Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 200 (2007) 276 – 282
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A note on the bounds of the error of Gauss–Turán-type quadratures ☆

Gradimir V. Milovanović[a], Miodrag M. Spalević[b,*]

[a]Department of Mathematics, Faculty of Electronic Engineering, University of Niš, P.O. Box 73, 18000 Niš, Serbia and Montenegro
[b]Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, P.O. Box 60, 34000 Kragujevac, Serbia and Montenegro

Received 21 September 2005; received in revised form 22 December 2005

Abstract

This note is concerned with estimates for the remainder term of the Gauss–Turán quadrature formula,

$$R_{n,s}(f) = \int_{-1}^{1} w(t) f(t)\,\mathrm{d}t - \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v),$$

where $w(t) = (U_{n-1}(t)/n)^2\sqrt{1-t^2}$ is the Gori–Michelli weight function, with $U_{n-1}(t)$ denoting the $(n-1)$th degree Chebyshev polynomial of the second kind, and f is a function analytic in the interior of and continuous on the boundary of an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$. The present paper generalizes the results in [G.V. Milovanović, M.M. Spalević, Bounds of the error of Gauss–Turán-type quadratures, J. Comput. Appl. Math. 178 (2005) 333–346], which is concerned with the same problem when $s = 1$.

MSC: Primary 65D30, 65D32; secondary 41A55

Keywords: Gauss–Turán quadrature formula; Gori–Michelli weight function; Error bounds for analytic functions

1. Introduction

Let w be an integrable weight function on the interval $(-1, 1)$. We consider the error term $R_{n,s}(f)$ of the Gauss–Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} w(t) f(t)\,\mathrm{d}t = \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) + R_{n,s}(f),$$

☆ The authors were supported in parts by the Swiss National Science Foundation (SCOPES Joint Research Project No. IB7320-111079 “New Methods for Quadrature”) and the Serbian Ministry of Science and Environmental Protection.
* Corresponding author. Tel.: +381 34 300 254; fax: +381 34 335 040.
E-mail addresses: grade@elfak.ni.ac.yu (G.V. Milovanović), spale@kg.ac.yu (M.M. Spalević).

0377-0427/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.12.021

which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$, and whose nodes are the zeros of the corresponding s-orthogonal polynomial $\pi_{n,s}(t)$ of degree n. For more details on Gauss–Turán quadratures and s-orthogonal polynomials see the book [1] and the survey paper [4].

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and D be its interior. If the integrand f is an analytic function in D and continuous on $\overline{D}$, then we take as our starting point the well-known expression of the remainder term $R_{n,s}(f)$ in the form of the contour integral

$$R_{n,s}(f) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} K_{n,s}(z) f(z)\, \mathrm{d}z. \tag{1.1}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1, 1], \tag{1.2}$$

where

$$\varrho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} w(t)\, \mathrm{d}t, \quad n \in \mathbb{N}, \tag{1.3}$$

and $\pi_{n,s}(t)$ is the corresponding s-orthogonal polynomial with respect to the weight function $w(t)$ on $(-1, 1)$.

The integral representation (1.1) leads to a general error estimate, by using Hölder inequality,

$$|R_{n,s}(f)| = \frac{1}{2\pi} \left| \oint_{\Gamma} K_{n,s}(z) f(z)\, \mathrm{d}z \right| \leqslant \frac{1}{2\pi} \left(\oint_{\Gamma} |K_{n,s}(z)|^{r} |\mathrm{d}z| \right)^{1/r} \left(\oint_{\Gamma} |f(z)|^{r'} |\mathrm{d}z| \right)^{1/r'},$$

i.e.,

$$|R_{n,s}(f)| \leqslant \frac{1}{2\pi} \|K_{n,s}\|_r \|f\|_{r'}, \tag{1.4}$$

where $1 \leqslant r \leqslant +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} \left(\oint_{\Gamma} |f(z)|^r |\mathrm{d}z|\right)^{1/r}, & 1 \leqslant r < +\infty, \\ \max\limits_{z \in \Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ ($r' = 1$) gives

$$|R_{n,s}(f)| \leqslant \frac{\ell(\Gamma)}{2\pi} \left(\max_{z \in \Gamma} |K_{n,s}(z)| \right) \left(\max_{z \in \Gamma} |f(z)| \right), \tag{1.5}$$

where $\ell(\Gamma)$ is the length of the contour Γ. On the other side, for $r = 1$ ($r' = +\infty$), the estimate (1.4) reduces to

$$|R_{n,s}(f)| \leqslant \frac{1}{2\pi} \left(\oint_{\Gamma} |K_{n,s}(z)||\mathrm{d}z| \right) \left(\max_{z \in \Gamma} |f(z)| \right), \tag{1.6}$$

which is evidently stronger than the previous, because of inequality

$$\oint_{\Gamma} |K_{n,s}(z)||\mathrm{d}z| \leqslant \ell(\Gamma) \left(\max_{z \in \Gamma} |K_{n,s}(z)| \right).$$

Also, the case $r = r' = 2$ could be of certain interest.

For getting the estimate (1.5) or (1.6) it is necessary to study the magnitude of $|K_{n,s}(z)|$ on Γ or the quantity

$$L_{n,s}(\Gamma) := \frac{1}{2\pi} \oint_{\Gamma} |K_{n,s}(z)|\, |\mathrm{d}z|,$$

respectively (see, e.g., [5,6]).

Error estimates (1.6) for Gauss–Turán quadratures with Gori–Micchelli weight function, and when Γ is taken to be a confocal ellipse, are considered for the general case ($s \in \mathbb{N}$) in Section 2. The particular case $s=1$ was considered in [7].

2. Error estimates for Gauss–Turán quadratures with Gori–Micchelli weight function for general s ∈ ℕ

Let the contour Γ be an ellipse with foci at the points ± 1 and sum of semi-axes $\varrho > 1$,

$$\mathscr{E}_\varrho = \{z \in \mathbb{C} : z = \tfrac{1}{2}(\varrho \mathrm{e}^{\mathrm{i}\theta} + \varrho^{-1}\mathrm{e}^{-\mathrm{i}\theta}),\ 0 \leqslant \theta \leqslant 2\pi\}. \tag{2.1}$$

In [7] we considered the error estimates (1.6) for Gauss–Turán quadrature formula with $s=1$ and for the Gori–Michelli weight function

$$w(t) = w_n(t) = \frac{U_{n-1}^2(t)}{n^2}\sqrt{1-t^2}, \tag{2.2}$$

where $U_{n-1}(\cos\theta) = \sin n\theta / \sin\theta$ is the Chebyshev polynomial of the second kind. Here we consider the general case with $s \in \mathbb{N}$.

It is well-known that for the weight function (2.2) the Chebyshev polynomials $T_n(t)$ of the first kind appear to be s-orthogonal ones (cf. [2]). For $z \in \mathscr{E}_\varrho$, i.e., $z = \frac{1}{2}(\xi + \xi^{-1})$, $\xi = \varrho \mathrm{e}^{\mathrm{i}\theta}$, we have $\pi_{n,s}(z) = T_n(z) = \frac{1}{2}(\xi^n + \xi^{-n})$ and, according to (1.3) and (2.2),

$$\varrho_{n,s}(z) = \frac{1}{n^2}\int_{-1}^{1} \frac{T_n(t)^{2s+1}U_{n-1}^2(t)}{z-t}\sqrt{1-t^2}\,\mathrm{d}t. \tag{2.3}$$

Since $|\mathrm{d}z| = 2^{-1/2}\sqrt{a_2 - \cos 2\theta}\,\mathrm{d}\theta$, where we put

$$a_j = a_j(\varrho) = \tfrac{1}{2}(\varrho^j + \varrho^{-j}), \quad j \in \mathbb{N},\ \varrho > 1, \tag{2.4}$$

we have, according to (1.2),

$$L_{n,s}(\mathscr{E}_\varrho) = \frac{1}{2\pi\sqrt{2}}\int_0^{2\pi} \frac{|\varrho_{n,s}(z)|(a_2 - \cos 2\theta)^{1/2}}{|T_n(z)|^{2s+1}}\,\mathrm{d}\theta. \tag{2.5}$$

Now, from (2.3), by substituting $t = \cos\theta$, we have, in view of $T_n(\cos\theta) = \cos n\theta$ and $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$,

$$\varrho_{n,s}(z) = \frac{1}{n^2}\int_0^{\pi} \frac{[\cos n\theta]^{2s+1}[\sin n\theta]^2}{z - \cos\theta}\,\mathrm{d}\theta.$$

We transform $[\cos n\theta]^{2s+1}$ by using a formula from [3, Eq. 1.320.7], while $[\sin n\theta]^2 = (1-\cos 2n\theta)/2$. Therefore,

$$\begin{aligned}\varrho_{n,s}(z) &= \frac{1}{n^2 2^{2s+1}}\int_0^{\pi} \frac{\sum_{k=0}^{s}\binom{2s+1}{k}\cos(2s+1-2k)n\theta(1-\cos 2n\theta)}{z-\cos\theta}\,\mathrm{d}\theta \\ &= \frac{1}{n^2 2^{2s+1}}\sum_{k=0}^{s}\binom{2s+1}{k}\left[\int_0^{\pi}\frac{\cos(2s+1-2k)n\theta}{z-\cos\theta}\,\mathrm{d}\theta - \int_0^{\pi}\frac{\cos(2s+1-2k)n\theta\cos 2n\theta}{z-\cos\theta}\,\mathrm{d}\theta\right],\end{aligned}$$

i.e.,

$$\begin{aligned}\varrho_{n,s}(z) = \frac{1}{n^2 2^{2s+1}}\sum_{k=0}^{s}\binom{2s+1}{k}\Bigg[&\int_0^{\pi}\frac{\cos(2s+1-2k)n\theta}{z-\cos\theta}\,\mathrm{d}\theta \\ &- \frac{1}{2}\int_0^{\pi}\frac{\cos(2s+3-2k)n\theta}{z-\cos\theta}\,\mathrm{d}\theta - \frac{1}{2}\int_0^{\pi}\frac{\cos(2s-1-2k)n\theta}{z-\cos\theta}\,\mathrm{d}\theta\Bigg].\end{aligned}$$

Furthermore, using [3, Eq. 3.613.1], one finds

$$\int_0^{\pi} \frac{\cos m\theta}{z-\cos\theta}\,\mathrm{d}\theta = \frac{\pi}{\sqrt{z^2-1}}\left(z-\sqrt{z^2-1}\right)^m, \quad m \in \mathbb{N}_0,$$

and we obtain

$$\begin{aligned} \varrho_{n,s}(z) = \frac{1}{2^{2s+1}n^2} \sum_{k=0}^{s} \binom{2s+1}{k} \Bigg\{ \Bigg[\frac{2\pi}{\xi-\xi^{-1}} \frac{1}{\xi^{2(s-k)n+n}} - \frac{1}{2} \frac{2\pi}{\xi-\xi^{-1}} \frac{1}{\xi^{2(s-k)n+3n}} \Bigg] \\ - \frac{1}{2} \Bigg[\sum_{k=0}^{s-1} \binom{2s+1}{k} \frac{2\pi}{\xi-\xi^{-1}} \frac{1}{\xi^{2(s-k)n-n}} + \binom{2s+1}{s} \frac{2\pi}{\xi-\xi^{-1}} \frac{1}{\xi^{n}} \\ + \binom{2s+1}{s} \frac{2\pi}{\xi-\xi^{-1}} \xi^{n} - \binom{2s+1}{s} \frac{2\pi}{\xi-\xi^{-1}} \xi^{n} \Bigg] \Bigg\}, \end{aligned}$$

where we used that $\sqrt{z^2-1} = \frac{1}{2}(\xi-\xi^{-1})$ and $z-\sqrt{z^2-1}=\xi^{-1}$. Finally, we obtain

$$\varrho_{n,s}(z) = \frac{\pi}{2^{2s+1}n^2} \cdot \frac{\xi^n-\xi^{-n}}{\xi-\xi^{-1}}\,(b-\alpha)\,, \tag{2.6}$$

where we used the notation

$$b \equiv b(s) = \binom{2s+1}{s}, \quad \alpha \equiv \alpha_{n,s}(\varrho,\theta) = \frac{\xi^n-\xi^{-n}}{\xi^n} \sum_{k=0}^{s} \binom{2s+1}{k} \frac{1}{\xi^{2(s-k)n}}.$$

Using (2.6) and

$$|T_n(z)| = (a_{2n}+\cos 2n\theta)^{1/2}/\sqrt{2}, \quad |\xi^k-\xi^{-k}| = \sqrt{2}(a_{2k}-\cos 2k\theta)^{1/2} \quad (k \in \mathbb{N}),$$

the quantity (2.5) reduces to

$$L_{n,s}(\mathscr{E}_\varrho) = \frac{1}{2^{s+2}n^2} \int_0^{2\pi} \sqrt{\frac{(a_{2n}-\cos 2n\theta)|b-\alpha|^2}{(a_{2n}+\cos 2n\theta)^{2s+1}}}\,\mathrm{d}\theta, \tag{2.7}$$

where $|b-\alpha|^2 = b^2 - 2b\,\Re\{\alpha\} + |\alpha|^2$ ($b \in \mathbb{R},\ \alpha \in \mathbb{C}$). It is not difficult to conclude that $|\alpha|^2 = \alpha \cdot \overline{\alpha} = h_2(2n\theta)$, where

$$h_2(\theta) = \frac{2(a_{2n}-\cos\theta)}{\varrho^{2n(s+1)}} |W_s(\varrho^n,\theta)|^2$$

and $W_s(\varrho,\theta) := \sum_{\nu=0}^{s} \binom{2s+1}{\nu} \varrho^{2\nu-s} \mathrm{e}^{\mathrm{i}(\nu-s/2)\theta}$ has been defined in [6, Eq. (4.12)].

Let $x=\varrho^{4n}$. Recall that $|W_s(\varrho^n,\theta)|^2 = \sum_{\ell=0}^{s} A_\ell \cos \ell\theta$ (cf. [6, Eqs. (4.13)–(4.15)]), where

$$A_0 = \frac{1}{x^{s/2}} \sum_{\nu=0}^{s} \binom{2s+1}{\nu}^2 x^\nu$$

and

$$A_\ell = \frac{2}{x^{(s-\ell)/2}} \sum_{\nu=0}^{s-\ell} \binom{2s+1}{\nu} \binom{2s+1}{\nu+\ell} x^\nu, \quad \ell = 1,\ldots,s.$$

Further, we have

$$\Re\mathrm{e}\{\alpha\} = \Re\mathrm{e}\left\{(1 - 1/\xi^{2n}) \sum_{v=0}^{s} \binom{2s+1}{v} \frac{1}{\xi^{2(s-v)n}}\right\} = h_1(2n\theta),$$

where

$$h_1(\theta) = \sum_{v=0}^{s} \binom{2s+1}{v} \varrho^{2(v-s)n} \cos(s-v)\theta - \sum_{v=0}^{s} \binom{2s+1}{v} \varrho^{2(v-s-1)n} \cos(s+1-v)\theta.$$

Therefore, (2.7) becomes

$$L_{n,s}(\mathscr{E}_\varrho) = \frac{1}{2^{s+2}n^2} \int_0^{2\pi} \sqrt{\frac{(a_{2n} - \cos 2n\theta)(b^2 - 2bh_1(2n\theta) + h_2(2n\theta))}{(a_{2n} + \cos 2n\theta)^{2s+1}}}\, \mathrm{d}\theta.$$

The last integrand depends in θ via $\cos 2n\ell\theta$ ($n \in \mathbb{N}, \ell \in \{1, \ldots, s+1\}, s \in \mathbb{N}_0$). It is a continuous function of the form $g(2n\theta)$, where

$$g(\theta) \equiv g(\cos\theta, \cos 2\theta, \ldots, \cos(s+1)\theta).$$

Because of periodicity, it is easy to prove that $\int_0^{2\pi} g(2n\theta)\,\mathrm{d}\theta = 2\int_0^{\pi} g(\theta)\,\mathrm{d}\theta$. Therefore, $L_{n,s}(\mathscr{E}_\varrho)$ reduces to

$$L_{n,s}(\mathscr{E}_\varrho) = \frac{1}{2^{s+1}n^2} \int_0^{\pi} \sqrt{\frac{(a_{2n} - \cos\theta)(b^2 - 2bh_1(\theta) + h_2(\theta))}{(a_{2n} + \cos\theta)^{2s+1}}}\, \mathrm{d}\theta. \tag{2.8}$$

Further, $h_1(\theta)$ can be written in the form

$$h_1(\theta) = x^{-s/2} \sum_{v=0}^{s} \binom{2s+1}{v} [x^{v/2} \cos(s-v)\theta - x^{(v-1)/2} \cos(s+1-v)\theta],$$

i.e., after expanding the sum and putting in order,

$$h_1(\theta) = \binom{2s+1}{s} - 2\sum_{\ell=1}^{s+1} \frac{\ell}{s+\ell+1} \binom{2s+1}{s+1-\ell} x^{-\ell/2} \cos \ell\theta.$$

Now, (2.8) obtains the form

$$L_{n,s}(\mathscr{E}_\varrho) = \frac{1}{2^{s+1}n^2} \int_0^{\pi} \sqrt{h_{n,s}(\varrho, \theta)}\, \mathrm{d}\theta, \tag{2.9}$$

where $h_{n,s}(\varrho, \theta) = \beta/(a_{2n} + \cos\theta)^{2s+1}$ and

$$\beta \equiv \beta_{n,s}(\varrho, \theta) = (a_{2n} - \cos\theta) \Bigg(2x^{-(s+1)/2}(a_{2n} - \cos\theta) \sum_{\ell=0}^{s} A_\ell \cos \ell\theta$$
$$- \binom{2s+1}{s}^2 + 4\binom{2s+1}{s} \sum_{\ell=1}^{s+1} \frac{\ell}{s+\ell+1} \binom{2s+1}{s+1-\ell} x^{-\ell/2} \cos \ell\theta\Bigg).$$

On the other hand, applying Cauchy's inequality to (2.9), we obtain

$$L_{n,s}(\mathscr{E}_\varrho) \leqslant \frac{\sqrt{\pi}}{2^{s+1}n^2} \left(\int_0^{\pi} h_{n,s}(\varrho, \theta)\, \mathrm{d}\theta\right)^{1/2}.$$

Since

$$\beta = -a_{2n}b^2 + 4ba_{2n}\sum_{\ell=1}^{s+1}\frac{\ell}{s+\ell+1}\binom{2s+1}{s+1-\ell}x^{-\ell/2}\cos\ell\theta$$
$$+ b^2\cos\theta - 4b\cos\theta\sum_{\ell=1}^{s+1}\frac{\ell}{s+\ell+1}\binom{2s+1}{s+1-\ell}x^{-\ell/2}\cos\ell\theta$$
$$+ 2x^{-(s+1)/2}(a_{2n}^2 - 2a_{2n}\cos\theta + \cos^2\theta)\sum_{\ell=0}^{s}A_\ell\cos\ell\theta,$$

we have that

$$\int_0^\pi h_{n,s}(\varrho,\theta)\,\mathrm{d}\theta = \int_0^\pi \frac{\beta}{(a_{2n}+\cos\theta)^{2s+1}}\,\mathrm{d}\theta$$
$$= -a_{2n}b^2J_0 + 4ba_{2n}\sum_{\ell=1}^{s+1}\frac{\ell}{s+1+\ell}\binom{2s+1}{s+1-\ell}x^{-\ell/2}J_\ell$$
$$+ b^2J_1 - 2b\sum_{\ell=1}^{s+1}\frac{\ell}{s+\ell+1}\binom{2s+1}{s+1-\ell}x^{-\ell/2}(J_{\ell-1}+J_{\ell+1})$$
$$+ x^{-(s+1)/2}\sum_{\ell=0}^{s}A_\ell\left[2a_{2n}^2J_\ell - 2a_{2n}(J_{|\ell-1|}+J_{\ell+1}) + J_\ell + \frac{1}{2}(J_{|\ell-2|}+J_{\ell+2})\right],$$

where by J_ℓ we denoted the following integrals (cf. [6, p. 127]):

$$J_\ell \equiv J_\ell(a_{2n}) = \int_0^\pi \frac{\cos\ell\theta}{(a_{2n}+\cos\theta)^{2s+1}}\,\mathrm{d}\theta.$$

It is well-known that (see [3, Eq. 3.616.7] or [6, Eq. 4.16])

$$J_\ell \equiv J_\ell(a_{2n}) = \frac{2^{2s+1}\pi(-1)^\ell x^{s-(\ell-1)/2}}{(x-1)^{4s+1}}\sum_{v=0}^{2s}\binom{2s+v}{v}\binom{2s+\ell}{\ell+v}(x-1)^{2s-v}.$$

Therefore, we have

$$L_{n,s}(\mathscr{E}_\varrho) \leqslant \frac{\sqrt{\pi\gamma}}{2^{s+1}n^2}, \tag{2.10}$$

where

$$\gamma \equiv \gamma_{n,s}(\varrho) = \binom{2s+1}{s}^2\left(J_1 - \frac{x+1}{2\sqrt{x}}J_0\right)$$
$$- 2\binom{2s+1}{s}\sum_{\ell=1}^{s+1}\frac{\ell}{s+\ell+1}\binom{2s+1}{s+1-\ell}x^{-\ell/2}\left(J_{\ell-1} - \frac{x+1}{\sqrt{x}}J_\ell + J_{\ell+1}\right)$$
$$+ x^{-(s+1)/2}\sum_{\ell=0}^{s}A_\ell\left[\left(1+\frac{(x+1)^2}{2x}\right)J_\ell - \frac{x+1}{\sqrt{x}}(J_{|\ell-1|}+J_{\ell+1}) + \frac{1}{2}\left(J_{|\ell-2|}+J_{\ell+2}\right)\right].$$

In this way, we have just proved the following result.

Theorem 2.1. *Let* $\mathscr{E}_\varrho$ $(\varrho > 1)$ *be given by* (2.1), a_{2n} *be defined by* (2.4), *and* $x = \varrho^{4n}$. *Then, for the weight function* (2.2), *the quantity* $L_{n,s}(\mathscr{E}_\varrho)$ *can be expressed in form* (2.9). *Furthermore, estimate* (2.10) *holds.*

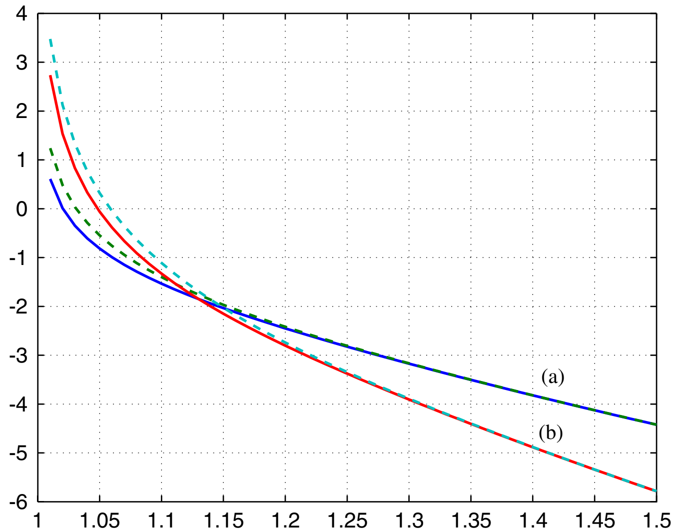

Fig. 1. Log_{10} of the values $L_{n,s}(\mathscr{E}_\varrho)$ (solid lines), with $n = 5$, given by (2.9) and its bound given by (2.10) (dashed lines) for $s = 1$ (the case (a)) and $s = 2$ (the case (b)).

Example 2.2. The function $\varrho \mapsto \log_{10}(L_{n,s}(E_\varrho))$, as well as its bound which appears on the right side in (2.10), are given in Fig. 1. Bound (2.10) are very precise especially for larger values of n, s, ϱ.

Acknowledgments

We are thankful to the referees for a careful reading of the manuscript and for their valuable comments.

References

[1] A. Ghizzetti, A. Ossicini, Quadrature Formulae, Akademie Verlag, Berlin, 1970.
[2] L. Gori, C.A. Micchelli, On weight functions which admit explicit Gauss–Turán quadrature formulas, Math. Comp. 65 (1996) 1567–1581.
[3] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980.
[4] G.V. Milovanović, Quadratures with multiple nodes, power orthogonality, and moment-preserving spline approximation, in: W. Gautschi, F. Marcellan, L. Reichel (Eds.), Numerical Analysis 2000, vol. V, Quadrature and orthogonal polynomials, J. Comput. Appl. Math. 127 (2001) 267–286.
[5] G.V. Milovanović, M.M. Spalević, Error bounds for Gauss–Turán quadrature formulae of analytic functions, Math. Comp. 72 (2003) 1855–1872.
[6] G.V. Milovanović, M.M. Spalević, An error expansion for Gauss–Turán quadratures and L^1-estimates of the remainder term, BIT 45 (2005) 117–136.
[7] G.V. Milovanović, M.M. Spalević, Bounds of the error of Gauss–Turán-type quadratures, J. Comput. Appl. Math. 178 (2005) 333–346.