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J. Math. Anal. Appl. 301 (2005) 384–385
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# On a fixed point theorem of Kirk

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Received 18 February 2004
Available online 2 September 2004
Submitted by W.A. Kirk

**Abstract**

W.A. Kirk [J. Math. Anal. Appl. 277 (2003) 645–650] first introduced the notion of asymptotic contractions and proved the fixed point theorem for this class of mappings. In this note we present a new short and simple proof of Kirk's theorem.


*Keywords:* Fixed point; Asymptotic contraction

Let $X$ be a nonempty set and $f: X \to X$ arbitrary mapping. $x \in X$ is a fixed point for $f$ if $x = f(x)$. If $x_0 \in X$, we say that a sequence $(x_n)$ defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of $f$ at point $x_0$ or that $(x_n)$ is the orbit of $f$ at point $x_0$. W.A. Kirk [1] introduced the notion of asymptotic contractions and proved the fixed point theorem for this class of mappings. Now we present a new short and simple proof of Kirk's theorem.

**Theorem 1** (W.A. Kirk [1]). *Let $(X, d)$ be a complete metric space, $f: X \to X$ continuous function and $(\varphi_i)$ sequence of continuous functions such that $\varphi_i: [0, \infty) \to [0, \infty)$ and for each $x, y \in X$, $d(f^i(x), f^i(y)) \leqslant \varphi_i(d(x, y))$. Assume also that there exists function $\varphi: [0, \infty) \to [0, \infty)$ such that for any $r > 0$, $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_i \to \varphi$ uniformly on the range of $d$. If there exists $x \in X$ such that orbit of $f$ at $x$ is bounded then $f$ has a unique fixed point $y \in X$ and all sequences of Picard iterates defined by $f$ converge to $y$.*

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.07.031

**Proof.** From the statement of the theorem it follows that $\varphi$ is continuous, because the sequence $(\varphi_i)$ is uniformly convergent. For any $x, y \in X$, $x \neq y$, we have

$$\overline{\lim}\, d\big(f^n(x), f^n(y)\big) \leqslant \overline{\lim}\, \varphi_n\big(d(x,y)\big) = \varphi\big(d(x,y)\big) < d(x,y).$$

If there exist $x, y \in X$ and $\varepsilon > 0$ such that $\overline{\lim}\, d(f^n(x), f^n(y)) = \varepsilon$ then there exists $k$ such that $\varphi(d(f^k(x), f^k(y))) < \varepsilon$, because $\varphi$ is continuous, and $\varphi(\varepsilon) < \varepsilon$. This implies that

$$\begin{aligned}\overline{\lim}\, d\big(f^n(x), f^n(y)\big) &= \overline{\lim_n}\, d\big(f^n\big(f^k(x)\big), f^n\big(f^k(y)\big)\big) \leqslant \overline{\lim_n}\, \varphi_n\big(d\big(f^k(x), f^k(y)\big)\big) \\ &= \varphi\big(d\big(f^k(x), f^k(y)\big)\big) < \varepsilon,\end{aligned}$$

which is a contradiction. So we obtain that

$$\lim d\big(f^n(x), f^n(y)\big) = 0, \tag{1}$$

for any $x, y \in X$, which implies that all sequences of Picard iterates defined by $f$, are equi-convergent and bounded.

Now let $a \in X$ be arbitrary, $(a_n)$ be a sequence of Picard iterates of $f$ at point $a$, $Y = \overline{(a_n)}$ and $F_n = \{x \in Y\colon\ d(x, f^k(x)) \leqslant 1/n,\ k = 1, \ldots, n\}$. $Y$ is bounded because $(a_n)$ is bounded. From (1) follows that $F_n$ is nonempty and since $f$ is continuous $F_n$ is closed, for any $n$. Also, we have $F_{n+1} \subseteq F_n$. Let $(x_n)$ and $(y_n)$ be arbitrary sequences, such that $x_n, y_n \in F_n$. Let $(n_j)$ be a sequence of integers, such that $\lim d(x_{n_j}, y_{n_j}) = \overline{\lim}\, d(x_n, y_n)$. Now we have

$$\begin{aligned}\lim d(x_{n_j}, y_{n_j}) &\leqslant \lim\big(d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) \\ &\quad + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)\big) \\ &= \lim \varphi_{n_j}\big(d(x_{n_j}, y_{n_j})\big) = \varphi\big(\lim d(x_{n_j}, y_{n_j})\big),\end{aligned}$$

and so $\lim d(x_{n_j}, y_{n_j}) = \varphi(\lim d(x_{n_j}, y_{n_j}))$ which implies that $\lim d(x_{n_j}, y_{n_j}) = 0$, because $Y$ is bounded. Thus $\overline{\lim}\, d(x_n, y_n) = 0$ and so $\lim d(x_n, y_n) = 0$. This implies that $\lim \operatorname{diam} F_n = 0$. By completeness of $Y$ follows that there exists $z \in X$ such that $\bigcap_{i=1}^{\infty} F_n = \{z\}$. Since $d(z, f(z)) \leqslant 1/n$ for any $n$, we have $f(z) = z$. From (1) follows that all sequences of Picard iterates defined by $f$ converge to $z$. □

**Remark.** In the statement of this Theorem in [1], assumption "$f$ is continuous" was inadvertently left out, but it was used in the proof of theorem.

## Acknowledgment

The author is grateful to the referee for helpful suggestions and to the Ministry of Science of Republic of Serbia for its support.

## Reference

[1] W.A. Kirk, Fixed points of asymptotic contractions, J. Math. Anal. Appl. 277 (2003) 645–650.