_0) + d(T^{n+1}x_0 T^{n+1}x_0)}{2}\Big\}\\
&= \max\Big\{d(T^n x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^{n+2}x_0), \frac{d(T^n x_0, T^{n+2}x_0)}{2}\Big\}.
\end{aligned}$$

Since

$$\begin{aligned}
d(T^n x_0, T^{n+2}x_0) &\le d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, T^{n+2}x_0)\\
&\le 2\max\{d(T^n x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^{n+2}x_0)\}
\end{aligned}$$

i.e.

$$\frac{d(T^n x_0, T^{n+2}x_0)}{2} \le \max\{d(T^n x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^{n+2}x_0)\}$$

we find that

$$d(T^{n+1}x_0, T^{n+2}x_0) < \max\{d(T^n x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^{n+2}x_0)\}$$

wherefrom

$$d(T^{n+1}x_0, T^{n+2}x_0) < d(T^n x_0, T^{n+1}x_0) \tag{6}$$

since $d(T^{n+1}x_0, T^{n+2}x_0) < d(T^{n+1}x_0, T^{n+2}x_0)$ is impossible.

This proves that the sequence $(d(T^n x_0, T^{n+1}x_0))_{n\in\mathbb{N}}$ is decreasing, and since it is non-negative, there exists

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0).$$

Since the iterated sequence $(T^n x_0)_{n\in\mathbb{N}}$ has a convergent subsequence $(T^{n_i}x_0)_{i\in\mathbb{N}}$, i.e. $\lim_{i\to\infty} T^{n_i}x_0 = u$ and since T is an orbitally continuous mapping, it follows that

$$\lim_{i\to\infty} T(T^{n_i}x_0) = Tu \text{ and } \lim_{i\to\infty} T(T^{n_i+1}x_0) = T(Tu)$$

so

$$\lim_{i\to\infty} d(T^{n_i}x_0, T^{n_i+1}x_0) = d(u, Tu) \text{ and } \lim_{i\to\infty} d(T^{n_i+1}x_0, T^{n_i+2}x_0) = d(Tu, T^2u).$$

Considering the fact that $(d(T^{n_i}x_0, T^{n_i+1}x_0))_{i\in\mathbb{N}}$ and $(d(T^{n_i+1}x_0, T^{n_i+2}x_0))_{i\in\mathbb{N}}$ are both subsequences of a convergent sequence $(d(T^n x_0, T^{n+1}x_0))_{n\in\mathbb{N}}$ we conclude that

$$d(u, Tu) = d(Tu, T^2u). \tag{7}$$

If $Tu \neq u$, applying the same method used to prove (6), we obtain

$$d(Tu, T^2u) < d(u, Tu)$$

which is impossible because of (7). Therefore, $Tu = u$. Fixed point uniqueness is proved the same way as in Theorem 2.1. □

References

[1] Lj. Ćirić, Some recent results in metrical fixed point theory, Beograd, 2003.

[2] M. Edelstein, On fixed and periodic point under contractive mappings, J. London Math. Soc., 3 (1962), 74–79.

[3] В. В. Немыцкий, Метод неподвижных точек в анализе, Усп. Мат. Наук,(1936), 141–174.

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

A Note on Generalized Operator Quasicontractions in Cone Metric Spaces

Ljiljana Paunović
University of Pristina-Kosovska Mitrovica
Faculty of Teacher Training Prizren-Leposavic, Serbia
ljiljana.paunovic76@gmail.com

Original scientific paper

Abstract

In a recent paper [*Fixed point theorem for generalized operator quasi-contractive mappings in cone metric spaces*, Afrika Mat., DOI 10.1007/s13370-012-0105-7], X. Zhang has used bounded positive definite linear operators on the given Banach space in an attempt to obtain a more general fixed point result in a normal cone metric space. We will show in this paper that most of the conditions in his theorem are superfluous and that the proof can be obtained in a much easier way, by reducing it to a well known Ćirić's result on quasicontractions in standard metric spaces.

Key words: metric space, solid cone, cone metric space, generalized operator quasicontractive mapping.

1 Introduction

L.G. Huang and X. Zhang introduced cone metric spaces in [4], replacing the set of real numbers by an ordered Banach space as the codomain of a metric. Thus, they reconsidered the notion of K-metric spaces that was used earlier (see, e.g., [12]). A lot of known metric fixed point and common fixed point results were subsequently extended to this new setting (for a review of these results till 2010 see [5]).

In a recent paper [13], X. Zhang has used bounded positive definite linear operators on the given Banach space in an attempt to obtain a more general fixed point result in a normal cone metric space. We will show in this note that most of the conditions in his theorem are superfluous and that the proof can be obtained in a much easier way, by reducing it to a well known Ćirić's ([1]) result on quasicontractions in standard metric spaces.

2 Preliminaries

We recall some properties of cones and cone metric spaces. The details and proofs can be found in [4, 5].

Let E be a real Banach space with the zero vector θ. A proper nonempty and closed subset K of E is called a (convex) *cone* if $K+K\subset K$, $\lambda K\subset K$ for $\lambda\geq 0$ and $K\cap(-K)=\{\theta\}$. We shall always assume that the cone K has a nonempty interior $\operatorname{int}K$ (such cones are called *solid*).

Each cone K induces a partial order $\preceq$ on E by $x\preceq y\Leftrightarrow y-x\in K$, $x\prec y$ will stand for $x\preceq y$ and $x\neq y$, while $x\ll y$ will stand for $y-x\in\operatorname{int}K$. The pair (E,K) is an *ordered Banach space.*

Definition 2.1. [3] A cone K in a Banach space $(E,\|\cdot\|)$ is called:

1° *normal* if

$$\inf\{\,\|x+y\|:x,y\in K,\ \|x\|=\|y\|=1\,\}>0;$$

2° *semi-monotone* if there exists $k>0$ such that, for all $x,y\in E$,

$$\theta\preceq x\preceq y \text{ implies } \|x\|\leq k\,\|y\|; \tag{1}$$

3° *monotone* if, for all $x,y\in E$,

$$\theta\preceq x\preceq y \text{ implies } \|x\|\leq\|y\|, \tag{2}$$

i.e. it is semi-monotone with $k=1$.

The next lemma contains results on cones in ordered Banach spaces that are rather old (1940, see [8]). It is interesting that most of the authors (working with normal cones after 2007) do not use these results, which can be applied to reduce a lot of results to the setting of ordinary metric spaces.

Lemma 2.1. *[8, 3] The following conditions are equivalent for a cone K in the Banach space $(E,\|\cdot\|)$:*

1° K is normal;

2° for arbitrary sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in E,

$$(\forall n)\ x_n\preceq y_n\preceq z_n \text{ and } \lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=x \text{ imply } \lim_{n\to\infty}y_n=x;$$

3° K is semi-monotone;

4° there exists a norm $\|\cdot\|_1$ on E, equivalent to the given norm $\|\cdot\|$, such that the cone K is monotone w.r.t. $\|\cdot\|_1$.

The smallest constant k satisfying the inequality (1) is called the normal constant of K. It is clear that it is always $k\geq 1$.

Example 2.1. [10] Let $E=C^1_{\mathbb{R}}[0,1]$, with $\|x\|=\|x\|_\infty+\|x'\|_\infty$, $K=\{x\in E:x(t)\geq 0\}$. This cone is non-normal. Consider, for example, $x_n(t)=\frac{t^n}{n}$ and $y_n(t)=\frac{1}{n}$. Then $\theta\preceq x_n\preceq y_n$, and $\lim_{n\to\infty}y_n=\theta$, but $\|x_n\|=\max_{t\in[0,1]}\left|\frac{t^n}{n}\right|+\max_{t\in[0,1]}|t^{n-1}|=\frac{1}{n}+1>1$; hence x_n does not converge to zero. It follows by 2° that K is a non-normal cone.

Let X be a nonempty set and (E, K) an ordered Banach space. A function $d : X \times X \to E$ is called a *cone metric* and (X, d) is called a *cone metric space* if the following conditions hold:

$(C1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

$(C2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(C3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in X. Then it is said that:

(i) $\{x_n\}$ cone converges to x if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $d\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \overset{d}{\to} x$ as $n \to \infty$;

(ii) $\{x_n\}$ is a cone Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$;

(iii) (X, d) is cone complete if every tvs-Cauchy sequence is tvs-convergent in X.

Suppose now that (X, d) is a cone metric space over a normal solid cone K in a Banach space E (with the normal constant satisfying $k \geq 1$). Then $D(x, y) = \|d(x, y)\|$ is a *symmetric* on the set X, that is, a mapping from $X \times X$ into $[0, +\infty)$ with the following properties:

(**s1**) $D(x, y) \geq 0$ and $D(x, y) = 0$ if and only if $x = y$;

(**s2**) $D(x, y) = D(y, x)$.

It satisfies also:

(**s3**) $D(x, y) \leq k(D(x, z) + D(z, y))$.

The cone metric d and the associated symmetric $D = \|d\|$ in X generate two topologies: t_d and t_D. Their bases of neighborhoods consist of the sets

$$B_c(y) = \{\, x \in X : d(x, y) \ll c \,\} \text{ and } B_\varepsilon(y) = \{\, x \in X : D(x, y) < \varepsilon \,\},$$

where y, c, ε are, respectively, a given point from X, a given interior point from K, and a given positive number (for details see [9]).

Theorem 2.1. *[9] Let (X, d) be a cone metric space with a normal solid cone K and let D be the associated symmetric. Then $t_d = t_D$.*

In other words, the spaces (X, d) and (X, D) have the same collections of open, closed, bounded and compact sets, and also the same convergent and Cauchy sequences, and the same continuous functions. Also, the interior of the cone is the same in both equivalent norms. If the normal constant $k = 1$, then the symmetric space (X, D) is a metric space.

Thus we obtain the following *principal remark*: From Lemma 2.1 and Theorem 2.1 it is obvious that, in the investigation of cone metric spaces with normal solid cones, we can assume that the normal constant can be taken to be $k = 1$. This follows from the fact that we can deal with the space E, equipped with the norm $\|\cdot\|_1$, which is equivalent to $\|\cdot\|$. Taking into account Lemma 2.1 and Theorem 2.1, it follows that results for normal solid cone metric spaces can be derived from the respective results for metric spaces. Namely, if d is a cone metric on X and the norm $\|\cdot\|$ is monotone on the cone, then the composition $\|\cdot\| \circ d = D$ is a usual metric on X. Clearly, non-expansive and contractive (with respect to d) self-mappings of X are also non-expansive and contractive (respectively) with respect to the metric $\|\cdot\| \circ d = D$.

3 Main results

Let $(E, \|\cdot\|)$ be a Banach space with a cone $K \subset E$. Consider the set $B(K)$ of bounded linear operators A on E that are positive-definite, i.e., $A(K) \subset K$ holds. Moreover, let $B(K,1)$ denote the subset of $B(K)$ containing operators A having the norm $\|A\| \le 1$. X. Zhang in [13], called a mapping $f : X \to X$ *generalized operator quasicontractive* on a cone metric space (X, d) (over K) if there exists $\lambda \in [0, 1)$ such that for all $x, y \in X$ there exist $a, b, c, d, e \ge 0$ satisfying $a+b+c+d+e \le 1$, and $A, B, C, D, E \in B(K, 1)$ such that the following inequality holds

$$d(fx, fy) \preceq \lambda(aAd(x, y) + bBd(x, fx) + cCd(y, fy) + dDd(x, fy) + eEd(y, fx)). \tag{3}$$

Note that the constants a, b, c, d, e, as well as the operators A, B, C, D, E depend on the points x, y, but we will not write $a_{x,y}, \dots$ in order to avoid cumbersome notation. The following example is inspired by [13, Example 1].

Example 3.1. Let $E = \mathbb{R}^2$ and let $K = \{(x, y)^\top \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$, which is a normal cone with $k = 1$. Let $\mathcal{X} = \{a, b, c\}$ and define a cone metric d on $\mathcal{X}$ by $d(x, x) = (0, 0)^\top$ for $x \in \mathcal{X}$, $d(a, b) = d(a, c) = (1, 1)^\top$, $d(b, c) = (0, 1.1)^\top$ and $d(x, y) = d(y, x)$ for $x, y \in \mathcal{X}$. It is easy to see that $(\mathcal{X}, d)$ is a cone metric space.

Let $f : \mathcal{X} \to \mathcal{X}$ be defined by $fa = b$, $fb = fc = c$. Take $A = \begin{bmatrix} 0 & 0 \\ 0.6 & 0.6 \end{bmatrix} \in B(K, 1)$ and $\lambda = \frac{11}{12}$. Then it is easy to check that the inequality $d(fx, fy) \preceq \lambda Ad(x, y)$ is fulfilled for all $x, y \in \mathcal{X}$. Hence, f is a generalized operator quasicontractive mapping on $\mathcal{X}$.

X. Zhang proved the following.

Theorem 3.1. *[13] Let (X, d) be a complete cone metric space over a normal cone K with normal constant k. Let $f : X \to X$ be a generalized operator quasicontractive mapping with contractive constant λ. If one of the following conditions hold:*

1. *f is continuous;*
2. *$\lambda k < 1$;*
3. *there exists $x_0 \in X$ such that*

$$\{\, A(x, y), B(x, y), C(x, y), D(x, y), E(x, y) : x, y \in \mathcal{O}(x_0, \infty) \,\}$$

is a compact set (here, $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ are operators A, B, C, D, E from (3.1) *and $\mathcal{O}(x_0, \infty)$ is the orbit of f at the point x_0),*

then f has a unique fixed point $x^* \in X$. Moreover, in the cases (1) *and* (2)*, any Picard sequence $\{f^n x\}$ converges to x^*; in the case* (3)*, the Picard sequence $\{f^n x_0\}$ converges to x^*.*

We shall prove that the previous result can be obtained without assuming any of the conditions (1)–(3).

Theorem 3.2. *Let (X,d) be a complete cone metric space over a normal cone K and let $f : X \to X$ be a generalized operator quasicontractive mapping. Then f has a unique fixed point $x^* \in X$ and any Picard sequence $\{f^n x\}$, $x \in X$ converges to x^*.*

Proof. According to Lemma 2.1, $3^\circ \Leftrightarrow 4^\circ$, there exists a norm on E (denoted by $\|\cdot\|$), equivalent to the original one, which is monotone, i.e., the implication (2) holds. As in Theorem 2.1, denote $D(x,y) = \|d(x,y)\|$. Then D is a standard metric on X, and (X,D) is a complete metric space. The inequality (3.1) implies that

$$\begin{aligned}
D(fx,fy) &= \|d(fx,fy)\| \\
&\le \lambda\|aAd(x,y) + bBd(x,fx) + cCd(y,fy) + dDd(x,fy) + eEd(y,fx)\| \\
&\le \lambda(a\|A\|\|d(x,y)\| + b\|B\|\|d(x,fx)\| + c\|C\|\|d(y,fy)\| \\
&\qquad + d\|D\|\|d(x,fy)\| + e\|E\|\|d(y,fx)\|) \\
&\le \lambda(a+b+c+d+e)\max\{\|d(x,y)\|, \|d(x,fx)\|, \|d(y,fy)\|, \\
&\qquad\qquad \|d(x,fy)\|, \|d(y,fx\|\} \\
&\le \lambda\max\{\|d(x,y)\|, \|d(x,fx)\|, \|d(y,fy)\|, \|d(x,fy)\|, \|d(y,fx)\} \\
&= \lambda\max\{D(x,y), D(x,fx), D(y,fy), D(x,fy), D(y,fx)\}
\end{aligned}$$

holds for all $x,y \in X$. In other words, f is a Ćirić-type quasicontraction ([1]) in a complete metric space (X,D). Hence, f has a unique fixed point $x^* \in X$, being the limit of an arbitrary Picard sequence $\{f^n x\}$. ⊣

Remark 3.1. Since the mapping f in Example 3.1 fulfills all the conditions of Theorem 3.2, it follows that f has a unique fixed point (which is $x^* = c$). As was shown in a similar situation in [13, Example 1], no scalar cone-quasicontractive condition is satisfied in this case. However, the statement from [13, Remark 1] does not hold, since Theorem 3.2 and its proof show that the result can still be obtained by reducing it to the standard metric arguments.

References

[1] Lj.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[2] D. Djukić, Lj. Paunović and S. Radenović,*Convergence of iterates with eror of uniformly quasi-Lipschitzian mappings in cone metric spaces*, Kragujevac J. Math., vol. 35, Number 3(2011), 399-410.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[4] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **323** (2007), 1468–1476.

[5] S. Janković, Z. Kadelburg, S. Radenović, *On cone metric spaces: A survey*, Nonlinear Anal. TM&A **74** (2011), 2591–2601.

[6] S. Radojević, Lj. Paunović and S. Radenović, *Abstract metric spaces and Hardy-Rogers-type theorems*, Appl. Math. Lett. 24(4) (2011), 553-558.

[7] M. Filipović, Lj. Paunović, S. Radenović and M. Rajović, *Remarks on "Cone Metric Spaces and Fixed Point Theorems of T-Kannan Contractive Mappings"*, Math. And Computer Modeling , vol. 54, Issue 5-6,(2011), 1467-1472.

[8] M. G. Krein, M. A. Rutman, *Linear operators leaving invariant a cone in a Banach spaces*, Uspekhi Math. Nauk (N.S.), **3**, 1 (1948), 3–95.

[9] S. Radenović, Z. Kadelburg, *Quasi-contractions on symmetric and cone symmetric spaces*, Banach J. Math. Anal. **5**, 1 (2011), 38–50.

[10] J. S. Vandergraft, *Newton's method for convex operators in partially ordered spaces*, SIAM J. Numerical Anal. **4**, 3 (1967), 406–432.

[11] Z. M. Fadail, A. G. B Ahmad and Ljiljana Paunović,*New fixed point results of single-valued mapping for c-distance in cone metric spaces*, Abstract and Applied Analysis, Article ID 639713 (2012) 12 pages.

[12] P.P. Zabreĭko, *K-metric and K-normed spaces: survey*, Collectanea Math. **48**, 4–6 (1997), 825–859.

[13] X. Zhang, *Fixed point theorem for generalized operator quasi-contractive mappings in cone metric spaces*, Afrika Mat., DOI 10.1007/s13370-012-0105-7.

SEKCIJA ZA PRIMIJENJENU MATEMATIKU

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Error Bounds of Gaussian Quadratures for One Class of Bernstein-Szegő Weight Functions

Miodrag M. Spalević
Department of Mathematics, University of Beograd, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia
mspalevic@mas.bg.ac.rs

Original scientific paper

Abstract

In this survey of our recent results, the kernels $K_n(z)$ in the remainder terms $R_n(f)$ of the Gaussian quadrature formulae for analytic functions f inside elliptical contours with foci at ∓ 1 and a sum of semi-axes $\rho > 1$, when the weight function w is of Bernstein-Szegő type

$$w(t) \equiv w_\gamma^{(\mp 1/2, \mp 1/2)}(t) = \frac{(1-t)^{\mp 1/2}(1+t)^{\mp 1/2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\, t^2}, \quad t \in (-1,1), \quad \gamma \in (-1,0),$$

are given. Sufficient conditions are found ensuring that the kernel attains its maximal absolute value at the intersection point of the contour with either the real or the imaginary axis. This leads to effective error bounds of the corresponding Gauss quadratures. The quality of the derived bounds is analyzed by a comparison with other error bounds intended for the same class of integrands.

1 Introduction

Let the weight function w be a nonnegative and integrable function on the interval $(-1,1)$. Consider the Gauss quadrature formula

$$\int_{-1}^{1} f(t)w(t)\,dt = G_n[f] + R_n(f), \quad G_n[f] = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) \quad (n \in \mathbb{N}) \tag{1}$$

which is exact for all algebraic polynomials of degree at most $2n-1$. The nodes τ_ν in (1) are zeros of the orthogonal polynomials π_n with respect to the weight function w.

In this paper w is the weight function of Bernstein-Szegő type

$$w(t) \equiv w_\gamma^{(\mp 1/2, \mp 1/2)}(t) = \frac{(1-t)^{\mp 1/2}(1+t)^{\mp 1/2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\, t^2}, \quad t \in (-1,1), \quad \gamma \in (-1,0). \tag{2}$$

The weight functions under consideration are special cases of the (more general) Bernstein-Szegő weight functions

$$w_{\alpha,\beta,\delta}^{(\mp1/2,\mp1/2)}(t) = \frac{(1-t)^{\mp1/2}(1+t)^{\mp1/2}}{\beta(\beta-2\alpha)\,t^2 + 2\delta(\beta-\alpha)\,t + \alpha^2 + \delta^2}, \quad t \in (-1,1), \tag{3}$$

where $0 < \alpha < \beta$, $\beta \neq 2\alpha$, $|\delta| < \beta - \alpha$, having in the denominator an arbitrary polynomial of exact degree 2 that remains positive on $[-1,1]$. Namely, if we set $\alpha = 1$, $\beta = 2/(1+\gamma)$, $-1 < \gamma < 0$, and $\delta = 0$, (3) reduces to (2). The weight function (3) have been studied extensively in [1], and therefore the results obtained there can be specialized in the case of (2).

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $\mathcal{D} = \operatorname{int}\Gamma$ its interior. If the integrand f is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term $R_n(f)$ in (1) admits the contour integral representation

$$R_n(f) = \frac{1}{2\pi i}\oint_\Gamma K_n(z) f(z)\,dz. \tag{4}$$

The *kernel* is given by

$$K_n(z) \equiv K_n(z,w) = \frac{\varrho_n(z)}{\pi_n(z)}, \quad z \notin [-1,1],$$

where

$$\varrho_n(z) \equiv \varrho_{n,w}(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\,w(t)\,dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_n(\overline{z})| = |K_n(z)|$.

The integral representation (4) leads to the error estimate

$$|R_n(f)| \le \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{5}$$

where $\ell(\Gamma)$ is the length of the contour Γ. In order to get estimate (5), one has to study the magnitude of $|K_n(z)|$ on Γ.

In many papers error bounds of $|R_n(f)|$, where f is an analytic function, are considered. Two choices of the contour Γ have been widely used:

- a circle C_r with a center at the origin and a radius $r\,(>1)$, i.e., $C_r = \{z \mid |z| = r\}$, $r > 1$ (cf. [2], [4], [5]), and
- an ellipse $\mathcal{E}_\rho$ with foci at the points ∓1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid \ z = \frac{1}{2}\left(\rho\, e^{i\theta} + \rho^{-1}e^{-i\theta}\right),\ 0 \le \theta \le 2\pi \right\}. \tag{6}$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1,1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1. In this paper we take Γ to be ellipse $\mathcal{E}_\rho$, then (5) has the form

$$|R_n(f)| \le \frac{\ell(\mathcal{E}_\rho)}{2\pi}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right). \tag{7}$$

Since the ellipse $\mathcal{E}_\rho$ has length $\ell(\mathcal{E}_\rho) = 4\varepsilon^{-1}E(\varepsilon)$, where ε is the eccentricity of $\mathcal{E}_\rho$, i. e., $\varepsilon = 2/(\rho+\rho^{-1})$, and $E(\varepsilon) = \int_0^{\pi/2}\sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta$ is the complete elliptic integral of the second kind, the estimate (7) reduces to

$$|R_n(f)| \le \frac{2E(\varepsilon)}{\pi\varepsilon}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\|f\|_\rho, \quad \varepsilon = \frac{2}{\rho+\rho^{-1}}, \tag{8}$$

where $\|f\|_\rho = \max_{z\in\mathcal{E}_\rho}|f(z)|$. As we can see, the bound on the right-hand side in (8) is a function of ρ, so it can be optimized with respect to $\rho > 1$.

The derivation of adequate bounds for $|R_n(f)|$ on the basis of (8) is possible only if good estimates for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$ are available. Especially useful is knowledge of the location of the extremal point $\eta\in\mathcal{E}_\rho$, at which $|K_n|$ attains its maximum. In such a case, instead of looking for upper bounds for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$ one can simply try to calculate $|K_n(\eta,w)|$. In general, this may not be an easy task, but in the case the Gauss-type quadrature formula (1) there exist effective algorithms for calculation of $K_n(z)$ at any point z outside $[-1,1]$ (see Gautschi and Varga [2]).

So far the approach (8) was discussed for Gaussian quadrature rules (1) with respect to the Chebyshev weight functions (see [2], [3])

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \quad w_2(t) = \sqrt{1-t^2}, \quad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \quad w_4(t) = \sqrt{\frac{1-t}{1+t}},$$

and later has been extended by Schira to symmetric weight functions under restriction of monotonicity type (either $w(t)\sqrt{1-t^2}$ is increasing on $(0,1)$ or $w(t)/\sqrt{1-t^2}$ is decreasing on $(0,1)$), including certain Gegenbauer weight functions (see [7]). Concerning error bounds and estimates for the Gauss-Turán quadrature formulae of analytic functions see [11] and reference therein.

With respect to the rational modification of the Chebyshev weight function of the second kind, i. e., the weight function of Bernstein-Szegő type $w_\gamma^{(1/2)}(t) = w_\gamma^{(1/2,1/2)}(t)$,

$$w_\gamma^{(1/2)}(t) = \sqrt{1-t^2}\left(1-\frac{4\gamma}{(1+\gamma)^2}\,t^2\right)^{-1}, \quad t\in(-1,1), \quad \gamma\in(-1,0),$$

we found in [8] sufficient conditions ensuring that there exists a $\rho^* = \rho_n^* = \rho_{n,\gamma}^*$ such that for each $\rho \ge \rho^*$ the kernel K_n attains its maximal absolute value at the intersection point of the ellipse with the imaginary axis. For this specialized case, we obtained much smaller values for $\rho = \rho_n^*$ than ones obtained by Schira (except for γ close to 0 and n even), especially for large values of n. Observe that the weight function $w_\gamma^{(1/2)}(t)$ belongs to the class considered by Schira [7] ($w_\gamma^{(1/2)}(t)/\sqrt{1-t^2}$ is decreasing on $(0,1)$).

In [9] the same problematic with respect to the weight function $w_\gamma^{(-1/2)}(t) = w_\gamma^{(-1/2,-1/2)}(t)$,

$$w_\gamma^{(-1/2)}(t) = (1/\sqrt{1-t^2}) \cdot \left(1 - \frac{4\gamma}{(1+\gamma)^2}\, t^2\right)^{-1}, \quad t \in (-1,1), \quad \gamma \in (-1,0),$$

has been considered. With respect to this symmetric weight function of Bernstein-Szegő type, sufficient conditions are found ensuring that there exists a $\rho^* = \rho_n^* = \rho_{n,\gamma}^*$ such that for each $\rho \geq \rho^*$ the kernel K_n attains its maximal absolute value at the intersection point of the ellipse with either the real or the imaginary axis. The corresponding analysis is much more complicated than the one in [8]. Observe that this weight function does not belong to the class of the ones considered by Schira [7].

The paper [10] is a continuation of the previous two [8], [9], as this methodology works good in the cases when the modulus of kernels have rather complicated forms. With respect to the weight function of Bernstein-Szegő type

$$w_\gamma^{(-1/2,1/2)}(t) = \sqrt{\frac{1+t}{1-t}} \cdot \left(1 - \frac{4\gamma}{(1+\gamma)^2}\, t^2\right)^{-1}, \quad t \in (-1,1), \quad \gamma \in (-1,0),$$

which is not symmetric and therefore does not belong to the class of the ones considered by Schira [7], sufficient conditions are found ensuring that there exists a $\rho^* = \rho_n^* = \rho_{n,\gamma}^*$ such that for each $\rho \geq \rho^*$ the kernel K_n attains its maximal absolute value at the intersection point of the ellipse with the positive real semi-axis. In an analogous way (using the substitution $t := -t$) a similar analysis could be derived with respect to the weight function of Bernstein-Szegő type

$$w_\gamma^{(1/2,-1/2)}(t) = \sqrt{\frac{1-t}{1+t}} \cdot \left(1 - \frac{4\gamma}{(1+\gamma)^2}\, t^2\right)^{-1}, \quad t \in (-1,1), \quad \gamma \in (-1,0).$$

In this way we rounded off this problematic with respect to the corresponding rational modification (of Bernstein-Szegő type) of all 4 classical Chebyshev weight functions. In this paper a survey of those results is presented.

2 Maximum of the modulus of kernel of the Gauss quadrature formula with the weight function $w_\gamma(t) = w_\gamma^{(1/2)}(t)$ $(\gamma \in (-1,0))$

For the weight function under consideration, the corresponding (monic) orthogonal polynomial $\pi_n(t) = \pi_{n,\gamma}(t)$ of the degree n has the form (see [1]):

$$\pi_n(t) = \pi_{n,\gamma}(t) = \frac{1}{2^n}[U_n(t) - \gamma U_{n-2}(t)], \quad n \geq 1, \tag{9}$$

where U_n denotes the Chebyshev polynomial of the second kind, characterized by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

As usual we use the substitution

$$z = \frac{1}{2}(\xi + \xi^{-1}), \quad \xi = \rho\, e^{i\theta}.$$

Using the well-known facts (cf. [2])

$$U_n(z) = \frac{\xi^{n+1} - \xi^{-(n+1)}}{\xi - \xi^{-1}}, \quad z = \frac{1}{2}(\xi + \xi^{-1}),$$

and

$$\int_{-1}^{1} \frac{U_n(t)}{z-t}\sqrt{1-t^2}\,dt = \int_0^\pi \frac{\sin(n+1)\theta\sin\theta}{z-\cos\theta}\,d\theta = \frac{\pi}{\xi^{n+1}},$$

on the basis of direct calculation, we obtain that the kernel can be expressed ($\gamma \in (-1,0),\ n \in \mathbb{N}$) in the following way

$$K_{n,\gamma}(z) = \frac{\pi(1+\gamma)^2(1-\gamma\xi^2)(\xi-\xi^{-1})}{\xi^{n+1}[(1+\gamma)^2 - \gamma(\xi+\xi^{-1})^2]\left[(\xi^{n+1}-\xi^{-(n+1)}) - \gamma(\xi^{n-1}-\xi^{-(n-1)})\right]}. \tag{10}$$

Namely,

$$\varrho_n(z) = \varrho_{n,\gamma}(z) = \int_{-1}^{1} \frac{2^n\pi_{n,\gamma}(t)}{z-t}\,\frac{\sqrt{1-t^2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\,t^2}\,dt.$$

We use the decomposition

$$\frac{1}{(z-t)\left(1 - \dfrac{4\gamma}{(1+\gamma)^2}\,t^2\right)} = \frac{A_1}{z-t} + \frac{A_2 t + A_3}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\,t^2}, \tag{11}$$

where

$$A_1 = \frac{(1+\gamma)^2}{(1+\gamma)^2 - 4\gamma z^2},\ A_2 = \frac{-4\gamma}{(1+\gamma)^2 - 4\gamma z^2},\ A_3 = \frac{-4\gamma z}{(1+\gamma)^2 - 4\gamma z^2}.$$

Multiplying (11) by $2^n\pi_{n,\gamma}(t)\sqrt{1-t^2}$ and integrating over the interval $[-1,1]$, we obtain

$$\begin{aligned}\varrho_{n,\gamma}(z) &= A_1\int_{-1}^{1} \frac{U_n(t) - \gamma U_{n-2}(t)}{z-t}\sqrt{1-t^2}\,dt \\ &+ 2^n\int_{-1}^{1} \pi_{n,\gamma}(t)(A_2 t + A_3)\,\frac{\sqrt{1-t^2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\,t^2}\,dt.\end{aligned}$$

If $n \geq 2$, in the last equality the second integral is equal to zero, then the formula (10) can be derived easily.

If $n = 1$, we have

$$\varrho_{1,\gamma}(z) = A_1 \int_{-1}^{1} \frac{U_1(t)}{z-t} \sqrt{1-t^2}\, dt + \frac{1}{2} A_2 \int_{-1}^{1} \frac{U_1^2(t)\sqrt{1-t^2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2} t^2}\, dt,$$

where $U_1(t) = 2t$. On the basis of [1, Eq. (3.22)], we have

$$\int_{-1}^{1} \frac{U_1^2(t)\sqrt{1-t^2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2} t^2}\, dt = \frac{(1+\gamma)^2\pi}{2}.$$

Finally, using the representation $U_1(z) = 2z = \xi + \xi^{-1}$, we obtain

$$K_{1,\gamma}(z) = \frac{\pi(1+\gamma)^2(1-\gamma\xi^2)}{\xi^2[(1+\gamma)^2 - \gamma(\xi+\xi^{-1})^2](\xi+\xi^{-1})},$$

which represents (10) for $n = 1$.

From here on, we use the usual notation (see for example [2])

$$a_j = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N}.$$

Using (10), and on the basis of

$$|1-\gamma\xi^2| = \left(1 + \gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta\right)^{1/2},$$

$$|\xi - \xi^{-1}| = \sqrt{2}\,(a_2 - \cos 2\theta)^{1/2},$$

$$\begin{aligned}\left|(1+\gamma)^2 - \gamma(\xi+\xi^{-1})^2\right| &= \big[(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta \\ &+ 2\gamma^2(a_4 + \cos 4\theta)\big]^{1/2},\end{aligned}$$

$$\begin{aligned}\left|\xi^{n+1} - \xi^{-(n+1)} - \gamma(\xi^{n-1} - \xi^{-n+1})\right| &= \sqrt{2}\,[a_{2n+2} - \cos(2n+2)\theta \\ &+ \gamma^2(a_{2n-2} - \cos(2n-2)\theta) \\ &- 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta)]^{1/2},\end{aligned}$$

we obtain (for $n \in \mathbb{N}$)

$$\begin{aligned}|K_{n,\gamma}(z)| &= \frac{\pi(1+\gamma)^2(a_2 - \cos 2\theta)^{1/2}\left(1+\gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta\right)^{1/2}}{\rho^{n+1}\,[(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4+\cos 4\theta)]^{1/2}} \\ &\times \frac{1}{[a_{2n+2} - \cos(2n+2)\theta + \gamma^2(a_{2n-2} - \cos(2n-2)\theta) - 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta)]^{1/2}}.\end{aligned} \tag{12}$$

Numerical experiments showed us that there exists a $\rho^* = \rho_n^* = \rho^*(n,\gamma) > 1$ so that $|K_{n,\gamma}(z)|$ attains its maximum value on the imaginary axis, i. e., at $\theta = \pi/2$, for each $\rho \geq \rho^*$. We stated and proved it in [8], as follows.

Theorem 2.1. *For the Gauss quadrature formula* (1), $n \in \mathbb{N}$, *with the weight function* $w_\gamma(t) = w_\gamma^{(1/2)}(t)$, $\gamma \in (-1, 0)$, *there exists a* $\rho^* \in (1, +\infty)$ ($\rho^* = \rho_n^* = \rho^*(n, \gamma)$) *such that for each* $\rho \geq \rho^*$ *the modulus of the kernel* $|K_{n,\gamma}(z)|$ *attains its maximum value on the imaginary axis* ($\theta = \pi/2$), *i. e.,*

$$\max_{z \in \mathcal{E}_\rho} |K_{n,\gamma}(z)| = \left| K_{n,\gamma}\left(\frac{i}{2}(\rho - \rho^{-1})\right)\right|.$$

Proof. On the basis of (12), in order to demonstrate the theorem we have to prove the following inequality for $\theta \in [0, \pi/2]$:

$$\begin{aligned} &\frac{(a_2 - \cos 2\theta)\left(1 + \gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta\right)}{(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4 + \cos 4\theta)} \\ &\times \frac{1}{a_{2n+2} - \cos(2n+2)\theta + \gamma^2(a_{2n-2} - \cos(2n-2)\theta) - 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta)} \\ \leq\ &\frac{(a_2 + 1)\left(1 + \gamma^2\rho^4 + 2\gamma\rho^2\right)}{(1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)} \\ &\times \frac{1}{a_{2n+2} + (-1)^n + \gamma^2(a_{2n-2} + (-1)^n) + 2\gamma(a_{2n} + (-1)^n a_2)}\ . \end{aligned} \tag{13}$$

First, we have

$$a_2 - \cos 2\theta \leq a_2 + 1. \tag{14}$$

Second, let us prove that for each $\rho > 1$ and $\gamma \in (-1, 0)$ there holds

$$\begin{aligned} &\frac{1 + \gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta}{(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4 + \cos 4\theta)} \\ \leq\ &\frac{1 + \gamma^2\rho^4 + 2\gamma\rho^2}{(1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)}. \end{aligned} \tag{15}$$

Let us denote

$$\begin{aligned} A &= 1 + \gamma^2\rho^4 + 2\gamma\rho^2\ (\geq 0), \quad A_1 = -4\gamma\rho^2\cos^2\theta, \\ B &= (1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)\ (\geq 0), \\ B_1 &= -8\gamma(1+\gamma^2)a_2\cos^2\theta - 4\gamma^2\sin^2 2\theta. \end{aligned}$$

The inequality (15) can now be written in the form

$$\frac{A + A_1}{B + B_1} \leq \frac{A}{B},$$

that is

$$AB_1 - BA_1 \geq 0,$$

i. e.

$$(1 + \gamma^2\rho^4 + 2\gamma\rho^2)\left(-8\gamma(1+\gamma^2)a_2\cos^2\theta - 4\gamma^2\sin^2 2\theta\right)$$

$$- \quad \big((1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)\big) \cdot (-4\gamma\rho^2\cos^2\theta) \geq 0.$$

The last inequality is satisfied for $\theta = \pi/2$. Therefore, let us consider the case when $\theta \in [0, \pi/2)$, and divide the last inequality by $\cos^2\theta$. We obtain

$$\begin{aligned}(1 \quad &+ \quad \gamma^2\rho^4 + 2\gamma\rho^2)\big(-8\gamma(1+\gamma^2)a_2 - 16\gamma^2\sin^2\theta\big)\\ &+ \quad 4\gamma\rho^2\big((1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)\big) \geq 0.\end{aligned}$$

The last inequality holds if

$$\begin{aligned}(1 \quad &+ \quad \gamma^2\rho^4 + 2\gamma\rho^2)\big((1+\gamma^2)a_2 + 2\gamma\big)\\ &- \quad \frac{1}{2}\rho^2\big((1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)\big) \geq 0.\end{aligned}$$

Now, we use $a_4 = 2a_2^2 - 1$, and conclude that the last inequality holds, since the left-hand side of it becomes

$$\begin{aligned}[(1 \quad &+ \quad \gamma^2)a_2 + 2\gamma](1+\gamma\rho^2)^2 - \frac{1}{2}\rho^2[(1+\gamma^2) + 2\gamma a_2]^2\\ &= \quad \frac{1}{2}[(1+\gamma^2)(\rho^2+\rho^{-2}) + 4\gamma](1+\gamma\rho^2)^2 - \frac{1}{2}\rho^2[(1+\gamma\rho^2) + \gamma(\gamma+\rho^{-2})]^2\\ &= \quad \frac{1}{2}\{[(1+\gamma^2)(\rho^2+\rho^{-2}) + 4\gamma](1+\gamma\rho^2)^2\\ &\quad\quad -\rho^2(1+\gamma\rho^2)^2 - 2\gamma\rho^2(1+\gamma\rho^2)(\gamma+\rho^{-2}) - \gamma^2\rho^2(\gamma+\rho^{-2})^2\}\\ &= \quad \frac{1}{2}\{[(1+\gamma^2)(\rho^2+\rho^{-2}) + 4\gamma](1+\gamma\rho^2)^2\\ &\quad\quad -\rho^2(1+\gamma\rho^2)^2 - 2\gamma(1+\gamma\rho^2)^2 - \rho^{-2}\gamma^2(1+\gamma\rho^2)^2\}\\ &= \quad \frac{1}{2}(1+\gamma\rho^2)^2(\gamma^2\rho^2 + 2\gamma + \rho^{-2})\\ &= \quad \frac{1}{2}(1+\gamma\rho^2)^2(\gamma\rho + \rho^{-1})^2.\end{aligned}$$

Therefore (15) holds, for each $\rho > 1$, $\gamma \in (-1, 0)$, $\theta \in [0, \pi/2]$.

Now, let n be EVEN. We put

$$\begin{aligned}C + C_1 = \quad & a_{2n+2} - \cos(2n+2)\theta + \gamma^2(a_{2n-2} - \cos(2n-2)\theta)\\ - \quad & 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta),\end{aligned} \tag{16}$$

where

$$\begin{aligned}C \quad &= \quad a_{2n+2} + 1 + \gamma^2(a_{2n-2}+1) + 2\gamma(a_{2n} + a_2),\\ C_1 \quad &= \quad -2\cos^2(n+1)\theta - 2\gamma^2\cos^2(n-1)\theta - 4\gamma a_{2n}\cos^2\theta - 4\gamma a_2\sin^2 n\theta.\end{aligned}$$

For the second fraction in (12) there holds

$$\frac{1}{C+C_1} \leq \frac{1}{C},$$

if $C_1 \geq 0$. This is satisfied if $\theta = \pi/2$, so we consider the case when $\theta \in [0, \pi/2)$. Using the well-known inequality

$$\left|\frac{\cos(n+1)\theta}{\cos\theta}\right| \leq n+1, \quad n \text{ even},$$

we conclude that

$$\frac{C_1}{\cos^2\theta} \equiv -2\frac{\cos^2(n+1)\theta}{\cos^2\theta} - 2\gamma^2\frac{\cos^2(n-1)\theta}{\cos^2\theta} - 4\gamma a_{2n} - 4\gamma a_2\frac{\sin^2 n\theta}{\cos^2\theta} \geq 0,$$

if there holds

$$-2(n+1)^2 - 2\gamma^2(n-1)^2 - 4\gamma a_{2n} \geq 0,$$

i. e., after dividing the last inequality by 2,

$$-(n+1)^2 - \gamma^2(n-1)^2 - 2\gamma a_{2n} \geq 0\,. \tag{17}$$

This is satisfied for each $\rho \geq \rho_E$ (> 1).

Finally, let n be ODD. In (16) we now take that

$$\begin{aligned} C &= a_{2n+2} - 1 + \gamma^2(a_{2n-2} - 1) + 2\gamma(a_{2n} - a_2), \\ C_1 &= 2\sin^2(n+1)\theta + 2\gamma^2\sin^2(n-1)\theta - 4\gamma a_{2n}\cos^2\theta + 4\gamma a_2\cos^2 n\theta. \end{aligned}$$

For the second fraction in (12) there holds $1/(C + C_1) \leq 1/C$, if $C_1 \geq 0$. This is satisfied if $\theta = \pi/2$, so we consider the case when $\theta \in [0, \pi/2)$. Similarly as in the previous case we conclude that

$$\frac{C_1}{\cos^2\theta} \equiv 2\frac{\sin^2(n+1)\theta}{\cos^2\theta} + 2\gamma^2\frac{\sin^2(n-1)\theta}{\cos^2\theta} - 4\gamma a_{2n} + 4\gamma a_2\frac{\cos^2 n\theta}{\cos^2\theta} \geq 0,$$

if there holds

$$-4\gamma a_{2n} + 4\gamma n^2 a_2 \geq 0,$$

i. e., after dividing it by -4γ, if there holds

$$a_{2n} - n^2 a_2 \geq 0\,. \tag{18}$$

If $n = 1$, then the expression $a_{2n} - n^2 a_2$ is equal to zero. If $n > 1$, let us write it in the form $h(x) = \cosh(nx) - n^2\cosh(x)$, where $x = \ln\rho^2$. We have that $h'(x) = ng(x)$, where $g(x) = \sinh(nx) - n\sinh(x)$. Since $g'(x) = n(\cosh(nx) - \cosh(x)) > 0$ for $x > 0$, g(0)=0, we conclude that the function g is positive for $x > 0$. For the function h we conclude that it is strongly increasing for $x > 0$, $h(0) < 0$. Therefore, the inequality (18) holds for each $\rho \geq \rho_O$ (> 1), $n = 3, 5, \ldots,$ and $\rho > 1$ for $n = 1$. Observe that (18) does not depend on γ.

Taking $\rho^* = \rho_E$ for n even and $\rho^* = \rho_O$ for n odd, because of (14) and (15), the inequality (13) holds on the interval $[\rho^*, +\infty)$. ⊣

3 Numerical results

The proof of Theorem 2.1 is of practical importance. Namely, on the basis of the conditions (17) and (18), we can determine the intervals $[\rho^*, +\infty)$ on which the modulus of the kernel $K_{n,\gamma}$ attains its maximum value on the imaginary axis. For some values of n, γ the values of ρ^* are displayed in the tables 1 and 2. Observe that the results become very satisfactory when n increases.

(n,γ)	ρ^*	(n,γ)	ρ^*	(n,γ)	ρ^*
$(2,-0.001)$	9.741	$(2,-0.1)$	3.081	$(2,-0.2)$	2.593
$(2,-0.3)$	2.346	$(2,-0.5)$	2.073	$(2,-0.7)$	1.917
$(2,-0.8)$	1.86	$(2,-0.9)$	1.814	$(2,-0.99)$	1.778
$(10,-0.001)$	1.796	$(10,-0.1)$	1.427	$(10,-0.2)$	1.38
$(10,-0.3)$	1.354	$(10,-0.5)$	1.327	$(10,-0.7)$	1.313
$(10,-0.8)$	1.309	$(10,-0.9)$	1.306	$(10,-0.99)$	1.305
$(30,-0.001)$	1.259	$(30,-0.1)$	1.166	$(30,-0.2)$	1.153
$(30,-0.3)$	1.146	$(30,-0.5)$	1.139	$(30,-0.7)$	1.135
$(30,-0.8)$	1.134	$(30,-0.9)$	1.134	$(30,-0.99)$	1.134
$(50,-0.001)$	1.16	$(50,-0.1)$	1.108	$(50,-0.2)$	1.1
$(50,-0.3)$	1.096	$(50,-0.5)$	1.092	$(50,-0.7)$	1.09
$(50,-0.8)$	1.09	$(50,-0.9)$	1.09	$(50,-0.99)$	1.089
$(100,-0.001)$	1.085	$(100,-0.1)$	1.06	$(100,-0.2)$	1.056
$(100,-0.3)$	1.054	$(100,-0.5)$	1.052	$(100,-0.7)$	1.052
$(100,-0.8)$	1.051	$(100,-0.9)$	1.051	$(100,-0.99)$	1.051

Table 1: The values of ρ^* for some $n \in 2\mathbb{N}$ and $\gamma \in (-1, 0)$

n	ρ^*	n	ρ^*	n	ρ^*
3	1.774	5	1.528	7	1.41
9	1.339	13	1.256	15	1.23
25	1.155	35	1.119	45	1.097
55	1.083	65	1.073	75	1.065
85	1.059	95	1.053	145	1.038

Table 2: The values of ρ^* for some $n \in 2\mathbb{N}+1$ and $\gamma \in (-1, 0)$

Remainder terms for quadrature formulas are traditionally expressed in terms of some high-order derivative of the involved function. This is a serious disadvantage, if such derivatives are not known, do not exist or are too complicated to be handled.

Let us consider numerical calculation of the integral

$$I(f) = \int_{-1}^{1} f(t) \frac{\sqrt{1-t^2}}{1 - \dfrac{4\gamma}{(1+\gamma)^2}\, t^2}\, dt, \tag{19}$$

with

$$f(t)=\frac{e^{e^t}}{(a+t)^k(b+t)^\ell(c+t)^m},$$

where $c\le b\le a<-1$; $k\in\mathbb{N}$, $\ell,m\in\mathbb{N}_0$.

Under the assumption that f is analytic inside $\mathcal{E}_{\rho_{\max}}$, from (8) we obtain the error bound

$$|R_n(f)|\le\tilde r_n(f), \tag{20}$$

where

$$\tilde r_n(f)=\inf_{\rho_n^*<\rho<\rho_{\max}}\left[\frac{\ell(\mathcal{E}_\rho)}{2\pi}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)\right],$$

and ρ_n^* is defined by Theorem 2.1. In the case under consideration $|a|=\frac12(\rho_{\max}+\rho_{\max}^{-1})$.

The length of the ellipse $\mathcal{E}_\rho$ can be estimated by (see [6, Eq. (2.2)])

$$\ell(\mathcal{E}_\rho)\le 2\pi a_1\left(1-\frac14 a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right), \tag{21}$$

where $a_1=(\rho+\rho^{-1})/2$.

For $z\in\mathcal{E}_\rho$, we have

$$e^{e^z}=e^{e^{a_1\cos\theta}\cdot\cos\left(\frac12(\rho-\rho^{-1})\sin\theta\right)}\cdot e^{i\,e^{a_1\cos\theta}\cdot\sin\left(\frac12(\rho-\rho^{-1})\sin\theta\right)},$$

and from this it follows that

$$\max_{z\in\mathcal{E}_\rho}|e^{e^z}|=e^{e^{a_1}}. \tag{22}$$

The above maximum is attained at $\theta=0$.

Further, we have

$$\frac{1}{|a+z|}=\frac{1}{\sqrt{a^2+\frac12(a_2-1)+2aa_1\cos\theta+\cos^2\theta}}\le\frac{1}{|a+a_1|},$$

where the equality holds for $\theta=0$. We have used the facts that the function under the squared root has minimum at $\theta=0$ and $a_2=2a_1^2-1$.

On the basis of the above analysis and (22), we have

$$\max_{z\in\mathcal{E}_\rho}\left|\frac{e^{e^z}}{(a+z)^k(b+z)^\ell(c+z)^m}\right|=\frac{e^{e^{a_1}}}{|a+a_1|^k\,|b+a_1|^\ell\,|c+a_1|^m},$$

where the maximum is attained at $\theta=0$. Now, $r_n(f)$ $(\ge\tilde r_n(f))$ has the form

$$\begin{aligned} r_n(f) &= \inf_{\rho_n^*<\rho<\rho_{\max}}\left\{\pi a_1\left(1-\frac14 a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\right.\\ &\times\ \frac{e^{e^{a_1}}}{|a+a_1|^k\,|b+a_1|^\ell\,|c+a_1|^m}\end{aligned}$$

$$
\begin{aligned}
&\times \quad \frac{(1+\gamma)^2(a_2+1)^{1/2}\left(1+\gamma\rho^2\right)}{\rho^{n+1}\left[(1+\gamma^2)^2+4\gamma(1+\gamma^2)a_2+2\gamma^2(a_4+1)\right]^{1/2}} \\
&\times \quad \left[a_{2n+2}+(-1)^n+\gamma^2(a_{2n-2}+(-1)^n)+2\gamma(a_{2n}+(-1)^n a_2)\right]^{-1/2}\Big\} \,.
\end{aligned}
$$

Let $-\sqrt{2}<a<-1,\ c\le b\le a$. This condition means that the function f is analytic inside the elliptical contour $\mathcal{E}_{\rho_{\max}}$, where $\rho_{\max}=1+\sqrt{2}$. Therefore, the results obtained by Schira [7] cannot be used here. Also, the classical error bound in this case is difficult to determine, since the derivatives $f^{(2n)}(t)$ for higher values n are too complicated to be handled. However, we can use the error bound (20) based on the results of Theorem 2.1.

The error bound (20) is valid for integrands analytic on a neighborhood of the interval of integration and should be compared with other error bounds intended for the same class of integrands. There are several classical error bounds for Gaussian quadrature rules of analytic functions. See Theorem 4 in [12] or Theorem 1 in [13], where the contour Γ is the ellipse $\mathcal{E}_\rho$ given by (6). We also take into account the error bounds appearing in [4], where the contour Γ is the circumference $C_r=\{z\in\mathbb{C}:\ |z|=r\}$ $(r>1)$.

Therefore, the error bound $\hat{r}_n(f)$ $(|R_n(f)|\le\hat{r}_n(f))$ of the Gauss quadrature formula (1) with respect to the weight function (2), for the integrand f under consideration, can be given by (see Stenger [12, Eq. (38)])

$$
\hat{r}_n(f)=\hat{r}_n^{(\mathrm{Sten})}(f)=\inf_{1<\rho<\rho_{\max}}\left\{\frac{16\mu_0}{\pi\rho^{2n}}\cdot\frac{e^{e^{a_1}}}{|a+a_1|^k\,|b+a_1|^\ell\,|c+a_1|^m}\right\},
$$

where $\mu_0=\pi(1+\gamma)/2$ (cf. [1, Eqs. (2.24),(2.27)]), or by (see von Sydow [13, Th. 1])

$$
\hat{r}_n(f)=\hat{r}_n^{(\mathrm{Syd})}(f)=\inf_{1<\rho<\rho_{\max}}\left\{\frac{4\mu_0}{(1-\rho^{-2})\rho^{2n}}\cdot\frac{e^{e^{a_1}}}{|a+a_1|^k\,|b+a_1|^\ell\,|c+a_1|^m}\right\},
$$

or by (see Notaris [4, Eq. (3.28)])

$$
\begin{aligned}
\hat{r}_n(f)=\hat{r}_n^{(\mathrm{Not})}(f) &= \inf_{1<r<r_{\max}}\left\{\frac{2\pi(1+\gamma)^2\,\tau^{2n+2}\,r\sqrt{r^2-1}}{(1-\gamma\tau^2)[1-\tau^{2n+2}-\gamma\tau^2(1-\tau^{2n-2})]}\right. \\
&\qquad\times\left.\frac{e^{e^r}}{|a+r|^k\,|b+r|^\ell\,|c+r|^m}\right\},
\end{aligned}
$$

where $\tau=r-\sqrt{r^2-1}$ and $r_{\max}=|a|$.

Let the integrand f be specialized by $k=1,\ell=5,m=10$, and

$$
a=-1.408333333333333,\quad b=-1.892857142857143,\quad c=-2.408695652173913,
$$

which means that $\rho_{\max}=2.4$.

We have been calculating the values of $\hat{r}_n^{(\mathrm{Sten})}(f),\hat{r}_n^{(\mathrm{Syd})}(f),\hat{r}_n^{(\mathrm{Not})}(f),r_n(f)$ for the corresponding integral $I(f)$ given by (19). The results do show the effectiveness of the error bound (20) as compared to, for instance, the error bounds given by $\hat{r}_n^{(\mathrm{Sten})}(f), \hat{r}_n^{(\mathrm{Syd})}(f)$. For some values of γ and $n = 15, 35$, the obtained results are displayed in Table 3. (Numbers in parentheses indicate decimal exponents.)

γ	$\hat{r}_{15}^{(\mathrm{Sten})}(f)$	$\hat{r}_{15}^{(\mathrm{Syd})}(f)$	$\hat{r}_{15}^{(\mathrm{Not})}(f)$	$r_{15}(f)$	$\hat{r}_{35}^{(\mathrm{Sten})}(f)$	$\hat{r}_{35}^{(\mathrm{Syd})}(f)$	$\hat{r}_{35}^{(\mathrm{Not})}(f)$	$r_{35}(f)$
-0.1	$1.63(-6)$	$1.61(-6)$	$2.64(-7)$	$3.66(-7)$	$5.04(-21)$	$4.84(-21)$	$8.31(-22)$	$1.11(-21)$
-0.2	$1.45(-6)$	$1.43(-6)$	$2.01(-7)$	$2.30(-7)$	$4.48(-21)$	$4.30(-21)$	$6.34(-22)$	$9.04(-22)$
-0.3	$1.27(-6)$	$1.26(-6)$	$1.48(-7)$	$2.41(-7)$	$3.92(-21)$	$3.76(-21)$	$4.69(-22)$	$7.19(-22)$
-0.4	$1.09(-6)$	$1.08(-6)$	$1.05(-7)$	$1.86(-7)$	$3.36(-21)$	$3.23(-21)$	$3.33(-22)$	$5.49(-22)$
-0.5	$9.04(-7)$	$8.93(-7)$	$6.97(-8)$	$1.35(-7)$	$2.80(-21)$	$2.69(-21)$	$2.24(-22)$	$3.97(-22)$
-0.6	$7.23(-7)$	$7.15(-7)$	$4.30(-8)$	$9.01(-8)$	$2.24(-21)$	$2.15(-21)$	$1.39(-22)$	$2.64(-22)$
-0.7	$5.42(-7)$	$5.36(-7)$	$2.34(-8)$	$5.32(-8)$	$1.68(-21)$	$1.62(-21)$	$7.54(-23)$	$1.55(-22)$
-0.8	$3.62(-7)$	$3.58(-7)$	$9.99(-9)$	$2.48(-8)$	$1.12(-21)$	$1.08(-21)$	$3.25(-23)$	$7.17(-23)$
-0.9	$1.81(-7)$	$1.79(-7)$	$2.42(-9)$	$6.51(-9)$	$5.60(-22)$	$5.37(-22)$	$7.87(-24)$	$1.88(-23)$

Table 3: The values of $\hat{r}_n^{(\mathrm{Sten})}(f), \hat{r}_n^{(\mathrm{Syd})}(f), \hat{r}_n^{(\mathrm{Not})}(f), r_n(f)$ for $n = 15, 35$ and some $\gamma \in (-1, 0)$

At the end, let us consider numerical calculation of the integral (19), with

$$f(t) = \bar{f}(t) = \cos t.$$

The function $\bar{f}(z) = \cos z$ is entire, and for it holds

$$\max_{z \in C_r} |\cos z| = \cosh(r),$$

and

$$\max_{z \in \mathcal{E}_\rho} |\cos z| = \cosh(b_1),$$

where $b_1 = \frac{1}{2}(\rho - \rho^{-1})$.

For some values of γ and $n = 5, 9$, the obtained results of $\hat{r}_n^{(\mathrm{Sten})}(\bar{f}), \hat{r}_n^{(\mathrm{Syd})}(\bar{f})$, $\hat{r}_n^{(\mathrm{Not})}(\bar{f}), r_n(\bar{f})$ are displayed in Table 4. The results do show the effectiveness of the error bound (20).

4 The corresponding results for $w_\gamma^{(-1/2)}(t)$ and $w_\gamma^{(-1/2,1/2)}(t)$

Theorem 4.1. *For the Gauss quadrature formula* (1) *with the weight function* $w_\gamma^{(-1/2)}(t)$ *there exists a* $\rho^* = \rho_n^* = \rho_{n,\gamma}^* \in (1, +\infty)$ *such that for each* $\rho > \rho^*$ *the modulus of the kernel* $\left|K_{n,\gamma}^{(-1/2)}(z)\right|$ *attains its maximum:*

a) on the real axis (positive real semi-axis $(\theta = 0)$*), i. e.,*

$$\max_{z \in \mathcal{E}_\rho} \left|K_{n,\gamma}^{(-1/2)}(z)\right| = K_{n,\gamma}^{(-1/2)}\left(\frac{1}{2}(\rho + \rho^{-1})\right),$$

if

γ	$\hat{r}_5^{(\mathrm{Sten})}(\bar{f})$	$\hat{r}_5^{(\mathrm{Syd})}(\bar{f})$	$\hat{r}_5^{(\mathrm{Not})}(\bar{f})$	$r_5(\bar{f})$	$\hat{r}_9^{(\mathrm{Sten})}(\bar{f})$	$\hat{r}_9^{(\mathrm{Syd})}(\bar{f})$	$\hat{r}_9^{(\mathrm{Not})}(\bar{f})$	$r_9(\bar{f})$
−0.1	7.57(−9)	5.97(−9)	1.41(−9)	1.35(−9)	2.27(−20)	1.78(−20)	4.11(−21)	4.01(−21)
−0.2	6.73(−9)	5.30(−9)	1.11(−9)	1.07(−9)	2.02(−20)	1.59(−20)	3.25(−21)	3.17(−21)
−0.3	5.89(−9)	4.64(−9)	8.48(−10)	8.13(−10)	1.76(−20)	1.39(−20)	2.49(−21)	2.43(−21)
−0.4	5.05(−9)	3.98(−9)	6.23(−10)	5.98(−10)	1.51(−20)	1.19(−20)	1.83(−21)	1.78(−21)
−0.5	4.21(−9)	3.32(−9)	4.32(−10)	4.16(−10)	1.26(−20)	9.88(−21)	1.27(−21)	1.24(−21)
−0.6	3.37(−9)	2.66(−9)	2.77(−10)	2.66(−10)	1.01(−20)	7.91(−21)	8.11(−22)	7.91(−22)
−0.7	2.53(−9)	1.99(−9)	1.56(−10)	1.50(−10)	7.55(−21)	5.93(−21)	4.56(−22)	4.45(−22)
−0.8	1.69(−9)	1.33(−9)	6.91(−11)	6.66(−11)	5.03(−21)	3.96(−21)	2.03(−22)	1.98(−22)
−0.9	8.42(−10)	6.63(−10)	1.73(−11)	1.67(−11)	2.52(−21)	1.98(−21)	5.07(−23)	4.95(−23)

Table 4: The values of $\hat{r}_n^{(\mathrm{Sten})}(\bar{f}), \hat{r}_n^{(\mathrm{Syd})}(\bar{f}), \hat{r}_n^{(\mathrm{Not})}(\bar{f}), r_n(\bar{f})$ for $n = 5, 9$ and some $\gamma \in (-1, 0)$

(i) $\gamma \in (-1/3, 0)$, $n = 2$,
(ii) $\gamma \in (-1/2, 0)$, $n \geq 3$,
(iii) $\gamma = -1/2$, $n \geq 5$;

b) on the imaginary axis ($\theta = \pi/2$), *i. e.,*

$$\max_{z \in \mathcal{E}_\rho} \left| K_{n,\gamma}^{(-1/2)}(z) \right| = \left| K_{n,\gamma}^{(-1/2)} \left(\frac{i}{2}(\rho - \rho^{-1}) \right) \right|,$$

if

(i) $\gamma \in (-1, 0)$, $n = 1$,
(ii) $\gamma \in (-1, -1/3)$, $n = 2$,
(iii) $\gamma \in (-1, -1/2)$, $n \geq 3$,
(iv) $\gamma = -1/2$, $n = 3, 4$,

c) on the line which bisect the angle between the axes ($\theta = \pi/4$) *i. e.,*

$$\max_{z \in \mathcal{E}_\rho} \left| K_{n,\gamma}^{(-1/2)}(z) \right| = \left| K_{n,\gamma}^{-1/2} \left(-\frac{1}{2\sqrt{2}}(\rho + \rho^{-1}) + \frac{i}{2\sqrt{2}}(\rho - \rho^{-1}) \right) \right|.$$

if $\gamma = -1/3$, $n = 2$.

Theorem 4.2. *For the Gauss quadrature formula* (1) *with the weight function* $w_\gamma^{(-1/2,1/2)}(t)$ *there exists a* $\rho^* = \rho_n^* = \rho_{n,\gamma}^* \in (1, +\infty)$ *such that for each* $\rho > \rho^*$ *the modulus of the kernel* $\left| K_{n,\gamma}^{(-1/2,1/2)}(z) \right|$ *attains its maximum on the real axis (positive real semi-axis* ($\theta = 0$)*), i. e.,*

$$\max_{z \in \mathcal{E}_\rho} \left| K_{n,\gamma}^{(-1/2,1/2)}(z) \right| = K_{n,\gamma}^{(-1/2,1/2)} \left(\frac{1}{2}(\rho + \rho^{-1}) \right),$$

for $\gamma \in (-1, 0)$.

Theorem 4.3. *If $n = 1$, for the Gauss quadrature formula* (1) *with the weight function $w_\gamma^{(-1/2,1/2)}(t)$ there exists a $\rho_1^* = \rho_{1,\gamma}^* \in (1, +\infty)$ such that for each $\rho > \rho_1^*$ the modulus of the kernel $\left|K_{1,\gamma}^{(-1/2,1/2)}(z)\right|$ attains its maximum on the real axis (positive real semi-axis $(\theta = 0)$), i. e.,*

$$\max_{z\in\mathcal{E}_\rho} \left|K_{1,\gamma}^{(-1/2,1/2)}(z)\right| = K_{1,\gamma}^{(-1/2,1/2)}\left(\frac{1}{2}(\rho+\rho^{-1})\right),$$

for $\gamma \in (-1, 0)$.

The proof of Theorem 4.1 can be found in [9], and the proofs of Theorems 4.2, 4.3 can be found in [10].

5 Acknowledgments

This work was supported in part by the Serbian Ministry of Education, Science and Technological Development (Research Project: "Methods of numerical and nonlinear analysis with applications" (#174002)).

References

[1] W. Gautschi and S.E. Notaris, *Gauss-Kronrod quadrature formulae for weight function of Bernstein-Szegő type*, J. Comput. Appl. Math. 25 (1989) 199–224; erratum in J. Comput. Appl. Math. 27 (1989) 429

[2] W. Gautschi and R. S. Varga, *Error bounds for Gaussian quadrature of analytic functions*, SIAM J. Numer. Anal. 20 (1983) 1170–1186

[3] W. Gautschi, E. Tychopoulos, and R. S. Varga, *A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rule*, SIAM J. Numer. Anal. 27 (1990) 219–224

[4] S.E. Notaris, *The error norm of Gaussian quadrature formulae for weight functions of Bernstein-Szegő type*, Numer. Math. 57 (1990) 271–283

[5] F. Peherstorfer, *On the remainder of Gaussian quadrature formulas for Bernstein-Szegő weight functions*, Math. Comput. 60 (1993) 317–325

[6] R. Scherer, T. Schira, *Estimating quadrature errors for analytic functions using kernel representations and biorthogonal systems*, Numer. Math., 84 (2000), pp. 497–518.

[7] T. Schira, *The remainder term for analytic functions of symmetric Gaussian quadratures*, Math. Comput. 66 (1997) 297–310

[8] M.M. Spalević and M.S. Pranić, *Error bounds of certain Gaussian quadrature formulae*, J. Comput. Appl. Math. 234 (2010) 1049–1057

[9] M.M. Spalević, M.S. Pranić, and A.V. Pejčev, *Maximum of the modulus of kernels of Gaussian quadrature formulae for one class of Bernstein-Szegő weight functions*, Appl. Math. Comput. 218 (2012) 5746–5756

[10] M.M. Spalević, *Error bounds of Gaussian quadrature formulae for one class of Bernstein-Szegő weights*, Math. Comput. 82 (2013) 1037–1056

[11] M.M. Spalević, *Error bounds and estimates for Gauss-Turán quadrature formulae of analytic functions*, submitted

[12] F. Stenger, *Bounds on the error of Gauss-type quadratures*, Numer. Math. 8 (1966) 150–160

[13] B. von Sydow, *Error estimates for Gaussian quadrature formulae*, Numer. Math. 29 (1977) 59–64

THIRD MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
Trebinje, 7 and 8 June 2013

Orbits of a $k-$sets of $\mathbb{Z}_n$

Vladimir Božović
Faculty of Natural Sciences and Mathematics, University of Montenegro
vladobozovic@yahoo.com
Srđan Kadić
Faculty of Natural Sciences and Mathematics, University of Montenegro
srdjankadic@gmail.com
Žana Kovijanić Vukićević
Faculty of Natural Sciences and Mathematics, University of Montenegro
zanak@t-com.me

Original scientific paper

Abstract

We consider the action of the automorphism group $\mathcal{I}(n)$ of $\mathbb{Z}_n$ on the set of $k-$sets of $\mathbb{Z}_n$ in the natural way. Although elementary in its nature, it has not been fully analyzed and understood yet. The vast class of enumerative and computational problems problems is related to this action. For example, the number of orbits on the set of $k-$sets of $\mathbb{Z}_n$ is one of them that we are interested in. Those enumerative problems are mainly resolved by application of Pólya's theory.

1 Introduction

Let $\mathcal{I}(n)$ be the automorphism group of cyclic additive group $\mathbb{Z}_n$. We consider the action of the group $\mathcal{I}(n)$ on the set of elements of $\mathbb{Z}_n$, given by

$$(x,t) \to tx \ \ (t \in \mathcal{I}(n),\ x \in \mathbb{Z}_n).$$

There is a natural way to induce this action on the set $\mathcal{O}_k$, that denotes the set of all subsets of $\mathbb{Z}_n$ of size k. In order to answer to some of the standard enumerative questions regarding this action, we need to determine the cycle index of $\mathcal{I}(n)$ acting on $\mathbb{Z}_n$. We first find the cycle index of the permutation groups $\mathcal{I}(p^m)$ acting on $\mathbb{Z}_{p^m}$, where p is a prime number and then use technique described in [4] in order to find the cycle index of the product of permutation groups. Consequently, our goal is to obtain the cycle index of $\mathcal{I}(n)$ acting on $\mathbb{Z}_n$.
Besides this combinatorial aspect of the described action, we were interested in some computational questions as finding the stabilizer of a $k-$set $A \subseteq \mathbb{Z}_n$. Once a stabilizer is found, there is a straightforward way to determine the orbit that a set A belongs to.

Motivation for this type of action is twofold. Firstly, it is very natural, elementary group action that, to the best of our knowledge, has not been fully analyzed and understood yet. There are couple of papers, like [5, 6] that deal with the very similar things but not in the same fashion.

Another motivation for the action of $\mathcal{I}(n)$ on the set $\mathcal{O}_k$ comes from one fact related to the factorizations of abelian groups. In the case of an abelian group G, a *factorization* is a collection of subsets

$$\alpha = [B_1, \dots, B_k]$$

such that that every element $g \in G$ has a unique representation $g = s_1 s_2 \dots s_k$, where $s_i \in B_i$ for $1 \leq i \leq k$. The subsets $B_i,\ \ 1 \leq i \leq k$ of G are called the *blocks* of the factorization.
Suppose the case of factorization of $\mathbb{Z}_{mn}$ into two blocks $[B_1, B_2]$ of size m and n respectively. Given $t, s \in \mathbb{Z}_{mn}$ such that $\gcd(t, mn) = 1$, $\gcd(s, mn) = 1$, it is proven that $[tB_1, sB_2]$ is also factorization of $\mathbb{Z}_{mn}$. That is why it is substantially enough to have just representatives of orbits of m and $n-$sets for describing all factorization of $\mathbb{Z}_{mn}$.

2 The notion of $(r, k)-$coprime residue set in $\mathbb{Z}_n$

In this section, we introduce the notion of a $(r, k)-$coprime residue set in $\mathbb{Z}_n$ and give their analysis from the algebraic and number theoretical point of view. Here, by *natural* number we assume positive integer.

Definition 2.1. Let r, k be natural numbers such that $\gcd(r, k) = 1$, $r < k$ and let k be a divisor of natural number n. A set of integers

$$\mathcal{I}_k^r(n) = \{x \in \mathcal{I}(n) |\ x \equiv r\ (mod\ k)\}$$

is called $(r, k)-$coprime residue set in $\mathbb{Z}_n$.

Firstly, we prove that any $(r, k)-$coprime set in $\mathbb{Z}_n$ is not empty.

Lemma 2.1. *Let r, k, ℓ, n be natural numbers such that $\gcd(r, k) = 1$, $r < k$ and $n = k\ell$. Then (r, k)-coprime set $\mathcal{I}_k^r(n)$ is nonempty.*

Proof. We prove for given r, k and n and $\gcd(r, k) = 1$, there exists t such that

$$\gcd(r + ut, n) = 1$$

Let $p_i^{v_i}$ be a general prime power divisor of n. Then, there exists t_i such that

$$\gcd(r + kt_i, p_i^{v_i}) = 1$$

Namely, if $p_i \mid k$, then $p_i \nmid r$ and $t_i = 0$ suffices. If $p_i \nmid k$, than any number t_i such that

$$t_i \not\equiv -r/k \pmod{p_i}$$

will work. By Chinese Reminder Theorem, there exists t such that

$$t \equiv t_i \pmod{p_i}$$

and $\gcd(r + kt, n) = 1$. We need to prove that there exists $x \in \mathcal{I}(n)$ such that $x \equiv r$ $(mod\ k)$. Let $x \equiv r + kt \pmod{n}$. Since $k \mid n$ then $x \equiv r \pmod{k}$. Also, it is easy to see that $\gcd(x, n) = 1$ and therefore $x \in \mathcal{I}(n)$.

⊣

Lemma 2.2. *Let r, k, ℓ be natural numbers such that $gcd(r, k) = 1$ and $r < k$. It follows that*

$$|\mathcal{I}_k^r(k\ell)| = |\mathcal{I}_k^1(k\ell)|.$$

Proof. According to Lemma 2.1, both sets $\mathcal{I}_k^r(k\ell)$ and $\mathcal{I}_k^1(k\ell)$ are nonempty. Let $x \in \mathcal{I}_k^r(k\ell)$. It follows that $x^{-1}\mathcal{I}_k^r(k\ell) \subseteq \mathcal{I}_k^1(k\ell)$. Hence, we have

$$|x^{-1}\mathcal{I}_k^r(k\ell)| = |\mathcal{I}_k^r(k\ell)|$$

and therefore

$$|\mathcal{I}_k^r(k\ell)| \leq |\mathcal{I}_k^1(k\ell)| \tag{1}$$

Similarly, $x\mathcal{I}_k^1(k\ell) \subseteq \mathcal{I}_k^r(k\ell)$ implies

$$|\mathcal{I}_k^1(k\ell)| \leq |\mathcal{I}_k^r(k\ell)|. \tag{2}$$

From inequalities 1 and 2, it follows that

$$|\mathcal{I}_k^1(k\ell)| = |\mathcal{I}_k^r(k\ell)|$$

⊣

Lemma 2.3. *Let k, ℓ be natural numbers and $k > 1$. Then $\mathcal{I}_k^1(k\ell)$ is a subgroup of $\mathcal{I}(k\ell)$.*

Proof. According to the definition of $\mathcal{I}_k^1(k\ell)$, it is clear that $\mathcal{I}_u^1(k\ell) \subseteq \mathcal{I}(k\ell)$. Apparently the identity, 1, is in $\mathcal{I}_k^1(k\ell)$. For any $x, y \in \mathcal{I}_u^1(k\ell)$, it holds $xy^{-1} \equiv 1$ $(mod\ k)$, i.e. $xy^{-1} \in \mathcal{I}_k^1(k\ell)$ that concludes the proof. ⊣

Lemma 2.4. *Let k and ℓ be relatively prime natural numbers and $k > 1$. Then, it holds*

$$\mathcal{I}_k^1(k\ell) \cong \mathcal{I}(\ell).$$

Proof. Let $\mathcal{A}$ be a mapping from $\mathcal{I}_k^1(k\ell)$ to $\mathcal{I}(\ell)$ defined by

$$\mathcal{A}(x) = x\ mod\ \ell$$

First, we show that $Im(\mathcal{A}) \subseteq \mathcal{I}(\ell)$. Let $x \in \mathcal{I}_k^1(k\ell)$. Then, $x = a\ell + b,\ \ 0 \leq b \leq \ell$. Since $x \in \mathcal{I}_k^1(k\ell)$, then by the definition of that set, it follows that $x \in \mathcal{I}(k\ell)$. Therefore $gcd(x, \ell) = 1$ and consequently $gcd(b, \ell) = 1$.

Thus, $b \in \mathcal{I}(\ell)$, so we have $\mathcal{A}(x) \in \mathcal{I}(\ell)$.

$\mathcal{A}$ is evidently homomorphism, according to properties of modulo operation.
$\mathcal{A}$ is one to one. Let $x, y \in \mathcal{I}_k^1(k\ell)$ and $\mathcal{A}(x) = \mathcal{A}(y)$. From the definition of $\mathcal{I}_k^1(k\ell)$, we have $x \equiv 1 \pmod{k}$ and $y \equiv 1 \pmod{k}$, so $x \equiv y \pmod{k}$. From $\mathcal{A}(x) = \mathcal{A}(y)$ it follows $x \equiv y \pmod{\ell}$. Since k and ℓ are relatively prime numbers, then $x \equiv y \pmod{k\ell}$, so $\mathcal{A}$ is one to one.

$\mathcal{A}$ is onto. Let $z \in \mathcal{I}(\ell)$. We have to find $x \in \mathcal{I}_k^1(k\ell)$ such that $\mathcal{A}(x) = \ell$, or in other words $x \equiv z \ (mod\ \ell)$. That x must be of the form $1 + kt$, so we should find such a t for which it holds $x \equiv z \ (mod\ \ell)$. From $gcd(k, \ell) = 1$, there exist $m, n \in \mathbb{Z}$ such that $mk + n\ell = 1$. Let us define $t = (z-1)m$, i.e. $x = 1 + (z-1)mk$. Clearly, $x \equiv 1 \ (mod\ k)$. Note that $x = 1 + (z-1)(1 - n\ell)$, that is $x = z + n\ell(1 - z)$, so $x \equiv z \pmod{\ell}$. Now, we need to prove that $gcd(x, \ell) = 1$. Let p be a prime divisor of x and l. Then, p divides z, from which we would have that $p \mid gcd(z, \ell)$ what is impossible since $z \in \mathcal{I}(\ell)$. Therefore, $gcd(x, k\ell) = 1$. At the end, we need to provide that $x < k\ell$. If $x = 1 + (z-1)mk$ is not less than $k\ell$ then we should take $x = 1 + (z-1)mk \pmod{k\ell}$ and all previously given arguments hold. ⊣

Corollary 2.1. *Let r, k, ℓ be natural numbers such that $r < k$, $\gcd(k, \ell) = 1$ and $\gcd(r, k) = 1$. Then, it holds*

$$|\mathcal{I}_k^r(k\ell)| = \phi(\ell).$$

Proof. It follows directly from Lemma 2.2 and Lemma 2.4. ⊣

Our goal is to find the cardinality of the set $\mathcal{I}_k^r(k\ell)$ when k and ℓ are not necessarily relatively prime numbers and when $gcd(r, k) = 1$. As we sow in the proof of Lemma 2.1 it holds $gcd(x, k\ell) = 1 \Leftrightarrow gcd(x, k\ell') = 1$ where ℓ' is the largest divisor of ℓ that is relatively prime to k. This gives us idea for the following lemma.

Lemma 2.5. *Let k, ℓ be natural numbers and $k > 1$. It follows that*

$$|\mathcal{I}_k^1(k\ell)| = \phi(\ell')\frac{\ell}{\ell'}$$

where ℓ' is the largest divisor of ℓ that is relatively prime to ℓ.

Proof. According to Lemma 2.3 $\text{¸}_k^1(k\ell)$ is a subgroup of $\text{¸}(k\ell)$. Let us define a homomorphism $\mathcal{S}$ from $\mathcal{I}_k^1(k\ell)$ to $\mathcal{I}_k^1(k\ell')$ in the following way

$$\mathcal{S}(x) = x \ \ mod\ k\ell'$$

This is evidently epimorphism and $Ker(\mathcal{S}) = \{1 + tk\ell' \mid 0 \le t < \frac{\ell}{\ell'}\}$. Therefore, we have that

$$|\mathcal{I}_k^1(k\ell)| = |\mathcal{I}_k^1(k\ell')|\frac{\ell}{\ell'}$$

By Corollary 2.1 it follows that $|\mathcal{I}_k^1(k\ell')| = \phi(\ell')$ and this concludes the proof. ⊣

Lemma 2.6. *Let k, ℓ be natural numbers and $k > 1$. Then it follows that*

$$|\mathcal{I}_k^1(k\ell)| = \frac{\phi(k\ell)}{\phi(k)}.$$

Proof. By Lemma 2.5 it holds that

$$\phi(\ell') = \frac{\ell' |\mathcal{I}_k^1(k\ell)|}{\ell}$$

where ℓ' is the largest divisor of ℓ that is relatively prime to k. Let $\ell = \ell'\ell''$. Clearly, $\ell'' \mid k$. Then $gcd(k\ell'', \ell') = 1$ and therefore $\phi(k\ell) = \phi(k\ell'')\phi(\ell')$. Since $\ell'' \mid k$ then

$$\phi(k\ell'') = k\ell'' \prod_{p|k} (1 - \frac{1}{p}) = \ell''\phi(k)$$

Therefore,

$$\phi(k\ell'')\phi(\ell') = \ell''\phi(k) \frac{\ell' |\mathcal{I}_k^1(k\ell)|}{\ell}$$

what implies

$$\phi(k\ell) = \frac{\phi(k)}{|\mathcal{I}_k^1(k\ell)|}$$

and

$$|\mathcal{I}_k^1(k\ell)| = \frac{\phi(k\ell)}{\phi(k)}$$

⊣

Corollary 2.2. *Let k, ℓ, r be natural numbers such that* $\gcd(r, k) = 1$ *and $r < k$. Then,*

$$|\mathcal{I}_k^r(k\ell)| = \frac{\phi(k\ell)}{\phi(k)}.$$

Proof. It follows directly from Lemma 2.2 and Lemma 2.6. ⊣

3 Action of $\mathcal{I}(n)$ on $k-$sets of $\mathbb{Z}_n$

It is well known fact [7] that automorphisms of $\mathbb{Z}_n$ are mappings $\pi_t : \mathbb{Z}_n \to \mathbb{Z}_n$ of the form

$$\pi_t(x) = tx, \quad (x \in \mathbb{Z}_n)$$

where $\gcd(t, n) = 1$. We denote the automorphism group of $\mathbb{Z}_n$ by $\mathcal{I}(n)$. There is actually obvious isomorphism between group $\mathcal{I}(n)$ and the multiplicative group of all positive integers that are coprime to n. Further we will denote by $\mathcal{I}(n)$ that group too, but it will be clear from the context which of those two we have in mind.

Consider the following, natural action of the group $\mathcal{I}(n)$ on the additive cyclic group $\mathbb{Z}_n$ by multiplication, $\mathbb{Z}_n \times \mathcal{I}(n) \to \mathbb{Z}_n$,

$$(x, t) \to tx \quad (t \in \mathcal{I}(n),\ x \in \mathbb{Z}_n)$$

Naturally, we can extend this action to $k-$ sets in $\mathbb{Z}_n$. By $k-$set we assume a set of size k. Let $\mathcal{O}_k$ be the collection of $k-$sets of $\mathbb{Z}_n$. Given a set $A = \{a_1, a_2, \ldots, a_k\}$, let us define multiplication of the set A by an element $t \in \mathbb{Z}_n$ in the usual way, that is

$$tA = \{ta_1, ta_2, \ldots, ta_k\}.$$

We consider the following type of group action $\mathcal{O}_k \times$ ¸$(n) \to \mathcal{O}_k$ defined as

$$(A, t) \to tA \quad (t \in \mathcal{I}(n),\ A \in \mathcal{O}_k)$$

If not specified differently, the term *action* will be further reserved for the one described above. By Ω_d we denote the set of elements of order d in the $\mathbb{Z}_n$. Obviously,

$$\mathbb{Z}_n = \biguplus_{d|n} \Omega_d$$

and $|\Omega_d| = \phi(d)$, where ϕ is Euler's phi function.

3.1 Cycle index of $\mathcal{I}(n)$ and orbits of $k-$sets of $\mathbb{Z}_n$

There is a class of interesting enumerative problems regarding the action of $\mathcal{I}(n)$ on $\mathcal{O}_k$ that principally could be answered by applying Pólya's theory. There is a lot of literature about PólyaвЂ™s counting theory. For instance see [1, 2, 3]. Frequently, the main combinatorial problem about the action of a permutation group acting on a set is to determine its cycle index. We determine cycle index of the group $\mathcal{I}(n)$ acting on $\mathbb{Z}_n$ and show that it contains information about the number of orbits of $k-$sets of $\mathbb{Z}_n$. Firstly, we give basic definitions.

Definition 3.1. (Type of a Permutation) Let M be a set with $|M| = m$. A permutation $\pi \in S_M$ is of the type $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, iff π can be written as the composition of λ_i disjointed cycles of length i, for $i = 1, \ldots, m$. Hence, by $\lambda_i(\pi)$ we mean the number of cycles of length i in the decomposition of π into disjoint cycles. Shortly, we write

$$ctype(\pi) = \left(\prod_{i=1}^{m} i^{\lambda_i} \right)$$

Definition 3.2. (Cycle Index) Let P be a set of $|P| = n$ elements and let Γ be a subgroup of S_P. The cycle index of Γ is defined as a polynomial in n indeterminates $x_1, \ldots, x_n$, defined as:

$$\mathcal{Z}_{(\Gamma, P)}(x_1, \ldots, x_n) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \prod_{i=1}^{n} x_i^{\lambda_i(\gamma)}.$$

The following famous theorem is Pólya's theorem [10], published in 1937.

Theorem 3.1. *Let P and F be finite sets with $|P| = n$, and let $\Gamma \leq S_P$. Furthermore let R be a commutative ring over the rationals Q and let w be a mapping $w : F \to R$. Two mappings $f_1, f_2 \in F^P$ are called equivalent, iff there exists some $\gamma \in \Gamma$ such that $f_1 \circ \gamma = f_2$. The equivalence classes are called mapping patterns and are written as $[f]$. For every $f \in F^P$ we define the weight $W(f)$ as product weight*

$$W(f) := \prod_{p \in P} w(f(p)).$$

Any two equivalent f's have the same weight. Thus we may define $W([f]) := W(f)$. Then the sum of the weights of the patterns is

$$\sum_{[f]} W([f]) = \mathcal{Z}_{(\Gamma,P)}\left(\sum_{y \in F} w(y), \sum_{y \in F} w(y)^2, \ldots, \sum_{y \in F} w(y)^n\right)$$

There are extensions of Pólya's theorem to cases of a different definition of the weight function and equivalence classes [2]. We will need the following elementary lemma. For proof, see for instance [9].

Lemma 3.1. *Let $G = \langle a \rangle$ be a cyclic group where $|G| = n$. Then $\langle a^{\frac{n}{d}} \rangle = \langle a^{k\frac{n}{d}} \rangle$ if and only if $gcd(k, d) = 1$.*

We prove that the typical orbit of the aforementioned group action is the set of all elements from $(\mathbb{Z}_n, +)$ of certain fixed order d.

Lemma 3.2. *Let $\Omega_d = \{a \in \mathbb{Z}_n \mid ord(a) = d\}$ where d is a divisor of n. We have*

$$\Omega_d = \frac{n}{d}\mathcal{I}(d)$$

Also, Ω_d is an orbit under the action of the group $\mathcal{I}(n)$ on $\mathbb{Z}_n$.

Proof. According to Lemma 3.1, it clearly follows that $\Omega_d = \frac{n}{d}\mathcal{I}(d)$. Let x and y be elements of order d. Then we have $x = (n/d)k_1$ and $y = (n/d)k_2$ where $k_1, k_2 \in \mathcal{I}(d)$. Therefore, there exist $k \in \mathcal{I}(d)$ and $k_1 = kk_2$. On the other hand, Lemma 2.1 claims the existence of an element $h \in \mathcal{I}(n)$ such that $h \equiv k \pmod{d}$. Clearly $k_1 \equiv hk_2 \pmod{d}$. By multiplying both sides by (n/d) we have $x = hy \pmod{n}$. Thus, $\mathcal{I}(n)$ is transitive on the set of elements of (additive) order d. ⊣

Note that mappings from $\mathcal{I}(n)$ keep fixed sets Ω_d for each $d \mid n$. In the following lemma, we implicitly prove that the group $\mathcal{I}(2^m)$ is generated by π_3 and π_{-1}. This result would be necessary for understanding of action of the group $\mathcal{I}(2^m)$ on $\mathbb{Z}_{2^m}$ and consequently finding the corresponding cycle index.

Lemma 3.3. *The order of element 3 is 2^{m-2} modulo 2^m if $m \geq 3$. The elements from Ω_{2^m} are represented by*

$$(-1)^a 3^b \text{ where } a \in \{0, 1\} \text{ and } b \in \{0, 1, \ldots, k-3\}.$$

Proof. We note

$$\begin{aligned}3^2 &= 1+8\\ 3^4 &= 1+16+32t\\ 3^8 &= 1+32+64u\end{aligned}$$

for integers t and u. By induction, 3 is of order 2^{m-2} modulo 2^m, if $m \geq 3$. For the second part, it is enough to show that

$$(-1)3^{b_1} \not\equiv 3^{b_2} \pmod{2^k}, \text{ for } b_1, b_2 \in \{0, 1, \ldots, k-3\}.$$

If not so, we would have

$$8 \mid 3^b + 1$$

where $b = |b_1 - b_2|$. This is not possible since $3^b + 1 \equiv 2$ or $4 \pmod 8$. ⊣

Lemma 3.4. *Let f_j, g_j be mappings from Ω_{2^m} to Ω_{2^m}, $m \geq 3$ defined by*

$$f_j(x) := 3^j x \text{ and } g_j(x) = -3^j x,$$

for $j \geq 1$. Let s_j be order of $f_j \in$ ¸(2^m). Then

$$ctype(f_j) = (s_j^{u_j})$$

where $u_j = 2^{m-1}/s_j$. Note that $ctype(g_j) = ctype(f_j)$.

Proof. Consider the mapping $f_j \in \mathcal{I}(2^m)$. Let $x_0 \in \Omega_{2^m}$ be arbitrary element. Since $f_j^s(x_0) \equiv x_0 \pmod{2^m}$ if and only if $s \equiv 0 \pmod{s_j}$, then it is clear that every cycle must contain exactly s_j elements. Since the order of g_j is equal to order f_j, conclusion $ctype(g_j) = ctype(f_j)$ follows easily. ⊣

The following lemma is explaining how the number of orbits of the permutation group $\mathcal{I}(n)$ acting on $\mathcal{O}_k$ can be determined, once the cycle index of $\mathcal{I}(n)$ acting on $\mathbb{Z}_n$ is determined.

Lemma 3.5. *The number of orbits in the action of $\mathcal{I}(n)$ on the $\mathcal{O}_k$ is equal to the coefficient of x^k in*

$$\mathcal{Z}_{(\mathcal{I}(n),\mathbb{Z}_n)}(1+x, 1+x^2, \ldots, 1+x^n)$$

Proof. We apply Pólya's theorem 3.1. Let $F = \{in, out\}$ and $P = \{0, 1, \ldots, n\}$. This means that functions $f \in F^P$ are actually characteristic functions of subsets in $\mathbb{Z}_n$. Let us define $w(in) = x$ and $w(out) = 1$. Then, the weight of characteristic function f of a $k-$set is

$$W(f) = x^k$$

Thus, the number of orbits of $k-$sets in the given group action is the coefficient of x^k in

$$\mathcal{Z}_{(\mathcal{I}(n),\mathbb{Z}_n)}(1+x, 1+x^2, \ldots, 1+x^n)$$

⊣

Therefore, we need to determine the cycle index of $\mathcal{I}(n)$ acting on $\mathbb{Z}_n$. The following lemma considers the case of $\mathcal{I}(p^m)$, where p is an od prime number. As it has been well known, this group is cyclic [7] and therefore it is trivial to find its cycle index.

Lemma 3.6. *Let p be an odd prime. The cycle type of the permutation group* $¸(p^m)$ *acting on* $\mathbb{Z}_{p^m}$ *is*

$$\mathcal{Z}_{(¸(p^m),\mathbb{Z}_{p^m})}(x_1, x_2, \ldots, x_{p^m}) = \sum_{k=1}^{r} x_1 \prod_{i=1}^{m} x_{u(i,k)}^{v(i,k)},$$

where $r = p^{m-1}(p-1)$, $v(i,k) = \gcd(k, p^{i-1}(p-1))$ *and*

$$u(i,k) = \frac{p^{i-1}(p-1)}{v(i,k)}.$$

Proof. It is well known that in the case of odd prime p, the automorphism group $\mathcal{I}(p^m)$ is cyclic [7]. Let β be a generator of $\mathcal{I}(p^m)$. According to Lemma 3.2, $\mathcal{I}(p^m)$ is transitive on each set Ω_d, for $d \mid p^m$. Note that $|\Omega_{p^i}| = p^{i-1}(p-1)$ and $|\mathcal{I}(p^m)| = p^{m-1}(p-1)$. Now it is easy to see that

$$ctype(\beta) = x_1 \prod_{i=1}^{m} x_{\phi(p^i)}^{1}.$$

Since every element in $\mathcal{I}(p^m)$ is a power of β, the rest of the conclusion follows trivially. ⊣

The much more complex case is the cycle index of the group $\mathcal{I}(2^m)$ when $m \geq 3$. This is the special case since $\mathcal{I}(2^m)$ for $m \geq 3$ is not cyclic group. That case remains unresolved yet, but the authors are getting very close to the answer.

3.2 Cycle index of direct product of permutation groups

When one find the cycle indices of all groups $\mathcal{I}(p^m)$ when p is a prime number, there is a natural question if there exists a way to combine them together in order to obtain the cycle index of $\mathcal{I}(n)$, where n is the product of those prime power components. Hence, we need something like the cycle index of the direct product of permutation groups.

Let G_1, G_2 be permutation groups acting on sets X_1, X_2 respectively. Let $G = G_1 \times G_2$ and $X = X_1 \times X_2$ be the direct product of corresponding groups and sets. For an element $x = (x_1, x_2)$ of X and an element $g = (g_1, g_2)$ of G, we define the action of g on x by

$$(g, a) \mapsto (g_1 x_1, g_2 x_2)$$

Evidently, G is a permutation group on X. Let P and Q be polynomials

$$P(x_1, x_2, \ldots, x_u) = \sum a_{i_1 i_2 \ldots i_u} x_1^{i_1} x_2^{i_2} \ldots x_u^{i_u},$$

$$Q(x_1, x_2, \dots, x_v) = \sum b_{j_1 j_2 \dots j_v} x_1^{j_1} x_2^{j_2} \dots x_v^{j_v}$$

In [4] the following product operator was defined

$$P \circledast Q = \sum a_{i_1 i_2 \dots i_u} b_{j_1 j_2 \dots j_v} \prod_{\substack{1 \le l \le u \\ 1 \le m \le v}} (x_l^{i_1} \circledast x_m^{j_m}),$$

where

$$x_l^{i_1} \circledast x_m^{j_m} = x_{\mathrm{lcm}(l,m)}^{i_1 j_m \gcd(l,m)}$$

We need the following lemma. For proof, see [4] and [5].

Lemma 3.7. *The cycle index of the natural action of permutation group $G_1 \times G_2$ on $X_1 \times X_2$ induced by actions G_1 on X_1 and G_2 on X_2 can be expressed as:*

$$\mathcal{Z}_{(G_1 \times G_2, X_1 \times X_2)} = \mathcal{Z}_{(G_1, X_1)} \circledast \mathcal{Z}_{(G_2, X_2)}.$$

Let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$. Applying the ring isomorphism

$$\mathbb{Z}_n \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{p_i^{\alpha_i}},$$

it follows that

$$\mathcal{I}(n) \cong \bigoplus_{i=1}^{s} \mathcal{I}_{p_i^{\alpha_i}}.$$

Hence, according to Lemma 3.7, we have

$$\mathcal{Z}_{(\mathcal{I}(n), \mathbb{Z}_n)} = \mathcal{Z}_{(\mathcal{I}_{p_1^{\alpha_1}}, \mathbb{Z}_{p_1^{\alpha_1}})} \circledast \mathcal{Z}_{(\mathcal{I}_{p_2^{\alpha_2}}, \mathbb{Z}_{p_2^{\alpha_2}})} \circledast \dots \circledast \mathcal{Z}_{(\mathcal{I}_{p_s^{\alpha_s}}, \mathbb{Z}_{p_s^{\alpha_s}})}.$$

Once the cylce index for the group the group ¸(2^m) is found, the cycle index $\mathcal{Z}_{(\mathcal{I}(n), \mathbb{Z}_n)}$ can be calculated as above. Hence, in order to find the number of orbits in the action of $\mathcal{I}(n)$ on $\mathcal{O}_k$ we just need to apply result from Lemma 3.5.

4 Conlusions

In this paper we studied problem of finding cycle index of the automorphism group $\mathcal{I}(n)$ of $\mathbb{Z}_n$ acting on the set of $k-$subsets of $\mathbb{Z}_n$. At the beginning we explained some algebraic and number theoretical questions regarding the group $\mathcal{I}(n)$. In the section 3.1 we deal with the problem of finding the number of orbits in the action we introduced before. That problem, as it has been explained, could be resolved *component wise* considering the action of the group $\mathcal{I}(p^t)$, where p is a prime number. In the Lemma 3.6 we resolved the case of $\mathcal{I}(p^t)$, where p is an odd prime number. However, the cycle index of the group $\mathcal{I}(2^t)$ still remains unresolved. Authors are very close to completing that case and that is the important part of the future research.

References

[1] G. Pólya and R.C. Read. Combinatorial Enumeration of Groups, Graphs and Chemical Compounds. Springer Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1987. ISBN 0-387-96413-4 or ISBN 3-540-96413-4.

[2] N.G. De Bruijn. Pólya's Theory of Counting. In Applied Combinatorial Mathematics, chapter 5, pages 144-184, Beckenbach, E.F., Wiley, New York, 1964.

[3] A Survey of Generalizations of Pólya's Enumeration Theorem. Nieuw Archief voor Wiskunde (2), XIX: pages 89-112, 1971.

[4] M.A. Harrison and R.G. High. On the Cycle Index of a Product of Permutation Groups. Journal of Combinatorial Theory, 4:277-299, 1968.

[5] Wan-Di Wei and Ju-Yong Xu. Cycle index of direct product of permutation groups and number of equivalence classes of subsets of Zv. Discrete Mathematics, 123, 179-188, 1993.

[6] Harald Fripertinger. Cycle indices of linear, affine and projective groups, Linear Algebra Appl. 263 (1997) 133-156.

[7] Hans J. Zassenhaus. The Theory of Groups. Dover, second edition, 145-146, 1958.

[8] Joseph A. Gallian and David J. Rusin. Factoring Groups of Integers Modulo n. Mathematics Magazine, Vol. 53, No. 1, 33-36, 1980.

[9] Dummit, D.S. and Foote, R.M., Abstract Algebra, Prentice-Hall, Upper Saddle River, NJ, 1999.

[10] G. Pólya, "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen" Acta Math. , 68 (1937) pp. 145-254

[11] Harvey Cohn, Advanced Number Theory, Dover, 1980, ISBN 048664023X.

[12] Daniel Shanks, Solved and Unsolved Problems in Number Theory (Fourth Edition), Chelsea, 1993, ISBN 082182824X.

[13] Zun Shan and Edward T. H. Wang. Mutual Multiples in $\mathbb{Z}_n$. Mathematics Magazine, Vol. 72, No. 2., pp. 143-145, (Apr., 1999).

[14] J. H. van Lint and R. M. Wilson. A Course in Combinatorics. Cambridge University Press, Cambridge, 1992.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Nelinearna metoda konačnih zapremina

Milan Dotlić
Institut Jaroslav Černi Beograd
milan.dotlic@jcerni.co.rs

Mladen Ignjatović
Tehnološki fakultet Zvornik
ignjatovic@gmail.com

Dragan Vidović
Institut Jaroslav Černi Beograd
draganvid@gmail.com

Boris Pokorni
Institut Jaroslav Černi Beograd
bpokorni.jci@gmail.com

Milenko Pušić
Rudarsko-geološki fakultet Beograd
milenko.pusic@jcerni.co.rsm

Originalni naučni rad

Abstract

U radu je predstavljena nelinearna metoda konačnih zapremina za diskretizaciju difuznog terma, sa osvrtom na primene u modeliranju bunara u jednačini podzemnog strujanja. Metode konačnih zapremina predstavljaju familiju metoda za numeričko rešavanje parcijalnih diferencijalnih jednacina zadatih u konzervativnoj formi. Ove metode garantuju lokalno održanje što je od značaja u velikom broju problema dinamike fluida.

Klasična metoda konačnih zapremina, gde je difuzni fluks aprok-simiran konačnom razlikom, zadovoljava princip maksimuma, ali je u opštem slučaju nekonzistentna. Numerički primeri pokazuju da je nelinearna metoda prikazana u ovom radu drugog reda tačnosti i u anizotropnom, diskontinualnom slučaju na proizvoljnim mrežama. Predstavljena metoda ne zadovoljava princip maksimuma, ali garantuje pozitivnost rešenja. Diskretizacijom se dobija M-matrica, što garantuje egzistenciju i jedinstvenost numeričkog rešenja.

U okolini bunara rešenje se ponaša logaritamski sa singularitetom u centru bunara, što rezultuje velikom greškom u odnosu proticaja i nivoa u bunaru. Predstavljena je metoda za diskretizaciju fluksa na bunarskim stranama koja znacajno smanjuje ovu grešku.

1 Uvod

Stacionarno stanje podzemnog strujanja na domenu Ω matemački opisujemo graničnim problemom:

$$\nabla \cdot \mathbf{u} = g_s, \tag{1}$$

$$\mathbf{u} = -\mathbb{K}\nabla h, \tag{2}$$

$$h = g_D \quad \text{na } \Gamma_D, \tag{3}$$

$$\mathbf{u} \cdot \mathbf{n} = g_N \quad \text{na } \Gamma_N, \tag{4}$$

gde je $\mathbf{u}$ brzina, g je term koji opisuje izvore/ponore, h je nepoznati hidraulički potencijal, a $\mathbf{n}$ je spoljna jedinična normala na $\partial\Omega$. Anizotropni tenzor koeficijenta filtracije $\mathbb{K}$ je simetričan i pozitivno definitan sa mogućim prekidima samo u stranama mreže. Jednačina (1) predstavlja održanje mase, dok je (2) Darsijev zakon.

Anizotropija i prekidnost tenzora koeficijenta filtracije predstavlja veliki problem za numeričko rešavanje jednačine podzemnog strujanja. Metode konačnih elemenata mogu dovesti do pojave oscilacija u rešenju. Ovi problemi ne postoje kod klasične metode konačnih zapremina, gde je numerički fluks aproksimiran konačnom razlikom. Medjutim, klasična metoda konačnih zapremina nije ni prvog reda tačnosti na proizvoljnim mrežama [10].

Nova familija nelinearnih metoda konačnih zapremina predstavljena u [3, 8, 9, 10, 16, 17, 18] je drugog reda tačnosti čak i u diskontinualnom neizotropnom slučaju na proizvoljnim mrežama. Ove metode generišu M-matrice uz pomoć nelinearne aproksimacije fluksa pomoću dve tačke. Zahvaljujući činjenici da je dobijena matrica M-matrica garantuje se pozitivnost rešenja. Drugi red tačnosti i pozitivnost rešenja su dobijeni po ceni rešavanja nelinearnog sistema jednačina.

U ovom radu razmatramo podzemno strujanje koje je najvećim delom posledica prisustva bunara. U svakom bunaru je zadat hidraulički potencijal ili proticaj (izdašnost).

Usled raznih mehaničkih, hemijskih i bioloških procesa dolazi do stvaranja kolmiranog (zapušenog) sloja duž zida bunara [4]. Kolmirani sloj stvara dodatnu hidrauličku otpornost, pa je proticaj kroz bunar odredjen jednačinom:

$$Q = A\Psi(h_r - h_w), \tag{5}$$

gde je h_r hidraulički potencijal na spoljašnjem zidu bunara, h_w je hidraulički potencijal u bunaru, Q je proticaj, A je površina filtera bunara, $\Psi = \mathbb{K}_c/d_c$ je transfer koeficijent, $\mathbb{K}_c$ je koeficijent filtracije kolmiranog sloja, a d_c je debljina kolmiranog sloja.

U okolini bunara rešenje jednačine (1) se ponaša logaritamski sa singularitetom u centru bunara, što rezultuje velikom greškom u odnosu proticaja i nivoa u bunaru. Pored toga u okolini bunara se gubi drugi red tačnosti.

Problem modeliranja bunara je često razmatran u literaturi [2, 5, 6, 11, 12, 14]. Najčešće korišćeni metod za diskretizaciju bunara je Pismanov model [11, 12].

Ovaj metod originalno formulisan za metodu konačnih razlika je preformulisan za različite metode [2]. U ovom radu je predložena jedna drugačija diskretizacija fluksa izmedju bunara i porozne sredine koja kao i Peaceman-ov model prevazilazi probleme sa tačnošću proticaja. Iako značajno smanjuje grešku u proticaju predložena diskretizacija nije drugog reda.

U poglavlju 2 predstavljena je diskretizacija koja garantuje pozitivnost rešenja zasnovana na radovima [3, 16]. Predložena korekcija za diskretizaciju fluksa na bunarskim stranama je opisana u poglavlju 3. Rešavanje nelinearnog sistema je opisano u poglavlju 4. Egzistencija i jedinstvenost rešenja u svakoj iteraciji dokazani su u poglavlju 5, dok je pozitivnost rešenja dokazana u poglavlju 6.

2 Diskretizacija

Domen je izdeljen na mrežu koja se sastoji od poliedarskih ćelija. Za svaku ćeliju T definišemo tačku kolokacije $\mathbf{x}_T$ u njenom težištu i pridružujemo joj diskretnu vrednost hidrauličkog potencijala h_T. Za tačke kolokacije u graničnim stranama uzimamo težišta i pridružujemo im diskretnu vrednost hidrauličkog potencijala h_f. Pomoćnim tačkama kolokacije nazivamo težišta strana na kojima postoji diskontinuitet u koeficijentu filtracije.

Integracijom (1) po svakoj od ćelija mreže T

$$\int_T \nabla \cdot \mathbf{u} d\Omega = \int_T g d\Omega \tag{6}$$

i primenom teoreme o divergenciji na levu stranu jednakosti dobija se:

$$\int_{\partial T} \mathbf{u} \cdot \mathbf{n} ds = \int_T g d\Omega, \tag{7}$$

gde je ∂T granica ćelije T i $\mathbf{n}$ spoljašnja normala na ∂T. Poslednju jednakost možemo zapisati u obliku

$$\sum_{f \in \partial T} u_f = \int_T g d\Omega, \tag{8}$$

gde je u_f fluks kroz stranu f dat sa:

$$u_f = \int_f \mathbf{u} \cdot \mathbf{n}_f ds = -\int_f (\mathbb{K} \nabla h) \cdot \mathbf{n}_f ds. \tag{9}$$

Pretpostavimo za početak da je tenzor $\mathbb{K}$ neprekidan u strani f. Kako je $\mathbb{K}$ simetričan tenzor važi da je $(\mathbb{K}\nabla h) \cdot \mathbf{n}_f = (\mathbb{K}\mathbf{n}_f) \cdot \nabla h$. Obeležimo sa $\boldsymbol{\ell}_f = \mathbb{K}\mathbf{n}_f$. Za takav vektor $\boldsymbol{\ell}_f$, skoro uvek je moguće je naći tri vektora $\mathbf{t}_i$, tako da su $\mathbf{t}_i = \mathbf{x}_i - \mathbf{x}_T$, gde su $\mathbf{x}_i$ tacke kolokacije, odnosno $\mathbf{t}_i = -\boldsymbol{\ell}_{f_{N_i}}$ u slučaju da je f_{N_i} granična strana na kojoj je zadat Nojmanov granični uslov. Prema tome:

$$\frac{\boldsymbol{\ell}_f}{|\boldsymbol{\ell}_f|} = \sum_{i=1}^{3} \alpha_i \frac{\mathbf{t}_i}{|\mathbf{t}_i|}, \tag{10}$$

pri čemu zahtevamo da su koeficijenti $\alpha_i \geq 0$. Koeficijente α_i dobijamo rešavanjem sistema (10). U retkim slučajevima nije moguće takve vektore da su koeficijenti $\alpha_i \geq 0$, tada je potrebno dodati tačke kolokacije u središtima ivica [3] ili čvorovima [16, 17].

Na osnovu jednakosti za izvod u pravcu

$$\frac{\partial h}{\partial \mathbf{v}} = \nabla h \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \tag{11}$$

i jednačine (10), fluks (9) kroz stranu f možemo predstaviti kao:

$$u_f = -\int_f (\boldsymbol{\ell}_f \cdot \nabla h) ds = -|\boldsymbol{\ell}_f| \sum_{i=1}^{3} \alpha_i \int_f \frac{\mathbf{t}_i}{|\mathbf{t}_i|} \cdot \nabla h ds = -|\boldsymbol{\ell}_f| \sum_{i=1}^{3} \alpha_i \int_f \frac{\partial h}{\partial \mathbf{t}_i} ds. \tag{12}$$

Parcijalne izvode u pravcu možemo aproksimirati konačnim razlikama:

$$\frac{\partial h}{\partial \mathbf{t}_i} \approx \frac{h_i - h_T}{|\mathbf{t}_i|}, \tag{13}$$

odnosno u slučaju $\mathbf{t}_i = -\boldsymbol{\ell}_{f_{N_i}}$

$$\frac{\partial h}{\partial \mathbf{t}_i} = -\nabla h \cdot \frac{\boldsymbol{\ell}_{f_{N_i}}}{|\boldsymbol{\ell}_{f_{N_i}}|} = -\frac{\bar{g}_N(\mathbf{x}_i)}{|l_{f_{N_i}}|}, \tag{14}$$

gde je $\bar{g}_N(\mathbf{x}_i)$ srednja vrednost funkcije g_N na strani sa težisštem $\mathbf{x}_i$. Zamenom aproksimacija (13) i (14) u jednačinu (12) dobijena je aproksimacija fluksa:

$$u_f \approx -|\boldsymbol{\ell}_f||f| \sum_i \frac{\alpha_i}{|\mathbf{t}_i|} (h_i - h_T) + r_f, \tag{15}$$

gde je r_f doprinos graničnih strana sa zadatim Nojmanovim uslovom:

$$r_f = -|\boldsymbol{\ell}_f||f| \sum_k \alpha_k \frac{\bar{g}_N(\mathbf{x}_k)}{|\boldsymbol{\ell}_{f_{N_k}}|}. \tag{16}$$

Svaka unutrašnja strana f pripada dvema ćelijama T^+ i T^-, pa prema tome postoje dve aproksimacije fluksa gde u svakoj učestvuje po četiri diskretne vrednosti hidrauličkog potencijala. Cilj je da se dobije aproksimacija koja koristi diskretne vrednosti hidrauličkog potencijala samo u ćelijama T^+ i T^-, a to se postiže linearnom kombinacijom aproksimacija fluksa kroz stranu f koje odgovaraju ćelijama T^+ i T^-:

$$\begin{aligned} u_f \approx \quad & -\mu_+ |\boldsymbol{\ell}_f||f| \textstyle\sum_i \frac{\alpha_i^+}{|\mathbf{t}_i^+|} \left(h_i^+ - h_{T^+}\right) + \\ & + \mu_- |\boldsymbol{\ell}_f||f| \textstyle\sum_j \frac{\alpha_j^-}{|\mathbf{t}_j^-|} \left(h_j^- - h_{T^-}\right) + \mu_+ r_f^+ - \mu_- r_f^-, \end{aligned} \tag{17}$$

pri čemu važi da je:

$$\mu_+ + \mu_- = 1. \tag{18}$$

Koeficijente $\mu_\pm$ biramo na takav način da se skrate sve diskretne vrednosti hidrauličkog potencijala osim onih u ćelijama T^+ i T^-. Drugim rečima zahtevamo da je:

$$-\mu_+ d_+ + \mu_- d_- = 0, \quad d_\pm = \sum_{i \neq T^\mp} \frac{\alpha_i^\pm}{|\mathbf{t}_i^\pm|} h_i^\pm. \tag{19}$$

Primetimo da su $d_\pm \geq 0$, ukoliko su $h_i \geq 0$.

Rešavanjem sistema jednačina (18) i (19) dobijamo da su

$$\mu_+ = \frac{d_-}{d_- + d_+}, \quad \mu_- = \frac{d_+}{d_- + d_+}, \tag{20}$$

ako je $d_- + d_+ > 0$. Ukoliko je $d_- + d_+ = 0$ postavljamo $\mu_\pm = 0.5$.

Sa ovako izabranim koeficijentima $\mu_\pm$ aproksimacija fluksa (17) koristi samo dve diskretne vrednosti hidrauličkog potencijala:

$$u_f \approx M_f^+ h_{T^+} - M_f^- h_{T^-} + r_f \tag{21}$$

$$M_f^\pm = \mu_\pm |\boldsymbol{\ell}_f||f| \sum_i \frac{\alpha_i}{|\mathbf{t}_i|}, \quad r_f = \mu_+ r_f^+ - \mu_- r_f^-. \tag{22}$$

Obratimo pažnju da koeficijenti $M_f^\pm$ zavise od diskretnih vrednosti hidrauli-čkog potencijala, te je stoga dobijena aproksimacija nelinearna.

Ukoliko koeficijent filtracije $\mathbb{K}$ ima prekid u strani f ili ako je to granična strana, ovu stranu možemo posmatrati kao ćeliju zapremine nula. Ekvivalentim izvodjenjem kao za jednačinu (15) dobijamo:

$$u_f \approx -|\boldsymbol{\ell}_f||f| \sum_i \frac{\alpha_i}{|\mathbf{t}_i|} (h_i - h_f) + r_f \tag{23}$$

pri čemu su u ovom slučaju vektori $\mathbf{t}_i = \mathbf{x}_i - \mathbf{x}_f$. Istim izvodjenjem kao i za jednačinu (21) dolazimo do:

$$u_f \approx N h_T - N_f h_f + r_f. \tag{24}$$

Ukoliko je $f \in \Gamma_D$ jednačina (24) se koristi za aproksimaciju fluksa, a diskretna vrednost hidrauličkog potencijala u takvoj strani se izračunava direktno iz graničnog uslova:

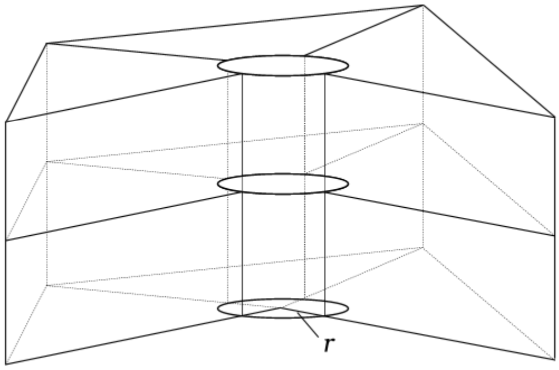
$$h_f = g_D(\mathbf{x}_f). \tag{25}$$

U slučaju da je f strana u kojoj je tenzor $\mathbb{K}$ ima prekid, postoje dve ovakve aproksimacije:

$$u_f \approx N^+ h_{T^+} - N_f^+ h_f + r_f^+, \quad u_f \approx -N^- h_{T^-} + N_f^- h_f - r_f^-. \tag{26}$$

Njihovim kombinovanjem možemo dobiti vrednost koncentracije u strani f:

$$h_f = \frac{N^+ h_{T^+} + N^- h_{T^-} + r_f^+ + r_f^-}{N_f^+ + N_f^-}. \tag{27}$$



Slika 1: Diskretizacija bunara

Eliminisanjem h_f iz jednačina (26) i njihovom linearnom kombinacijom opet dobijamo jednačinu (21), pri čemu su:

$$M_f^{\pm} = \frac{N^{\pm} N^{\mp}}{N_f^+ + N_f^-}, \quad r_f = \frac{N_f^- r_f^+ - N_f^+ r_f^-}{N_f^+ + N_f^-}. \tag{28}$$

Fluks kroz stranu $f \in \Gamma_N$ dobijamo iz:

$$u_f = \bar{g}_N(\mathbf{x}_f)|f|, \tag{29}$$

Diskretne vrednosti hidrauličkog potencijala h_f, kada je $f \in \Gamma_N$, dobijamo iz jednačina (24) i (29).

3 Diskretizacija bunara

Bunar je predstavljen cilindričnim ćelijama (Slika 1) čiji je poluprečnik jednak poluprečniku bunara. Granični uslov u bunaru je zadat na donjoj kružnoj strani najniže ćelije. Tok kroz cev bunara je modeliran Hagen-Puivilovim zakonom [15], odnosno fluks kroz strane izmedju bunarskih ćelija je dat sa:

$$u_f = |f| \frac{r^2 \rho g}{8\mu} \frac{h_{T^+} - h_{T^-}}{\|\mathbf{x}_{T^+} - \mathbf{x}_{T^-}\|}, \tag{30}$$

pri čemu je r poluprečnik bunara, ρ je gustina vode, g je gravitaciona konstanta, a μ je dinamička viskoznost vode. Jednačinu (30) možemo predstaviti kao (21) pri čemu su:

$$M_f^+ = M_f^- = \frac{r^2 \rho g}{8\mu} \frac{|f|}{\|\mathbf{x}_{T^+} - \mathbf{x}_{T^-}\|}, \quad r_f = 0. \tag{31}$$

Kada je na graničnoj strani bunara zadat hidraulički potencijal, fluks je aproksimiran sa (24), pri čemu je:

$$N = N_f = \frac{r^2 \rho g}{8\mu} \frac{|f|}{\|\mathbf{x}_T - \mathbf{x}_f\|}, \quad r_f = 0. \tag{32}$$

Kada je na graničnoj strani bunara zadat ukupni proticaj kroz bunar, diskretnu vrednost hidrauličkog potencijala u toj strani nalazimo iz prethodne jednačine.

Fluks na stranama izmedju bunara i porozne sredine možemo aproksimirati na način opisan za strane gde postoji prekid u koeficijentu filtracije (26). Pri čemu je fluks izmedju strane i bunarske ćelije aproksimiran konačnom razlikom. Ovakvom aproksimacijom fluksa dolazi do velike greške hidrauličkog potencijala odnosno proticaja u bunaru, ako je u bunaru zadat proticaj odnosno hidraulički potencijal.

Poznato je da je u analitičkom slučaju homogenog izotropnog cilindra [7] proticaj kroz bunar zadat sa:

$$Q = AK\frac{h_R - h_r}{r\ln\frac{R}{r}}, \tag{33}$$

gde je h_r hidrulički potencijal u poroznoj sredini tik do kolmiranog sloja bunara, h_R hidrulički potencijal na udaljenosti R od bunara i K je izotropni koeficijent filtracije porozne sredine.

Ideja je da se iskoristi proticaj (33) za računanje fluksa kroz strane izmedju bunara i porozne sredine. Neka se strana f nalazi izmedju bunarske ćelije W i ćelije porozne sredine T. Tada je na osnovu (33) fluks kroz stranu f:

$$u_f \approx |f|K\frac{h_T - h_r}{r\ln\frac{\rho(\mathbf{x}_T)}{r}}, \tag{34}$$

gde je r poluprečnik bunara, a $\rho(\mathbf{x}_T)$ udaljenost tačke $\mathbf{x}_T$ od centralne ose bunara. Sa druge strane na osnovu (5) fluks je:

$$u_f = |f|\Psi(h_r - h_W). \tag{35}$$

Kombinacijom jednačina (34) i (35) dobijamo jednačinu oblika (21), odnosno:

$$u_f \approx M_f^+ h_T - M_f^- h_W, \tag{36}$$

gde su:

$$M_f^+ = M_f^- = \frac{|f|\Psi K}{r\ln\frac{\rho(\mathbf{x}_T)}{r}\Psi + K}. \tag{37}$$

U slučaju homogene anizotropne sredine potrebno je koordinatnim transformacijama tenzor koeficijenta filtracije svesti na jediničnu matricu

4 Pikarove iteracije

Vrednosti $M_f^\pm$ i r_f u jednačini (21) zavise od diskretnih vrednosti koncentracije, jer zavise od koeficijenata $\mu_\pm$ preko jednačine (22). Prema tome diskretizacija opisana u odeljcima 2 i 3 daje sistem nelinearnih jednačina:

$$A(\mathbf{h})\mathbf{h} = b(\mathbf{h}), \tag{38}$$

gde je $\mathbf{h}$ vektor nepoznatih diskretnih vrednosti h_T.

Ovakav sistem je moguće rešiti na različite načine, recimo Pikarovim metodom:

$$A(\mathbf{h}^n)\mathbf{h}^{n+1} = b(\mathbf{h}^n). \tag{39}$$

Rešavanjem linearnog sistema (39) počevši od nekog početnog rešenja $\mathbf{h}^0$ dolazimo do sledeće Pikarove iteracije. Ovaj postupak nastavljamo sve dok ne bude ispunjen kriterijum konvergencije:

$$r(\mathbf{h}^n) = \frac{\|A(\mathbf{h}^n)\mathbf{h}^n - b(\mathbf{h}^n)\|}{\|\mathbf{h}^n\|} < \epsilon, \tag{40}$$

za zadato ϵ, ili dok se ne dostigne maksimalni broj iteracija.

5 Linearni sistem jednačina

Matrica sistema $A(\mathbf{h})$ u jednačini (38) je dobijena sastavljanjem (asembliranjem) matrica 2×2:

$$\mathbb{M}_f = \begin{pmatrix} M_f^+ & -M_f^- \\ -M_f^+ & M_f^- \end{pmatrix}, \tag{41}$$

za unutrašnje strane i 1×1 matrica $\mathbb{M}_f = N$ za granične strane sa zadatim Dirihleovim uslovom. Pri tome su $M_f^{\pm}$ koeficijenti iz jednačine (21), a N je koeficijent iz jednačine (24). Iz jednačine (22) sledi da su ti koeficijenti pozitivni ukoliko važi $h_i \geq 0$, za svako i. Prema tome dijagonalni elementi matrice $A(\mathbf{h})$ su pozitivni, dok van glavne dijagonale nema pozitivnih elemenata. Kako je matrica A nastala sastavljanjem matrica $\mathbb{M}_f$, zbir svake vrste je nenegativan. Osim toga matrica $\mathbb{M}_f$ je nesvodljiva za povezane mreže, pa je i matrica A nesvodljiva. Ovim su ispunjeni uslovi za poznatu teoremu linearne algebre [1]:

Teorema 5.1. Neka je A nesvodljiva kvadratna matrica koja van glavne dijagonale nema pozitivnih elemenata. Tada je A M-matrica ako je

$$A\mathbf{x} > 0, \tag{42}$$

za neki strogo pozitivni vektor $\mathbf{x}$.

Egzistencija i jedinstvenost rešenja linearnog sistema (39) sledi iz činjenice da postoji inverz M-matrice [13].

6 Pozitivnost rešenja

Elementi desne strane sistema (38) su:

$$b_T(\mathbf{h}) = \int_T g d\Omega + \sum_{f \in \Gamma_D} N_f \bar{g}_{D,f} - \sum_{f \in \Gamma_N} |f| \bar{g}_{N,f}, \tag{43}$$

za svaku ćeliju mreže. Ukoliko su $g \geq 0$, $g_D \geq 0$ i $g_N \leq 0$, svi elementi vektora $b(\mathbf{h})$ su nenegativni. Za M-matrice važi da su svi elementi njenog inverza nenegativni [13]. Prema tome važi da je $\mathbf{h}^k \geq 0$, $k \geq 1$, ako za zadato početno rešenje važi $\mathbf{h}^0 \geq 0$. Time je dokazana pozitivnost rešenja.

7 Numerički primeri

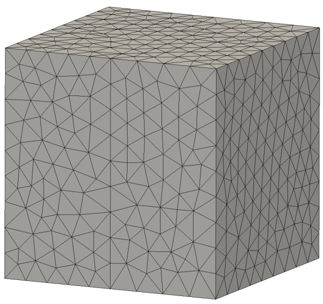
Primer 7.1. Prikazan je problem iz [16] gde je koeficijent filtracije anizotropan i prekidan. U jediničnoj kocki u delu $\Omega_1 = \{(x, y, z) \in \Omega | x < 0.5\}$ dat je koeficijent filtracije $\mathbb{K}_1$, dok je u delu $\Omega_2 = \{(x, y, z) \in \Omega | x > 0.5\}$ dat koeficijent filtracije $\mathbb{K}_2$:

$$\mathbb{K}_1 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{K}_2 = \begin{pmatrix} 10 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{44}$$

Term g_s je izabran tako da je analitičko rešenje dato sa:

$$h(x, y, z) = \begin{cases} 1 - 2y^2 + 4xy + 2y + 6x, & x < 0.5, \\ 3.5 - 2y^2 + 2xy + x + 3y, & x \geq 0.5. \end{cases} \tag{45}$$

Na granici $x = 0$ zadat je analitički hidraulicki potencijal, dok se na svim ostalim granicama zadaje analitički fluks. Za ocenu greške hidrauličkog potencijala



Slika 2: Mreža h=1/10 u primeru 7.1.

i fluksa korišćene su L_2 i maksimalna norma:

$$\varepsilon_2^h = \left(\frac{\sum_T (h(\mathbf{x}_T) - h_T)^2 |T|}{\sum_T (h(\mathbf{x}_T))^2 |T|} \right)^{1/2}, \tag{46}$$

$$\varepsilon_2^{\mathbf{u}} = \left(\frac{\sum_f (\mathbf{u}(\mathbf{x}_f) \cdot \mathbf{n}_f - u_f)^2 |f|}{\sum_f (\mathbf{u}(\mathbf{x}_f) \cdot \mathbf{n}_f)^2 |f|} \right), \tag{47}$$

$$\varepsilon_{max}^h = \frac{\max_T |h(\mathbf{x}_T) - h_T|}{\left(\sum_T (h(\mathbf{x}_T))^2 |T| / \sum_T |T|\right)^{1/2}}, \tag{48}$$

$$\varepsilon^{\mathbf{u}}_{max} = \frac{\max_f |\mathbf{u}(\mathbf{x}_f) \cdot \mathbf{n}_f - u_f|}{\left(\sum_f (\mathbf{u}(\mathbf{x}_f) \cdot \mathbf{n}_f)^2 |f| / \sum_f |f|\right)^{1/2}}. \tag{49}$$

Tabela 1: Greške u hidrauličkom potencijalu i fluksu u primeru 7.1.

h	ε^h_2	ε^h_{max}	$\varepsilon^{\mathbf{u}}_2$	$\varepsilon^{\mathbf{u}}_{max}$
1/10	0.00205541	0.0292998	0.00519191	0.0349371
1/20	0.000463163	0.00683567	0.00235716	0.0144156
1/40	8.86811e-05	0.0019212	0.0011501	0.00822886
1/80	2.16327e-05	0.00061031	0.000560393	0.00564582

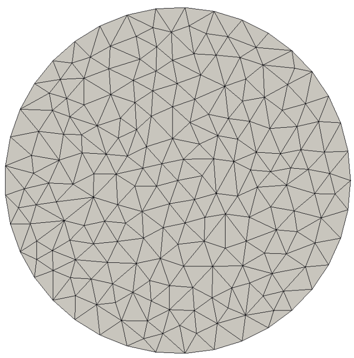
Rezultati u tabeli 1 pokazuju da je nelinearna metoda drugog reda tačnosti, bez obzira na anizotropiju i diskontinuitet u koeficijentu fitracije.

Primer 7.2. Neka se bunar poluprečnika r nalazi u centru izotropnog cilindra poluprečnika R. Neka su zadati hidraulički potencijal na omotaču cilindra h_R i hidraulički potencijal u bunaru h_w. Fluks je nula na bazama cilindra. Tačno rešenje je dato sa:

$$h(\rho) = \frac{h_r \ln \frac{R}{\rho} + h_R \ln \frac{\rho}{r}}{\ln \frac{R}{r}}, \tag{50}$$

pri čemu je ρ udaljenost od centralne ose bunara, a h_r je hidraulički potencijal u poroznoj sredini tik do bunara koji se nalazi izjednačavnjem jednačina (5) i (33).

Zadate su vrednosti $R = 200m$, $h_R = 100m$, $h_w = 50$, a transfer koeficijent bunara je izabran tako da je $h_r = 60m$. Izotropni koeficijent filtracije je $K = 1e{-}4$. Visina filtra je $H = 10m$. U tabeli 2 su prikazani rezultati dobijeni na istoj



Slika 3: Mreža korišéna za dobijanje rezultata u tabeli 2.

Tabela 2: Proticaji u bunaru

r	Analitički proticaj	Numerički proticaj	
		bez korekcije	sa korekcijom
0.5	0.04195	0.00768	0.04253
0.1	0.03307	0.00161	0.03343
0.05	0.03030	0.00081	0.03060

mreži sa različitim poluprečnikom bunara. Rezultati pokazuju da korišćenjem korekcije fluksa (poglavlje 3) proticaji u bunaru imaju grešku manju od jednog procenta. Bez korišćenja korekcije greške u proticajima su ogromne.

8 Zaključak

U radu je prikazana jedna varijanta nelinearne metode konačnih zapremina. Dokazana je nenegativnost rešenja, kao i egzistencija i jedinstvenost rešenja linearnog sistema. U primeru 7.1 numerički je pokazana tačnost drugog reda.

Zbog logaritamske prirode rešenja, standardne numeričke metode daju netačan odnos proticaja i hidrauličkog nivoa u bunaru. Takodje, u okolini bunara se gubi drugi red tačnosti. U primeru 7.2 prikazano je značajno smanjenje greške u proticaju u bunaru. Iako predložena metoda značajno smanjuje grešku, gubi se drugi red tačnosti.

9 Zahvalnost

Milan Dotlić, Dragan Vidović i Milenko Pušić se zahvaljuju Ministarstvu prosvete, nauke i tehnološkog razvoja za finansijsku podršku kroz projekat TR37014.

Literatura

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, 1994.

[2] Zhangxin Chen and Youqian Zhang, Well flow models for various numerical methods, *Int. J. Numer. Anal. Mod.*, 6(3):375–388, 2009.

[3] A.A. Danilov and Yu.V. Vassilevski, A monotone nonlinear finite volume method for diffusion equations on conformal polyhedral meshes, *Russ. J. Numer. Anal. Math. Modelling*, 24(3):207–227, 2009.

[4] M. Dimkić, M. Pušić, V. Obradović, and D. Djurić, Several natural indicators of radial well aging at the Belgrade Groundwater Source, part 2, *Water Science and Technology*, 2011. submitted WST-S-10-02140[1].

[5] Yu Ding and L. Jeannin, A new methodology for singular modeling in flow simulations in reservoir engineering, *Computat. Geosci.*, 5(2):93–119, 2001.

[6] L. J. Durlofsky, An approximate model for well productivity in heterogeneous porous media, *Math. Geol.*, 32(4):421–438, 2000.

[7] H. M. Haitjema, *Analytic Element Modeling of Groundwater Flow*, Academic Press, Inc, San Diego, 1995.

[8] Christophe Le Potier, Schema volumes finis monotone pour des operateurs de diffusions fortement anisotropes sur des maillages de triangle non structures, *C.R. Math. Acad. Sci. Paris*, 341:787–792, 2005.

[9] K. Lipnikov, M. Shashkov, D. Svyatskiy, and Yu. Vassilevski, Monotone finite volume schemes for diffusion equations on unstructured triangular and shape-regular polygonal meshes, *J. Comp. Phys.*, 227(1):492–512, 2007.

[10] K. Lipnikov, D. Svyatskiy, and Yu. Vassilevski, Interpolation-free monotone finite volume method for diffusion equations on polygonal meshes, *J. Comp. Phys.*, 228(3):703–716, 2009.

[11] D.W. Peaceman, Interpretation of well-block pressures in numerical reservoir simulation, *SPE J.*, 18(3):183–194, 1978.

[12] D.W. Peaceman, Interpretation of wellblock pressures in numerical reservoir simulation with nonsquare grid blocks and anisotropic permeability, *SPE J.*, 23(3):531–543, 1983.

[13] H.S. Price, Monotone and oscillation matrices applied to finite difference approximation, *Math. Comput.*, 2(103):489–516, 1968.

[14] Sissel Slettemark, Eirik Keilegavlen, and Ivar Aavatsmark, Simulation of anisotropic heterogeneous near-well flow using MPFA methods on flexible grids, *Computat. Geosci.*, 14(4):509–525, 2010.

[15] S. P. Sutera and R. Skalak, The history of poiseuille's law, *Annu. Rev. Fluid Mech.*, 25:1–19, 1993.

[16] D. Vidović, M. Dimkić, and M. Pušić, Accelerated non-linear finite volume method for diffusion, *J. Comp. Phys.*, 230(7):2722–2735, 2011.

[17] D. Vidović, M. Dotlić, M. Dimkić, M. Pušić, and B. Pokorni, Convex combinations for diffusion schemes, *J. Comp. Phys.*, 264:11–27, 2013.

[18] A. Yuan and Z. Sheng, Monotone finite volume schemes for diffusion equations on polygonal meshes, *J. Comp. Phys.*, 227(12):6288–6312, 2008.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Primjena Ermitovih polinoma za određivanje Furijeove transformacije

Tomislav B. Šekara
Elektrotehnički fakultet, Univerzitet u Beogradu, Srbija
tomi@etf.rs

Vidan Govedarica
Elektrotehnički fakultet, Univerzitet u I. Sarajevu, Republika Srpska
vidangov@yahoo.com

Milan R. Rapaić
Fakultet tehničkih nauka, Univerzitet u Novom Sadu, Srbija
rapaja@uns.ac.rs

Originalni naučni rad

Apstrakt

Furijeova transformacija (FT) ima široku primjenu u mnogim oblastima nauke i tehnike gdje se proučava frekvencijski spektar signala, prostiranje oscilacija ili zračenja koje su funkcija promjene amplitude neke veličine (veoma često električnog signala) u zavisnosti od vremena. Zbog svoje velike primjene u obradi signala razvijeni su brzi algoritmi za njeno računanje (brza Furijeova transformacija). Brzo računanje FT je i danas predmet naučnog interesovanja kao i osnovna motivacija ovoga rada. Pošto su Ermitovi polinomi, odnosno Ermitove funkcije, jedne od sopstvenih funkcija FT, to znači da se određivanje FT signala može svesti na linearnu interpolaciju signala Ermitovim funkcijama (čine bazu ortogonalnih funkcija), na osnovu kojih direktno slijedi FT signala. Na ovaj način se otvara pitanje brzine i tačnosti linearne interpolacije signala Ermitovim funkcijama. U ovom radu biće prikazana simulaciona analiza dobijenih rezultata FT signala primjenom Ermitovih polinoma sa analitičkim vrijednostima FT signala.

1 Osnovni pojmovi

1.1 Furijeova transformacija

Definicija 1.1. [Furijeova transformacija.] Furijeova transformacija $\mathcal{F}\{f(t)\} = F(\mathrm{i}\omega)$ funkcije $t \mapsto f(t)$ definisana je integralom

$$F(\mathrm{i}\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}dt, \text{ pri čemu je i imaginarna jedinica.} \quad (1)$$

Dakle ako postoji nesvojstveni integral sa desne strane relacije (1), funkcija F predstavlja Furijeovu transformaciju funkcije f. Označimo sa $G(\mathbb{R})$ familiju apsolutno itegrabilnih, dio-po-dio neprekidnih funkcija†. Pretpostavimo, takođe, da funkcija f ima ograničen broj ekstrema (maksimuma i minimuma), odnosno da zadovoljava Dirihleove uslove. Funkcija može imati beskonačan broj tačaka prekida, ali samo konačan broj na svakom konačnom intervalu.

Može se definisati i inverzna Furijeova transformacija koja, pod određenim uslovima, vraća originalnu funkciju

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)e^{\mathrm{i}\omega t}d\omega. \tag{2}$$

Napomenimo da prethodne definicije Furijeove transformacije i inverzne Furijeove transformacije (Furijeov par) nisu jedinstvene. U literaturi mogu naći drugačije definicije u odnosu na definicija (1) i (2). Tako Furijeova transformacija može biti definisana sa ([1])

$$F(\mathrm{i}\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}dt, \tag{3}$$

a njena inverzna FT sa

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)e^{\mathrm{i}\omega t}d\omega. \tag{4}$$

Napomena 1.1. Može se definisati i uopštena Furijeova i inverzna Furijeova transformacija koristeći proizvoljne konstante α i β

$$F(\mathrm{i}\omega) = \sqrt{\frac{|\beta|}{(2\pi)^{1-\alpha}}}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\beta\omega t}dt\ , \tag{5}$$

$$f(t) = \sqrt{\frac{|\beta|}{(2\pi)^{1+\alpha}}}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)e^{\mathrm{i}\beta\omega t}d\omega\ . \tag{6}$$

Prethodno udevedene definicije ne mijenjaju smisao FT, već su u osnovi samo oblici pogodni za određenu klasu primjena.

1.2 Ermitovi polinomi i Ermitove funkcije

Definicija 1.2. [Ermitovi polinomi] Posmatrajmo funkciju $Y(s) = e^{2st-s^2}$, gdje je s kompleksna promjenljiva a t realan parametar. Funkcija Y nema singulariteta u konačnoj s-ravni, pa se može razviti u Tejlorov red, u okolini taške $s = 0$, koji je konvergentan za svako konašno s. Prema tome, imamo razvoj ([2,3])

†Funkcija f je apsolutno integrabilna ukoliko je $\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$. Funkcija je dio po dio neprekidna na cijelom $\mathbb{R}$ ako je dio po dio neprekidna na svakom konačnom intervalu (a, b).

$$e^{2st-s^2} = \sum_{n=0}^{\infty} H_n(t) \frac{s^n}{n!}, \tag{7}$$

gdje su

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}(e^{-t^2}) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2t)^{n-2k}, \tag{8}$$

Ermitovi polinomi, dok je funkcija $s \mapsto Y(s)$ generatrisa Ermitovih polinoma.

Napomena 1.2. Ermit je 1864. godine definisao polinome H_n. Međutim, još 1859. godine je Čebišev definisao ove polinome, pa se zato u literaturi nazivaju i Čebišev-Ermitovi polinomi. Interesantno je napomenuti da se u literaturi i polinomi definisani izrazom

$$He_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n}(e^{-t^2/2}) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!} \frac{x^{n-2k}}{2^k} \tag{9}$$

takođe zovu Ermitovi polinomi.

Napomena 1.3. Ermitovi polinomi $H_n(t)$ predstavljaju takođe jedno partikularno rješenje diferencijalne jednačine

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2ny = 0, \tag{10}$$

koja se naziva Ermitovom diferencijalnom jednačinom.

Navedimo sada prvih nekoliko Ermitovih polinoma (8), i njihovu rekurentnu relaciju:

$$H_0(t) = 1, \tag{11}$$
$$H_1(t) = 2t, \tag{12}$$
$$H_2(t) = -2 + 4t^2, \tag{13}$$
$$H_3(t) = -12t + 8t^3, \tag{14}$$
$$H_4(t) = 12 - 48t^2 + 16t^4, \tag{15}$$
$$H_5(t) = 120t - 160t^3 + 32t^5, \tag{16}$$

i Ermitovi polinomi zadovoljavaju rekurentnu relaciju

$$H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t)\ , \quad n > 1\,. \tag{17}$$

Definicija 1.3. [Ermitove funkcije] Funkcija $t \mapsto \psi_n(t)$ definisana sa

$$\psi_n(t) = H_n(t)e^{-t^2/2}\ , \tag{18}$$

gde je H_n n-ti Ermitov polinom, naziva se Ermitovom funkcijom.

Napomena 1.4. Ermitove funkcije $\psi_0, \psi_1, \cdots, \psi_n, \cdots$ čine ortogonalan skup na intervalu $(-\infty, \infty)$, jer važi

$$\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)dt = \int_{-\infty}^{\infty} H_n(t)H_m(t)e^{-t^2}dt = \sqrt{\pi}2^n n!\delta_{nm} \tag{19}$$

gdje je $\delta_{nm} = 0$, $n \neq m$ i $\delta_{nn} = 1$. Relacija (19) takođe pokazuje i ortogonalnost Ermitovih polinoma sa težinskom funkcijom $w(t) = e^{-t^2}$. Zapravo, može se pokazati da Ermitove funkcije čine bazu na familiji kvadratno integrabilnih funkcija ([2]).

Napomena 1.5. Ermitove funkcije $\psi_n(t)$ predstavljaju takođe jedno partikularno rješenje diferencijalne jednačine

$$\frac{d^2z}{dt^2} + (2n + 1 - t^2)z = 0. \tag{20}$$

1.3 Furijeova transformacija Ermitove funkcije

Lema 1.1. Ermitova funkcija $t \mapsto \psi_n(t)$ predstavlja jednu od sopstvenih funkciju Furijeove transformacije

$$\mathcal{F}\{\psi_n\} = \int_{-\infty}^{\infty} \psi_n(t)e^{-\mathrm{i}\omega t}dt = \lambda\psi_n(\omega),\ \lambda{=}\sqrt{2\pi}(-\mathrm{i})^n. \tag{21}$$

Ova osobina je interesantna jer omogućava direktno određivanje $F(f(t))$ bez računanja integrala ako je funkcija f predstavljena u vidu linearne superpozicije Ermitovih funkcija $\psi_0, \psi_1, \ldots, \psi_{n-1}$.

Dokaz 1.1. Da bi dokazali relaciju (21) primijenimo FT na diferencijalnu jednačinu (20). Na osnovu elementarnih svojstava FT kojih dobijamo da je

$$(\mathrm{i}\omega)^2 Z(\mathrm{i}\omega) + (2n+1) - (-i)^2 \frac{d^2 Z(\mathrm{i}\omega)}{d\omega^2} = 0\ ,$$

odakle neposredno slijedi

$$\frac{d^2 Z(\mathrm{i}\omega)}{d\omega^2} + (2n + 1 - \omega^2)Z(\mathrm{i}\omega) = 0\ .$$

čime je jednačina (21) dokazana: $F(\psi_n(t)) = \lambda\psi_n(\omega)$. Nije teško pokazati da je $\lambda{=}\sqrt{2\pi}(-\mathrm{i})^n$.

1.4 Linearna interpolacija funkcije

Svaka funkcija $t \mapsto f(t)$ se može predstaviti kao linearna superpozicija pogodno izabranih elementarnih funkcija $h_0(t), h_1(t), \cdots h_{n-1}(t)$ na sledeći način

$$f = c_0h_0 + c_1h_1 + \cdots + c_{n-1}h_{n-1} + \varepsilon\ . \tag{22}$$

Pri tome su c_k, $k = \overline{0, n-1}$, konstante koje se mogu dobiti iz minimuma kvadratnog kriterijuma $J = \int_a^b \varepsilon^2(t)dt$. Dobijeni izraz (22) predstavlja jednu od linearnih interpolacija funkcije f funkcijama h_k na intervalu (a, b). Iz n uslova $\partial J/\partial c_k = 0$, se dobija n jednačina sa n nepoznatih konstanti c_k, $k = \overline{0, n-1}$,

$$\sum_{m=0}^{n-1} c_m \int_a^b h_k(t)h_m(t)dt = \int_a^b f(t)h_m(t)dt \; . \tag{23}$$

Ako se u prethodnoj relaciji (23) pretpostavi da funkcije $h_0(t), h_1(t), \cdots, h_{n-1}(t)$ čine bazu ortogonalnih funkcija na intervalu (a, b), tj. pretpostavimo li da važi

$$\int_a^b h_k(t)h_m(t)dt = \delta_{km} \int_a^b h_k^2(t)dt \; , \tag{24}$$

tada rješenje jednačina (23) dobija jednostavnu formu

$$c_k = \frac{\int_a^b f(t)h_k(t)dt}{\int_a^b h_k^2(t)dt} \; , \quad k = \overline{0, n-1} \; . \tag{25}$$

Napomenimo da u graničnom procesu za $n \to \infty$, $(\varepsilon \to 0)$ relacija (22) postaje $h = \sum_{k=0}^{\infty} c_k h_k$ pod određenim uslovima. Kao što se da primijetiti, ortogonalne funkcije odnosno ortogonalni polinomi ([2,3]) imaju veoma veliku primjenu, ne samo u matematici, nego u mnogim oblastima nauke i tehnike.

2 Primjena Ermitovih polinoma za određivanje FT

Neka postoji Furijeov par F, f dobijen na osnovu definicija (1) i (2). Koristeći linearnu interpolaciju funkcije f, odnosno F, pomoću Ermitovih funkcija koje čine bazu $\psi_0, \psi_1, \cdots, \psi_n$ ortogonalnih funkcija u intervalu $(-\infty, \infty)$ dobijamo, na osnovu relacija (21)– (25),

$$f(t) = \sum_{n=0}^{\infty} C_{\mathrm{n}} \psi_n(t) = \sum_{n=0}^{\infty} C_{\mathrm{n}} H_n(t) e^{-t^2/2} \; , \tag{26}$$

čija je $F(f(t))$, na osnovu relacije (21), jednaka

$$F(\mathrm{i}\omega) = \sqrt{2\pi} \sum_{n=0}^{\infty} (-\mathrm{i})^n C_{\mathrm{n}} \psi_n(\omega) = \sqrt{2\pi} \sum_{n=0}^{\infty} (-\mathrm{i})^n C_{\mathrm{n}} H_n(\omega) e^{-\omega^2/2} \tag{27}$$

gdje su

$$C_{\mathrm{n}} = \frac{\int_{-\infty}^{\infty} f(t)\psi_n(t)dt}{\sqrt{\pi} 2^n n!} = \frac{\int_{-\infty}^{\infty} f(t)H_n(t)e^{-t^2/2}dt}{\sqrt{\pi} 2^n n!} \; . \tag{28}$$

Primjenom Ermitovih polinoma za računanje inverzne FT funkcije F

$$F(\mathrm{i}\omega) = \sum_{n=0}^{\infty} \underline{C}_{\mathrm{n}} \psi_n(\omega) = \sum_{n=0}^{\infty} \underline{C}_{\mathrm{n}} H_n(\omega) e^{-\omega^2/2}, \tag{29}$$

na osnovu relacije (21) dobijamo da je $F^{-1}(F(\mathrm{i}\omega))$ jednako

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (\mathrm{i})^n \underline{C}_\mathrm{n} \psi_n(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (\mathrm{i})^n \underline{C}_\mathrm{n} H_n(t) e^{-t^2/2}, \tag{30}$$

gdje su

$$\underline{C}_\mathrm{n} = \frac{\int_{-\infty}^{\infty} F(\mathrm{i}\omega)\psi_n(\omega)d\omega}{\sqrt{\pi}2^n n!} = \frac{\int_{-\infty}^{\infty} F(\mathrm{i}\omega)H_n(\omega)e^{-\omega^2/2}d\omega}{\sqrt{\pi}2^n n!}. \tag{31}$$

Očigledno je, na osnovu relacija (26) i (30), da postoji veza između konstanti

$$C_n = \frac{1}{\sqrt{2\pi}} (\mathrm{i})^n \underline{C}_n. \tag{32}$$

Ako je Furijeov par definisan sa (3) i (4), jednostavno dobijamo Furijeov par iskazan preko Ermitovih polinoma

$$F(\mathrm{i}\omega) = \sum_{n=0}^{\infty} (-\mathrm{i})^n C_\mathrm{n} H_n(\omega) e^{-\omega^2/2} \tag{33}$$

$$f(t) = \sum_{n=0}^{\infty} (\mathrm{i})^n \underline{C}_\mathrm{n} H_n(t) e^{-t^2/2} \tag{34}$$

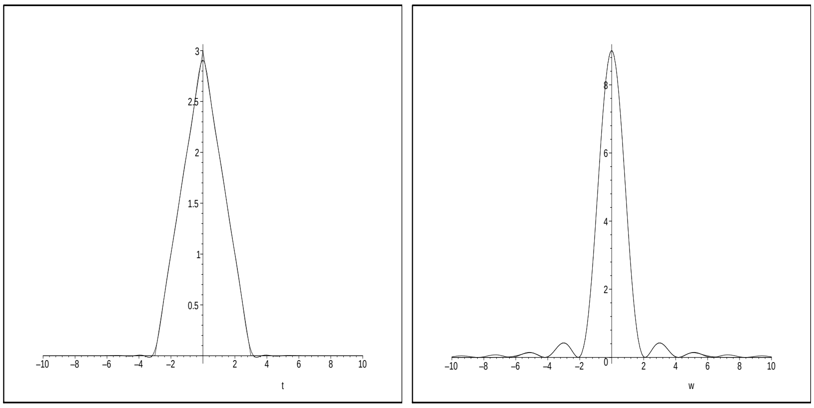
dok su konstante C_n i $\underline{C}_n$ vezane relaciom $C_n = (\mathrm{i})^n \underline{C}_n$.

Kao što se da primijetiti, glavno računanje Furijeovog para primjenom Ermitovih polinoma se svodi na računanje interpolacionih konstanti C_n ili $\underline{C}_n$.

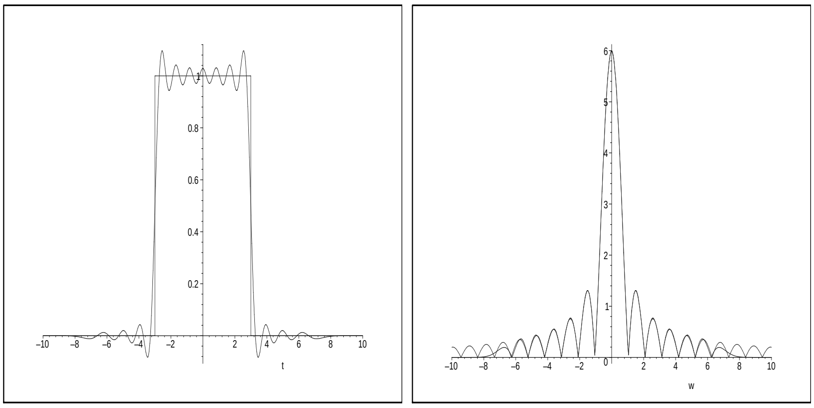
3 Simulaciona analiza

Neka je signal $f(t)$ zbog praktične primjene ograničen u vremenskom intervalu, na primjer, pravougaonim prozorom $\hat{f}(t) = f(t)(u(t+t_0) - u(t-t_0))$, gdje je funkcija u Hevisajdova jedinična funkcija. Uporedimo numeričke rezultate $F(\hat{f}(t))$ primjenom Ermitovih polinoma konačnog reda n sa analitički dobijenim vrijenostima $F(\hat{f}(t))$.

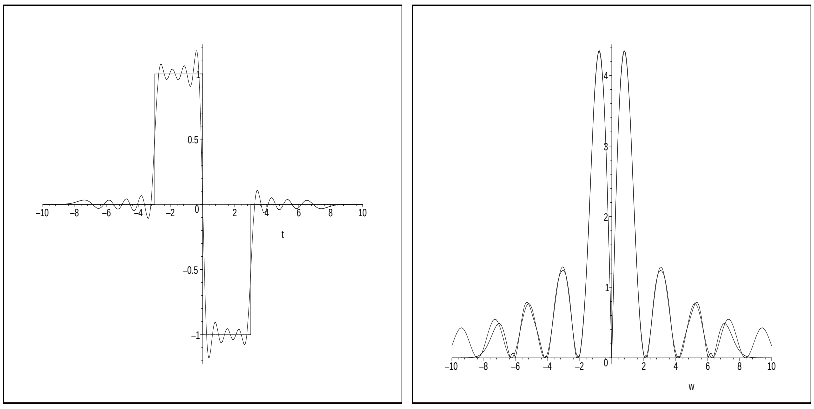
Napomenimo da se za ograničenje signala u vremenskom intervalu mogu koristiti i druge prozorske funkcije $\hat{f}(t) = f(t)\Pi(t)$ koje nemaju dodatne zahtjeve u smislu određivanja $F(\hat{f}(t))$ primjenom Ermitovih polinoma. Ovo važi i kod računanja inverzne Furijeove transformacije, stim da se radi sve u frekvencijskom domenu. Uporedni prikaz Furijeove transformacije različitih karakterističnih signala računatih analitičkim putem i pomoću Ermitovih polinoma prikazan je na slikama 1, 2 i 3.



Slika 1: (Lijevo) vremenski oblik trougaonog signala i odgovarajuća njegova interpolacija Ermitovim funkcijama reda $n = 20$ i (Desno) frekvencijski spektar trougaonog signala dobijen analitičkim putem i pomoću Ermitovih polinoma za $n = 20$.



Slika 2: (Lijevo) vremenski oblik pravougaonog signala i odgovarajuća njegova interpolacija Ermitovim funkcijama reda $n = 30$ i (Desno) frekvencijski spektar pravougaonog signala dobijen analitičkim putem i pomoću Ermitovih polinoma za $n = 30$.



Slika 3: (Lijevo) vremenski oblik bipolarnog signala i odgovarajuća njegova interpolacija Ermitovim funkcijama reda $n = 30$ i (Desno) frekvencijski spektar pravougaonog signala dobijen analitičkim putem i pomoću Ermitovih polinoma za $n = 30$.

4 Zaključak

Računanje Furijeove transformacije i njene inverzne Furijeove transformacije (Furijeov par) korišćenjem Ermitovih polinoma ima praktični smisao jer omogućava direktno određivanje Furijeovog para bez računanja integrala pri direktnoj i inverznoj Furijeovoj transformaciji. Dalja istraživanja u ovoj problematici mogu da se odnose na podjelu domena funkcije, kako u vremenskom tako i u frekvencijskom domenu, u niz odgovarajućih prozora na kojima će se primijeniti linearna interpolacija i na taj način smanjiti visoki red Ermitovih polinoma u cilju veće efikasnosti računanja Furijeovog para.

Zahvalnica

Autori T. B. Šekara i M. R. Rapaić se zahvaljuju Ministarstvu za nauku i tehnološki razvoj Republike Srbije za finansijsku podršku ovom radu u okviru tehnološkog projekta TR33020 (T.B.Š) i TR32018 i TR33013 (M.R.R).

Literatura

[1] Ja. S. Ufland, *Integralne transformacije sa primenom u teoriji elastičnosti*, (na ruskom), Nauka, Lenjingrad, 1968.

[2] Dragoslav S. Mitrinović, *Uvod u specijalne funkcije*, Građevinska knjiga Beograd, 1975.

[3] G. Mastroianni, G. Milovanović, *Interpolation processes basic theory and applications*, Springer-Verlag, 2008.

TREĆA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
Trebinje, 7. i 8. juni 2013. godine

Pravila frakcionog diferenciranja i integracije Laplasovog lika signala

Milan R. Rapaić
Fakultet tehničkih nauka, Univerzitet u Novom Sadu
rapaja@uns.ac.rs

Tomislav B. Šekara
Elektrotehnički fakulteta, Univerzitet u Beogradu
tomi@etf.rs

Originalni naučni rad

Apstrakt

Laplasova transformacija ima široku primenu u različitim oblastima fizike, tehnike i nauke uopšte. Uistinu, svaka pojava koja se odvija u "kontinualnom vremenu", a koja pri tome zadovoljava princip superpozicije može se analizirati pomoću Laplasove trasnformacije. Među dobro poznate osobine Laplasove transformacije svakako spada i ta da se diferenciranja preslikavaju u množenja stepenom funkcijom, ali i obrnuto, da se množenje stepenom funkcijom preslikava u diferenciranje. Slično važi i za operaciju integraljenja: integracija se preslikava u deljenje stepenom funkcijom, i obrnuto. U okviru ovog rada razmatramo opštu vezu između množenja, odnosno deljenja, funkcijom proizvoljnog, pozitivnog realnog stepena i operacija frakcionog diferenciranja, odnosno integracije.

1 Uvod

Laplasovom transformacijom realne funkcije $f : [0, \infty) \to \mathbb{R}$ nazivamo kompleksnu funkciju kompleksne promenljive s, u oznaci $\mathcal{L}\{f(t)\}(s)$, definisanu nesvojstvenim integralom

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt \; . \tag{1}$$

U prethodnom izrazu, integraciju treba razumeti u Lebegovom smislu. Laplasova transformacija funkcije je definisana za sve one vrednosti kompleksne (Laplasove) promenljive s za koje integral (1) konvergira (apsolutno i uniformno). Dobro je poznato da, ukoliko je funkcija f eksponencijalno ograničena, tj. ukoliko postoje realne konstante $M > 0$ i γ takve da je $|f(t)| < M\,e^{\gamma t}$, tada integral (1) konvergira nad oblašću $\Re\{s\} > \gamma$.

Napomena 1.1. Laplasovu transformaciju je obično moguće analitički produžiti na celu kompleksnu ravan, isključujući eventualno diskretan skup tačaka. U ovom radu, kada god govorimo o Laplasovoj transformaciji mislimo na ovo analitičko produženje.

Napomena 1.2. S obzirom da su nam od prevashodnog interesa primene Laplasove transformacije u oblastima upravljanja i obrade signala, funkciju f ćemo najčešće nazivati signalom, njenu nezavisnu promenljivu t ćemo interpretirati kao vreme, a samu Laplasovu transformaciju signala nazivaćemo njegovim kompleksnim likom.

Napomena 1.3. Laplasova transformacija se može definisati i nad prostorom temperiranih distribucija. U tom slučaju, definiciju Laplasove transformacije moguće je proširiti na znatno širu klasu signala, koja između ostalog uključuje Dirakov δ-signal i njegove izvode. Iako svi dobijeni rezultati važe i u ovom slučaju, u nastavku izlaganja ćemo se ograničiti na "klasičnu" definiciju (1).

U nastavku ćemo se baviti uopštenjem sledećih dobro poznatih tvrđenja

$$\mathcal{L}\{tf(t)\}(s) = -\frac{d}{ds}\mathcal{L}\{f(t)\}(s) \ , \tag{2}$$

$$\mathcal{L}\{\frac{1}{t}f(t)\}(s) = \int_s^{\infty} F(u)du \ . \tag{3}$$

koja važe unutar oblasti apsolutne konvergencije razmatranih transformacija (za detalje videti [2]).

Dakle, množenje signala stepenim signalom u vremenskom domenu ekvivalentno je diferenciranju (sa negativnim predznakom) u kompleksnom domenu. Slično, množenje signala recipročnim stepenim signalom preslikava se u integraciju njegovog kompleksnog lika, pri čemu se integracija vrši nad intervalom (s, ∞). Poznato je da se pod relativno blagim uslovima (videti [2]) gornje osobine neposredno uopštavaju na slučaj kada je diferenciranje višestruko, a integracija ponovljena.

Posmatraćemo slučaj koji nastaje kada se signal u vremenskom domenu množi ili deli stepenim signalom *necelog* reda. Posmatramo dakle Laplasove transformacije signala $t^{\pm\alpha}f(t)$, gde je $\alpha > 0$. Razumno je pretpostaviti da se u ovom slučaju na kompleksne likove signala moraju primeniti operacija frakcionog diferenciranja i integracije. Pri tome, predznak u izrazu (2), te granice integracije u izrazu (3) sugerišu da diferenciranje i integraciju treba shvatiti u desnom smislu.

2 Diferenciranje i integracija necelog stepena

Diferenciranje i integracija necelog (frakcionog) stepena se može definisati na veći broj načina. U okviru ovog rada oslonićemo se na tzv. Riman-Ljuvilovu definiciju prema kojoj se frakcioni integral definiše neposrednim uopštenjem Košijeve formule

$$\begin{aligned} {}_aI_t^n f &= \int_a^t \int_0^{t_1} \dots \int_0^{t_{n-1}} f(t_n)dt_n dt_{n-1} \dots dt_1 \\ &= \frac{1}{(n-1)!}\int_a^t f(\tau)(t-\tau)^{n-1}d\tau \ , \end{aligned} \tag{4}$$

gde je n pozitivan, ceo broj, dok je a proizvoljna realna donja granica. Formalnom smenom n sa α, gde je α pozitivan, realan broj odmah nalazimo

$$ {}_aI_t^{\alpha}f = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \ , \tag{5} $$

gde smo faktorijel uopštili pomoću dobro poznate Ojlerove Gama funkcije. Izraz (5) predstalja definiciju **Levog Riman-Ljuvilovog integrala reda** α sa donjom granicom a.

S obzirom da je funkcija $(t-\tau)^{\alpha-1}$ neintegrabilna u okolini tačke $t=\tau$ za $\alpha < 0$, operaciju diferenciranja nije moguće definisati izrazom (5) u kome bi se dopustile i negativne vrednosti reda α. Umesto toga, izvod necelog stepena se može definisati pomoću klasičnog, celobrojnog izvoda i operacije integracije necelog stepena, i to na sledeći način

$$ {}_aD_t^{\alpha}f = \left(\frac{d}{dt}\right)^n {}_aI_t^{n-\alpha}f, \tag{6} $$

gde je n najmanji pozitivan ceo broj koji je veći ili jednak od α, $n-1 < \alpha \leq n$. Izrazom (6) definiše se **Levi Riman-Ljuvilov izvod reda** α sa donjom granicom a.

Desni operatori necelog diferenciranja i integracije definišu se na sličan način. Neka je $b \geq t$ proizvoljno izabrana gornja granica integracije. Tada je **Desni Riman-Ljuvilov integral reda** α sa gornjom granicom b

$$ {}_tI_b^{n}f = \int_t^b f(\tau)(\tau-t) d\tau \ . \tag{7} $$

Desni Riman-Ljuvilov izvod reda α sa gornjom granicom b definiše se izrazom

$$ {}_tD_b^{n}f = \left(-\frac{d}{dt}\right)^n {}_tI_b^{n-\alpha}f \ , \tag{8} $$

gde je, kao i ranije, n najmanji pozitivan ceo broj koji je veći ili jednak od α.

Napomena 2.1. Iako su u opštem slučaju donja i gornja granica integracije, a i b, proizvoljni realni brojevi, mi ćemo se ograničiti na slučaj kada je $a=0$ i $b=\infty$. Pri tome, svi signali koje posmatramo biće kauzalni, definisani nad intervalom $[a,b) = [0,\infty)$.

Napomena 2.2. Ukoliko se prethodno uvedeni definicioni izrazi (5), (6), (7) i (8) shvate u distributivnom smislu, može se pokazati da je

$$ \lim_{\alpha\to 0^+} {}_aI_t^{\alpha}f = f(t) \ , $$
$$ \lim_{\alpha\to 0^+} {}_tI_b^{\alpha}f = f(t) \ , $$

gde date granične vrednosti treba shvatiti u tzv slabom ili distributivnom smislu. Drugim rečima, kada je red integracije približava nuli Riman-Ljuvilovi integratori se ponašaju kao operator identiteta, te preslikavaju signal u sebe samog. Na osnovu ove činjenice, te na osnovu Košijeve formule (4), neposredno sledi da su svi Riman-Ljuvilovi integrali celobrojnog stepena jednaki odgovarajućim klasičnim, celobrojnim integralima, te da su svi Levi Riman-Ljuvilovi izvodi celobrojnog stepena jednaki odgovarajućim klasičnim izvodima. Desni Riman-Ljuvilovi izvodi celobrojnog stepena jednaki su odgovarajućim klasičnim izvodima pomnoženim sa $(-1)^n$.

Prethodna razmatranja u potpunosti opravdavaju sledeću definiciju.

Definicija 2.1 (Riman-Ljuvilovi integro-diferencijalni operatori necelog stepena). Neka je f dati signal definisan nad intervalom $[0, \infty)$, tada je

- **Levi RL integral reda α.**

$$ {}_0I_t^\alpha f = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, & \alpha > 0 \ , \\ f(t), & \alpha = 0 \ . \end{cases} \tag{9} $$

- **Desni RL integral reda α.**

$$ {}_tI_\infty^\alpha f = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^\infty f(\tau)(\tau-t)^{\alpha-1} d\tau, & \alpha > 0 \ , \\ f(t), & \alpha = 0 \ . \end{cases} \tag{10} $$

- **Levi RL izvod reda α.**

$$ {}_0D_t^\alpha f = \left(\frac{d}{dt}\right)^n {}_0I_t^{n-\alpha} f \tag{11} $$

- **Desni RL izvod reda α.**

$$ {}_tD_\infty^\alpha f = \left(-\frac{d}{dt}\right)^n {}_tI_\infty^{n-\alpha} f \tag{12} $$

U svim prethodnim izrazima n je najmanji ceo broj veći ili jednak od α.

Među mnogobrojnim osobinama diferencijalnih operatora necelog reda izdvajamo sledeća, za nas posebno interesantna svojstva

Lema 2.1 ([1])**.** Laplasova transformacija levog RL integrala signala f, ukoliko postoji, jednaka je

$$ \mathcal{L}\{{}_0I_t^\alpha f\} = \frac{1}{s^\alpha} F(s) \ . \tag{13} $$

(Drugim rečima, $s^{-\alpha}$ se može interpretirti kao operator frakcione integracije reda α u kompleksnom domenu.)

Lema 2.2 ([1])**.** Desni Riman-Ljuvilov izvod reda β stepenog signala $t^{\rho-1}$, ukoliko postoji, jednak je

$$ {}_tD_\infty^\beta t^{\rho-1} = \frac{\Gamma(1+\beta-\rho)}{\Gamma(1-\rho)} t^{\rho-1-\beta} \ . \tag{14} $$

3 Osnovni rezultat

Teorema 3.1. Neka je f dati signal i neka njegov kompleksni lik $\mathcal{L}\{f(t)\}(s)$ postoji za svako $s \in \mathbb{C}$ takvo da je $\Re(s) > \gamma$, za neko $\gamma \in \mathbb{R}$. Tada nad istim domenom važe sledeća tvrđenja:

1. Laplasova transformacija signala $t^{-\alpha} f(t)$, ukoliko postoji, jednaka je desnom Riman-Ljuvilovom frakcionom integralu reda α kompleksnog lika signala $f(t)$,

$$\mathcal{L}\{t^{-\alpha} f(t)\} = {}_sI_\infty^\alpha \mathcal{L}\{f(t)\}(s) = \frac{1}{\Gamma(\alpha)} \int_s^\infty F(u)(u-s)^{\alpha-1} du \,. \tag{15}$$

2. Laplasova transformacija signala $t^\alpha f(t)$ jednaka je desnom Riman-Ljuvilovom frakcionom izvodu reda α kompleksnog lika signala $f(t)$,

$$\begin{aligned} \mathcal{L}\{t^\alpha f(t)\} &= {}_sD_\infty^\alpha \mathcal{L}\{f(t)\}(s) = \left(-\frac{d}{ds}\right)^n {}_sI_\infty^{n-\alpha} \mathcal{L}\{f(t)\}(s) \\ &= (-1)^n \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_s^\infty F(u)(u-s)^{n-\alpha-1} du \,, \end{aligned} \tag{16}$$

gde je n najmanji prirodan broj veći ili jednak redu diferenciranja, $n-1 < \alpha \leq n$.

Dokaz 3.1. Dokažimo najpre tvrđenje po 1. Polazimo od krajnjeg izraza (15) u koji ćemo uvrstiti definicioni izraz Laplasove transformacije (1) u kome je izvršena formalna smena promenljive s promenljivom u,

$$\frac{1}{\Gamma(\alpha)} \int_s^\infty F(u)(u-s)^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_s^\infty \left(\int_0^\infty f(t) e^{-ut} dt \right) (u-s)^{\alpha-1} du.$$

Prema uvedenim pretpostavkama, svi nesvojstveni integrali konvergiraju apsolutno i uniformno, stoga je moguće izvršiti smenu redosleda integracije. Tako dobijamo

$$\frac{1}{\Gamma(\alpha)} \int_s^\infty F(u)(u-s)^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(t) \left(\int_s^\infty (u-s)^{\alpha-1} e^{-ut} du \right) dt \,.$$

Uvođenjem smene $\xi = u - s$ u unutrašnji integral, nalazimo da važi

$$\begin{aligned} \int_s^\infty (u-s)^{\alpha-1} e^{-ut} du &= \int_0^\infty \xi^{\alpha-1} e^{-(s+\xi)t} d\xi \\ &= e^{-st} \int_0^\infty \xi^{\alpha-1} e^{-\xi t} d\xi \\ &= e^{-st} \frac{\Gamma(\alpha)}{t^\alpha} \,. \end{aligned}$$

Poslednja jednakost je posledica dobro poznatog izraza za Laplasovu transformaciju stepene funkcije,

$$\int_0^\infty \tau^{\alpha-1} e^{-s\tau} d\tau = \frac{\Gamma(\alpha)}{s^\alpha} \ ,$$

u kome je, formalno, izvršena smena promenljive s promenljivom t, a promenljive ξ promenljivom τ. Konačno je

$$\frac{1}{\Gamma(\alpha)} \int_s^\infty F(u)(u-s)^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(t) e^{-st} \frac{\Gamma(\alpha)}{t^\alpha} dt = \int_0^\infty \frac{f(t)}{t^\alpha} e^{-st} dt \ .$$

Time je Tvrđenje pod 1. dokazano.

Dokaz Tvrđenja 2. ćemo izvesti oslanjajući se na Tvrđenje pod 1. Naime, na osnovu prethodno dokazanog izraza važi da je

$$\frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{ds}\right)^n \int_s^\infty F(u)(u-s)^{n-\alpha-1} du = \left(-\frac{d}{ds}\right)^n \int_0^\infty \frac{f(t)}{t^{n-\alpha}} e^{-st} dt \ .$$

Kako prema uvedenim pretpostavkama operacija diferenciranja po Laplasovoj promenljivoj može da zameni mesto sa operacijom integracije po vremenu, zaključujemo da je

$$\begin{aligned} \left(-\frac{d}{ds}\right)^n \int_0^\infty \frac{f(t)}{t^{n-\alpha}} e^{-st} dt &= \int_0^\infty \frac{f(t)}{t^{n-\alpha}} \left(-\frac{d}{ds}\right)^n e^{-st} dt \\ &= \int_0^\infty \frac{f(t)}{t^{n-\alpha}} t^n e^{-st} dt = \int_0^\infty t^\alpha f(t) e^{-st} dt \ , \end{aligned}$$

čime je Tvrđenje dokazano.

4 Zaključak

U okviru rada izvršeno je jedno uopštenje dobro poznatih osobina diferenciranja i integracije Laplasovih likova signala. U radu je razmatran slučaj kada su operacije diferenciranja i integracije necelobrojnog (frakcionog) stepena. Pokazano je da su ove operacije ekvivalentne množenju polaznog signala stepenom funkcijom odgovarajućeg stepena.

U daljim istraživanjima bavićemo se primenom prikazanih rezultata u oblasti projektovanja optimalnih regulatora sa frakcionim astatizmom.

Literatura

[1] A. A. Kilbas, H. M. Srivastava, J. J. Truhillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.

[2] J. L. Schiff, The Laplace Transform: Theory and Application. Springer, 1999

CIP - Каталогизација у публикацији
Народна и универзитетска библиотека
Републике Српске, Бања Лука

51(082)

MATHEMATICAL Conference of the Republic of Srpska (3 ; 2013 ; Trebinje)

Proceedings. Volume I / Third Mathematical Conference of the Republic of Srpska, Trebinje, 07-08 June 2013 = Zbornik radova. tom I / Treća matematička konferencija Republike Srpske ; [glavni urednik Milenko Pikula]. - 1. izd. - Trebinje : Fakultet za proizvodnju i menadžment, 2014 (Trebinje : Grafokomerc). - 340 str. : ilustr. ; 25 cm

Tiraž 300. - Bibliografija uz sve radove. - Abstracts.

ISBN 978-99976-600-0-8

COBISS.RS-ID 4289048